Lattice Points and Polygons

This problem session is modeled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su, Mohamed Omar, and Nick Pippenger.

A1: Suppose you have a polygon of perimeter 12, whose vertices are all lattice points in the plane and whose sides all have integer lengths. Show that its area can be $3, 4, 5, 6, 7, 8$ or 9.
(Inspired by Wagon)

A2: (a) Show there is no equilateral triangle whose vertices are all lattice points in the plane.

(b) Show that there are equilateral triangles with vertices that are lattice points in three dimensions.

A3: (a) A group of explorers live on the planet Qubit which is in the shape of a perfect cube with a side of length q. They wish to walk from the north pole (which is a vertex of the cube) to the south pole (which is the opposite vertex). What is the minimum distance they need to walk?

(b) The moon of Qubit is a rectangular solid with sides s, $2s$ and $3s$. What is the minimum distance one needs to walk to get from one vertex of the moon to the vertex opposite it?

A4: (a) Prove that any pentagon whose vertices are lattice points must have an area greater than or equal to $3/2$. Can you find an example where the area is $3/2$?

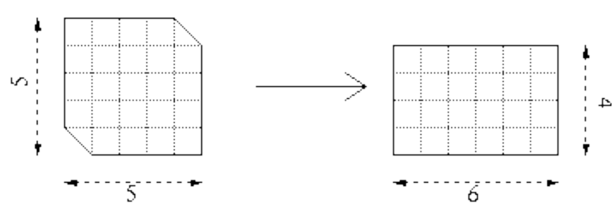
(b) Show that among any five lattice points in the plane that there are a pair whose midpoint is a lattice point also.

(c) Prove that any convex pentagon whose vertices (no three of which are collinear) are lattice points must have area greater than or equal to $5/2$. (Putnam 1990)

A5: Suppose a cube has vertices that are lattice points. Show that the length of its side must be an integer. (Gelca & Andreescu)

And for a little bit of variety...

A6: Paper & Scissors You have a 5×5 piece of paper and a pair of scissors. Two diagonally opposite corners of this paper are truncated as shown in the diagram below. Show how to cut up the 5×5 paper into two pieces, that can be arranged to form a 6×4 rectangle. (Wu)



Hints:

1. Start with rectangles and triangles with integer length sides
2. (a) Can you remember/derive the formula for the area of an equilateral triangle in terms of its side length?
   (b) Useful fun fact: A regular tetrahedron can be nicely inscribed in a cube.
3. Think about unfolding a map of the planet. Why does the shortest path between the vertices look like a line segment between two lattice points?
4. (a) Pick's Theorem is your friend here. (b) Think about the parity (even/odd) of the coordinates of the vertices. (c) Use part (b).
5. Can you compute the volume of the cube in two different ways? The triple product formula for the volume of a parallelepiped is useful.