Recall the construction of $\mathbb{R}$.

An element of $\mathbb{R}$ is an equivalence class of Cauchy sequences $\{x_i \in \mathbb{Q}\}_i$ i.e.

$x_i - x_j \to 0$ as $i, j \to \infty$

Here $\{x_i\}$, $\{y_i\}$ are equivalent if $x_i - y_i \to 0$

<u>Def</u> An <u>absolute value</u> on a field K is a function

$|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.

1) $|x+y| \leq |x| + |y|$

2) $|xy| = |x||y|$

3) $|x| = 0 \iff x = 0$.

For example, have usual absolute value on $\mathbb{Q}$.

Given an absolute value, K becomes a metric space (where $d(x,y) = |x-y|$). Say that K is complete if this metric space is complete (i.e., every Cauchy sequence converges).

<u>Construction</u>: Given a field K w/ absolute value $|\cdot|$, can form the <u>completion</u> $\hat{K}$ which is an extension of K & $|\cdot|$ which is complete.

$\hat{K}$ is explicitly constructed as the metric space completion (i.e., as equivalence classes of Cauchy sequences in K.

Ex) $\mathbb{R}$ are the completion of $\mathbb{Q}$ w/ usual absolute value.

Ex) Consider $\mathbb{Q}(i)$

Define an absolute value

$$|a+ib| = \sqrt{a^2+b^2}.$$

The completion is given by $\mathbb{C}$.

Observation: There are many more absolute values on $\mathbb{Q}$.

Ex) Define an absolute value

$$|x| = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Def Let p be a prime.
Given $x \in \mathbb{Q}$, define the ***p-adic valuation***

$\mathrm{ord}_p(x)$ = # of factors of p in numerator
— # of factors in denominator.

(if $x = p^r \frac{m}{n}$ m, n coprime to p
$\mathrm{ord}_p(x) = r$.

$\mathrm{ord}_p(0) = \infty$.

$\mathrm{ord}_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$.

1) $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$
2) $\mathrm{ord}_p(x+y) \geq \min(\mathrm{ord}_p(x), \mathrm{ord}_p(y))$

3) $\mathrm{ord}_p(x) = \infty \iff x = 0$.

Define $|x|_p = p^{-\mathrm{ord}_p(x)}$

<u>p-adic absolute value</u>. We have

1) $|xy|_p = |x|_p |y|_p$
2) $|x+y|_p \leq \max(|x|_p, |y|_p)$
3) $|x|_p = 0 \iff x = 0$.

These are close to the axioms of an absolute value, except instead of triangle inequality, have $|x+y|_p \leq \max(|x|_p, |y|_p)$.

(called <u>nonarchimedean property</u>).

e.g. $|\cdot|_p$ is ≤ 1 on $\mathbb{Z}$.

<u>**Def**</u> The p-adic numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.

Entirely analogous to the construction of $\mathbb{R}$ as completion w.r.t. usual absolute value.

In $\mathbb{R}$, can represent any real number via a decimal expansion

<u>Prop</u> Any p-adic number has a <u>unique</u> p-adic expansion

$$\sum_{i \gg -\infty}^{\infty} a_i p^i \qquad a_i \in \{0, 1, \ldots, p-1\}.$$

- Note that the expansion goes in the opposite direction as decimal expansions on $\mathbb{R}$
- This <u>converges</u> b/c $|p|_p = 1/p$.
- Need to use base p (or $p^2, \ldots$)
- Unique (unlike decimal expansion).

How to produce p-adic expansion?

Suppose given a p-adic number,

⇝ Cauchy sequence of rational numbers (in p-adic topology).

⇝ Suppose I have a Cauchy sequence of integers.

⇝ Any integer has a <u>finite</u> base p expansion.

⇝ Given a Cauchy sequence, the collection of base p expansions stabilizes in any range → gives well-defined p-adic expansion

Ex) $\frac{1}{1-p} = 1 + p^1 + p^2 + p^3 + \cdots$

Comparison between $\mathbb{Q}_p$ & $\mathbb{R}$.

1) Both are locally compact.

2) $\mathbb{Q}_p$ contains a subring

$$\mathbb{Z}_p = \left\{ \begin{array}{l} x \in \mathbb{Q}_p \text{ s.t.} \\ |x|_p \leq 1 \end{array} \right\}.$$

(the unit disk).

$\rightsquigarrow$ those p-adic expansions going from 0 to ∞.

$\mathbb{Z}_p$ is a commutative ring

which is the inverse limit

$\varprojlim \mathbb{Z}/p^n \equiv \left\{ \begin{array}{l} \text{compatible systems} \\ \text{of congruence} \\ \text{classes mod } p^n, \\ \text{each } n. \end{array} \right\}$

3) $\mathbb{R}$ is connected.

$\mathbb{Q}_p$ is totally disconnected.

ex) $\mathbb{Z}_p$ = unit disk in $\mathbb{Q}_p$ is both open & closed

$\mathbb{Z}_p$ is homeomorphic to a Cantor set.

4) Only complete archimedean fields are $\mathbb{R}$ & $\mathbb{C}$.

Lots of complete nonarchimedean fields.
Lots of ways to extend $\mathbb{Q}_p$ to complete NA field.

For some questions, one can regard $\mathbb{R}$ and $\{\mathbb{Q}_p\}_{p \text{ prime}}$ on a similar footing and consider them together.

<u>Theorem</u> (Ostrowski)
Any nontrivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ or $|\cdot|_\infty$.
(archimedean)

the same up to raising to a power

Ex) Product formula:
If $x \in \mathbb{Q}^\times$, then

$$\prod_{p=2,3,5,7,\dots,\infty} |x|_p = 1.$$

(all but finitely many factors are 1).

Later: quadratic reciprocity is a product formula over all p (including ∞).

<u>Theorem</u> (Hasse-Minkowski)
A quadratic form over $\mathbb{Q}$

is isotropic $\Longleftrightarrow$ isotropic over $\mathbb{Q}_p$, each p & over $\mathbb{R}$.

Two quadratic forms over $\mathbb{Q}$ are $\cong$ if & only if they are isomorphic over each $\mathbb{Q}_p$ and over $\mathbb{R}$.

Ex) Given a polynomial equ $f(x) = 0$ $f(x) \in \mathbb{Z}[x]$

Suppose you can solve $f(x) \equiv 0 \mod p^n$ each n.

Then you can solve $f(x) = 0$ in $\mathbb{Z}_p$.