$\mathbb{A}^1$-homotopy theory and the Weil Conjectures IV:

Relating $X(\mathbb{R})$ and $|X(\mathbb{F}_{q^m})|$

new work joint with Bachmann, Bilu, Ho, Srinivasan, Vogt

Spectra

Real realization: $r^{\mathbb{R}}: SH(\mathbb{R}) \longrightarrow Sp$

constructed as follows

$$\begin{tikzcd} Sm_{\mathbb{R}} \arrow[r, "\otimes"] \arrow[d, "\otimes\ Y"'] & Spaces \\ Fun(Sm_{\mathbb{R}}^{op}, Spaces) \arrow[ur, dashed] & \end{tikzcd}$$

$X \longmapsto X(\mathbb{R})$

LKE lax $\otimes$

preserves colimits

So $LKE(\Delta^n_+ \wedge X_+) = \Delta^n_+ \wedge X(\mathbb{R})_+$

$\otimes$ on representables $\quad r^{\mathbb{R}}(X) \otimes r^{\mathbb{R}}(Y) \xrightarrow{\sim} r^{\mathbb{R}}(X \otimes Y)$

$\Rightarrow$ LKE $\otimes$

Bousfield localization imposes additional weak equivalences

$\mathcal{C} \to \mathcal{C}[W^{-1}] \qquad Fun(\mathcal{C}[W^{-1}], \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$

Fully faithful

essential image consists of functors sending W to weak equivalences

$\supset$

$$\mathrm{Fun}(\mathrm{Sm}_{\mathbb{R}}^{op}, \mathrm{Spaces}) \xrightarrow{L_{Mot}} \mathrm{Spc}(\mathbb{R})$$

Bousfield localization

Nis descent

$\mathbb{A}^1$-invariance

$r^{\mathbb{R}}(\mathbb{A}^1) \simeq *$ and topological excision

$\rightsquigarrow$ $r^{\mathbb{R}}: \mathrm{Spc}(\mathbb{R}) \longrightarrow \mathrm{Spaces}$ $\otimes$

$r^{\mathbb{R}}(\mathbb{P}^1) \simeq S^1$

$\rightsquigarrow$ $r^{\mathbb{R}}: \mathrm{SH}(\mathbb{R}) = \mathrm{Spc}(\mathbb{R})[(\mathbb{P}^1)^{\otimes -1}] \xrightarrow{\otimes} \mathrm{Sp} = \mathrm{Spaces}[(S^1)^{\otimes -1}]$

Existence of realizations/

Universal property SH: (Robalo)

B q.c.q.s. base scheme $\mathrm{Sm}_B \to \mathrm{SH}(B)$

$$\mathrm{Fun}^{\otimes}(\mathrm{SH}(B), \mathcal{D}^{\otimes}) \longrightarrow \mathrm{Fun}(\mathrm{Sm}_B, \mathcal{D}^{\otimes})$$

is fully faithful. It's essential image consists of $\mathcal{F}: \mathrm{Sm}_B \to \mathcal{D}$ satisfying

1. Nisnevich descent
2. $\mathbb{A}^1$-htpy invariance
3. $\mathrm{cofib}(\mathcal{F}(\infty: B \to \mathbb{P}^1))$ is $\otimes$-invertible in $\mathcal{D}$

$\Rightarrow$ <u>Existence of $\mathbb{R}$-realization:</u>

$$Sm_k \longrightarrow \text{Spaces}$$
$$X \longmapsto X(\mathbb{R})$$

induces a symmetric monoidal functor

$$r^{\mathbb{R}}: SH(\mathbb{R}) \longrightarrow Sp$$

<u>Rmks:</u> i) $x: \operatorname{Spec}\mathbb{R} \to B$ $\quad B$ q.c.q.s.

