Last week:

- Introduced the field $\mathbb{Q}_p$ of p-adic numbers.
- $p > 2$, discussed the classification of quadratic forms over $\mathbb{Q}_p$.

Main result: (for $p > 2$)

$$W(\mathbb{Q}_p) \cong W(\mathbb{F}_p) \oplus W(\mathbb{F}_p)$$

$$\langle u_1, \dots, u_r, pv_1, \dots pv_s \rangle \longmapsto \left(\langle \bar{u}_1, \dots, \bar{u}_r \rangle, \langle \bar{v}_1, \dots, \bar{v}_s \rangle \right).$$

$u_i, v_j \in \mathbb{Z}_p^\times$

- Used the structure of squares in $\mathbb{Q}_p$, in particular that any $x \equiv 1 \mod p$ is a square in $\mathbb{Z}_p$. $(p > 2)$.

($\implies$ map from right to left well defined).

- Cor: any anisotropic form over $\mathbb{Q}_p$ $(p>2)$ has dim ≤ 4. $(u(\mathbb{Q}_p) \leq 4.)$

- First goal: Describe $W(\mathbb{Q}_p)$ in a more uniform way ($p=2$ ok).

<u>Def</u> A <u>local field</u> is a field E w/ a nontrivial absolute value (up to $\cong$), s.t. E is locally compact (as a metric space $\Rightarrow$ complete in particular)

Ex) $\mathbb{Q}_p$ is a local field
$\mathbb{R}$ is a local field

Classification of local fields

Let $\mathcal{E}$ be a local field.

1) If $\mathcal{E}$ is archimedean, then $\mathcal{E} \cong \mathbb{R}$ or $\mathbb{C}$.

2) If $\mathcal{E}$ has positive char, then
$\mathcal{E} \cong \mathbb{F}_q((t))$ $\quad q = p^i$ some i

("t-adic absolute value")

(unit disk $= \mathbb{F}_q[[t]]$ as a top space $\cong \prod_{\mathbb{N}} \mathbb{F}_q$)

3) If $\operatorname{char}(\mathcal{E}) = 0$ but $\mathcal{E}$ nonarch, then $\mathcal{E}$ is a finite ext of some $\mathbb{Q}_p$. (e.g. $\mathcal{E} = \mathbb{Q}_p(\sqrt{p})$.)

(Informally, $\mathbb{Q}_p$ is like $\mathbb{F}_p((t))$

except the variable is p and the rules for adding & multiplying are more complicated).

In general, there is a pretty explicit description of quadratic forms over local fields.

Uses an ingredient called the "Hilbert symbol"

Thm: Let E be a nonarchimedean local field, w/ $\mathrm{char}(E) \neq 2$. Then any quadratic form in ≥ 5 variables is isotropic and there exists a unique anisotropic form of dimension 4.

Goal: say something about Witt ring

of $\check{\mathcal{E}}$.

In general, if F any field, can consider $W(F)$. The ring $W(F)$ has an ideal $I \subseteq W(F)$ of even-dimensional forms.

I is additively generated by

$$\langle 1, -a \rangle_{= 1 - \langle a \rangle} \quad a \in F^\times.$$

Fact: There is an isomorphism

$$I/I^2 \cong F^\times/F^{\times 2}$$

which carries

$$\langle 1, -a \rangle \longmapsto a.$$

Explicitly, this isomorphism

sends the class of an even-dimensional form

$$(V, q) \longmapsto (-1)^{\dim/2} \det V.$$

also called disc

So this map

$$\mathrm{disc}: I \longrightarrow F^\times / F^{\times 2}.$$

I^2 is generated by $\{\langle 1, -a, -b, ab \rangle\}$ which has discriminant 1.

