(Fix a field F, char $\neq 2$.)

(Witt)
__Theorem__ Let V, W, Z be quadratic forms over F. Suppose $V \oplus Z \cong W \oplus Z$ (as quadratic spaces). Then $V \cong W$.

Pf: Z is diagonalizable. By induction, can assume $Z = \langle a \rangle$.

So suppose given an isomorphism

$$T: V \oplus \underset{\substack{\uparrow \\ e_1}}{\langle a \rangle} \cong W \oplus \underset{\substack{\uparrow \\ f_1}}{\langle a \rangle}$$

In general, $T(e_1) \neq f_1$.

(If $T(e_1) = f_1$, then T takes $V = e_1^\perp$ into $W = f_1^\perp$ ✓.

Observe $T(e_1)$ & $f_1 \in W \oplus \langle a \rangle f_1$
are two vectors w/ same self-dot product.

Apply prop below to $W \oplus \langle a \rangle$ to obtain $U \in O(W \oplus \langle a \rangle)$ s.t. $U(T(e_1)) = f_1$.
So $U \circ T : V \oplus \langle a \rangle \cong W \oplus \langle a \rangle$ which carries $e_1 \rightsquigarrow f_1$ & have indeed $V \cong W$. ✓

Prop: If (V, q) is any quadratic space, then $O(V,q)$ acts transitively on the set $\{v \in V \mid v \cdot v = a\}$ for any $a \neq 0$ in F.

Pf: Suppose $v, w \in V$
s.t. $v \cdot v = w \cdot w = a \neq 0$.
Want: $T \in O(V,q)$ s.t. $Tv = w$.
Consider the vectors $v \pm w$.
Observe $(v+w) \cdot (v-w) = 0$,

$v+w \perp v-w$.

As a consequence, either $v+w$ or $v-w$ must be anisotropic (or $2v$ would be isotropic). Suppose $v-w$ is anisotropic.

Then: consider reflection through $v-w$.

$$R_{v-w}: \begin{array}{l} v-w \longmapsto w-v \\ v+w \longmapsto v+w \end{array}$$

$\rightsquigarrow$ carries $v \longmapsto w$ as desired.

If $v-w$ is isotropic, then $v+w$ is anisotropic & compose reflection w/ -1. ✓

Witt's theorem is fundamental b/c enables one to write quadratic form in a "normal form"

Hyperbolic forms

The hyperbolic plane (H_2) is the quadratic form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Has a basis e_1, e_2 which are isotropic & s.t. $e_1 \cdot e_2 = 1$.

Prop: Let (V, q) be an isotropic form. Then $V \cong H_2 \oplus V'$ for some V'.
(Note: V' determined up to iso.)

Pf: Let $v \in V$ be isotropic & nonzero.
Find $w \in V$ s.t. $w \cdot v \neq 0$.
Then v, w span a hyperbolic plane,

(replace w with $w - \lambda v$ for some $\lambda \in F$).

This gives $H_2 \subseteq V$, then $V' = H_2^{\perp}$.

<u>Cor</u> Any quadratic form

$$(V, q) \cong H_2^{\oplus r} \oplus (V', q')$$

where (V', q') is <u>anisotropic</u> & determined up to isomorphism.

(Recall: V is <u>isotropic</u> if $\exists\, v \in V \setminus \{0\}$ w/ $v \cdot v = 0$, <u>anisotropic</u> otherwise.)

The fundamental question in the theory of quadratic forms is to determine when a quadratic form is isotropic. (Stronger than classifying forms up to $\cong$.)

Why?

Suppose (V, q) & (W, q') are quadratic forms, want to know if they're isomorphic.

If V, W are isotropic $\rightsquigarrow$ write them in normal form
$\Rightarrow V \cong W$ iff anisotropic parts are iso & # of hyperbolic parts iso.

May as well assume V, W anisotropic.
Pick $v \in V$ and let $a = v.v.$
By construction, $V \oplus \langle -a \rangle$ isotropic.
Q: Is $W \oplus \langle -a \rangle$ isotropic? If not, $V \not\cong W$.
If yes, split off hyperbolic planes $W \oplus \langle -a \rangle$ & from $V \oplus \langle -a \rangle$, ask if complements are isomorphic.

Indirectly determing when $V \cong W$.

Application of idea:
Let's classify quadratic forms over $\mathbb{F}_q$.

<u>Def</u> Given a field F, the <u>u-invariant</u> of F is the largest dimension of an anisotropic form over F (or ∞).

Roughly, the smaller the u-invariant, the easier it is to classify quadratic forms.

<u>Theorem</u>: Any quadratic form of dimension ≥ 3 over $\mathbb{F}_q$ is

isotropic. (So the u-invariant is 2.)

Pf: Assume the form is $\langle a, b, c \rangle$.

Want: a nonzero soln to the equation

$$ax^2 + by^2 + cz^2 = 0.$$

Take $z = 1$, so we get

$$ax^2 = -c - by^2.$$

Make a counting argument:
LHS depends on x, RHS depends on y.

But $\{ax^2\}_{x \in \mathbb{F}_q}$ has size $\frac{q+1}{2}$

Similarly $\{-c - by^2\}_{y \in \mathbb{F}_q}$ has size $\frac{q+1}{2}$!

So by pigeonhole, can solve eqn.

Ex) Let $v \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$.
Then $x^2 - vy^2 = 0$ has no solution.

(More generally
Finite fields are C_1,
meaning $\Longrightarrow$ if $P(X_1, \ldots, X_n)$
is homogeneous of deg d in
n variables, & $n > d$, then
there is a nontrivial zero.

Now let's classify quadratic forms over $\mathbb{F}_q$ (or over any field of u-invariant ≤ 2).

Def: Let (V, q) be a quadratic form (over F). The *discriminant* is defined by choosing a basis $\{e_i\}_{1 \le i \le n}$ of V,

$$\mathrm{disc}(V, q) = \det(e_i \cdot e_j) \in F^{\times}/F^{\times 2}.$$

Ex) $F = \mathbb{F}_q$, then $\mathbb{F}_q^{\times}/\mathbb{F}_q^{\times 2} = \mathbb{Z}/2$.

Prop: Over a field of u-invariant ≤ 2 (i.e. 3-dim forms are isotropic), then quadratic forms are classified by dimension & discriminant.

Pf: Can assume dim ≤ 2.

Suppose V, W are quadratic spaces of dim 2 w/ the same discriminant. Then $V \cong W$.

If $d =$ discriminant, then

$$V \cong \langle 1, d \rangle \cong W.$$

Why?: Because V has u-invariant ≤ 2, can find a vector $v \in V$ w/ $v \cdot v = 1$.

Let $v' \in v^{\perp}$.

Diagonalize.

Can get any dim & discriminant

b/c

$\langle 1, 1, \ldots, 1, Q \rangle$