So factors over a map

$$\mathrm{disc}: I/I^2 \xrightarrow{\sim} F^\times / F^{\times 2}$$

$$F^\times / F^{\times 2} \longrightarrow I/I^2, \quad a \longmapsto \langle 1, -a \rangle$$

General facts: for any field F, consider $I \subseteq W(F)$ of even-dim forms

& the filtration by powers of I,

$$W(F) \supseteq I \supseteq I^2 \supseteq \ldots,$$

Always: $I/I^2 \cong F^\times/F^{\times 2}$

$$W(F)/I \cong \mathbb{Z}/2.$$

For a local field $\mathcal{E}$,

$I^3 = 0$. (coming from the fact that $u(\mathcal{E}) = 4$)

and $I^2/I^3 \simeq \mathbb{Z}/2$.

So for $\mathcal{E}$ local field, have a finite filtration on Witt ring

$$W(\mathcal{E}) \supseteq I \supseteq \underset{\underset{\mathbb{Z}/2}{\shortparallel}}{I^2} \supseteq 0.$$

$W(\mathcal{E})/I = \mathbb{Z}/2$ $\qquad$ $I/I^2 = \mathcal{E}^\times/\mathcal{E}^{\times 2}$.

For $\mathbb{Q}_2$, can show directly that every 5-dim form is isotropic & that any anisotropic 4-dim form

$\overset{\text{13}}{\cong} \langle 1, -5, 2, -10 \rangle.$

Need to look at congruences mod 8 (b/c any elt of $\mathbb{Z}_2$ which is $\equiv 1 \bmod 8$ is a square)

$$\Rightarrow \mathbb{Z}_2^{\times 2} = \left\{ \begin{array}{l} x \in \mathbb{Z}_2 \\ \text{w/ } x \equiv 1 \bmod 8 \end{array} \right\}.$$

$W(\mathcal{E})$ has a filtration

$\mathcal{E}$ local nonarch fld

$\mathrm{char}(\mathcal{E}) \neq 2$

where

$W(\mathcal{E})/I = \mathbb{Z}/2$

$$I/I^2 = \varepsilon^x/\varepsilon^{x2}$$

$$I^2/I^3 = 2\!\!\!/4/2.$$

$$I^3 = 0.$$

Multiplication:

$$I/I^2 \times 8/I^2 \longrightarrow I^2/I^3 = I^2 = 4/2.$$

This is a pairing

$$\varepsilon^x/\varepsilon^{x2} \times \varepsilon^x/\varepsilon^{x2} \longrightarrow 4/2$$

Hilbert symbol:

<u>Def</u> Let $\mathcal{E}$ be a local field.

Given $a, b \in \mathcal{E}^\times$, define the Hilbert symbol

$$(a,b)_{\mathcal{E}} = \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a nontrivial soln} \\ & \text{i.e. if } \langle 1, -a, -b \rangle \text{ is isotropic} \\ -1 & \text{otherwise} \end{cases}$$

$$(a,b)_{\mathcal{E}} = \Big\{ \langle 1 \underset{\substack{\uparrow \\ I}}{,} -a \rangle \otimes \langle 1, \underset{\substack{\uparrow \\ I}}{-b} \rangle$$

$\in I^2 = \mathbb{Z}/2$

$$(a,b)_\varepsilon = \begin{cases} 1 & \text{if the form } \langle 1, -a, -b, ab \rangle \text{ is hyperbolic} \\ -1 & \text{if } \langle 1, -a, -b, ab \rangle \text{ is anisotropic} \end{cases}$$

The Hilbert symbol satisfies various properties

$(a, 1)_\varepsilon = 1.$

$(a, 1-a)_\varepsilon = 1$ $\qquad 1 = a(1) + (1-a)1$

$(a, -a)_\varepsilon = 1.$

The Hilbert symbol is <u>bilinear</u>: gives a symmetric bilinear pairing

$$\mathcal{E}^\times/\mathcal{E}^{\times 2} \times \mathcal{E}^\times/\mathcal{E}^{\times 2} \to \mathbb{Z}/2$$

which is nondegenerate

Encoded in the Witt ring

b/c

$$I/I^2 \times I/I^2 \to I^2/I^3$$

$$\mathcal{E}^\times/\mathcal{E}^{\times 2} \times \mathcal{E}^\times/\mathcal{E}^{\times 2} \qquad \mathbb{Z}/2$$