Defines a real realization

$$SH(B) \xrightarrow{x^*} SH(\mathbb{R}) \xrightarrow{r^{\mathbb{R}}} Sp$$

ii) Can define complex realization replacing $\mathbb{R}$ by $\mathbb{C}$

<u>Thm</u> (Morel) k field $\quad \operatorname{End}_{SH(k)}(\mathbb{1}) \cong GW(k)$

$$\text{sign}: GW(\mathbb{R}) \longrightarrow \mathbb{Z}$$
$$\begin{bmatrix} \overbrace{1 \cdots}^{a} & \\ & \overbrace{-1 \cdots -1}^{b} \end{bmatrix} \longmapsto a-b$$

For $k \subset \mathbb{R}$

$$\begin{tikzcd}
\mathrm{End}_{sp}(1) \arrow[d, "\deg"'] & \mathrm{End}_{SH(k)}(1) \arrow[l, "r^{\mathbb{C}}"'] \arrow[r, "r^{\mathbb{R}}"] \arrow[d, "\substack{\text{Morel's} \\ \mathbb{A}^1\deg}"] & \mathrm{End}_{sp}(1) \arrow[d, "\deg"] \\
\mathbb{Z} & GW(k) \arrow[l, "\mathrm{rank}"'] \arrow[r, "\mathrm{Sign}"] & \mathbb{Z}
\end{tikzcd}$$

<u>Def</u>: (Bilu – Ho – Srinivasan – Vogt – W.)

For $X \in Sm_B$ dualizable and

$F: X \to X$ endomorphism, let

$$\mathrm{End}_{SH(B)}(1)[[t]] \ni d\log Z_X^{\mathbb{A}^1}(t) = \sum_{m=1}^{\infty} \mathrm{Tr}_{SH(B)}(F^m)\, t^{m-1}$$

<u>Prop</u>: For $f: X \to X$ with $X \in Sm_{\mathbb{R}}$

$$\mathrm{Sign}\ d\log Z_X^{\mathbb{A}^1}(t) = \frac{d}{dt} \log \prod_{i=0}^{d} \det(1 - tF|_{H^i(X(\mathbb{R});\mathbb{Q})})^{(-1)^{i+1}}$$

<u>pf</u>:

$$\mathrm{Sign}\ d\log Z_X^{\mathbb{A}^1}(t) = \sum_{m=1}^{\infty} \mathrm{Sign}\, \mathrm{Tr}_{SH(\mathbb{R})}(F^m)\, t^{m-1}$$

$$= \sum_{m=1}^{\infty} \mathrm{Tr}_{Sp}\left(r^{\mathbb{R}}(F^m)\right) t^{m-1}$$

$$= \sum_{m=1}^{\infty} H^* \mathrm{Tr}_{Sp}\left(r^{\mathbb{R}}(F^m)\right) t^{m-1}$$

$$= \sum_{m=1}^{\infty} \mathrm{Tr}\, H^* r^{\mathbb{R}}(F^m)\, t^{m-1}$$

$$= \sum_{m=1}^{\infty} \sum_{i} (-1)^i \,\mathrm{Tr}\, F^m\big|_{H^i(X(\mathbb{R}),\mathbb{Q})} t^{m-1}$$

$$= \sum_{i} (-1)^i \sum_{m=1}^{\infty} \mathrm{Tr}\, F^m\big|_{H^i(X(\mathbb{R}),\mathbb{Q})} t^{m-1}$$

$$= \sum_{i} (-1)^{i+1} \frac{d}{dt} \log \det\left(1 - tF\big|_{H^i}\right)$$

$\square$

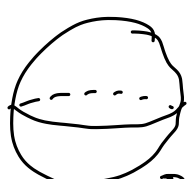
<u>Def</u>: Restriction of scalars $\quad X \in Sm_k \quad k \subset L$ étale

$$\mathrm{Res}_{L/k} X(E) = X(E \otimes_k L)$$

$f: S' \to S \quad f_* h_X = h_{\mathrm{Res}_{S'/S} X}$

Yoneda

<u>ex ex</u> $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1$



Ex (i): $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1 \circlearrowleft 1$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft 1$

$\mathbb{C}$-points $\quad S^2 \times S^2 \qquad S^2 \times S^2$

$\mathbb{R}$-points $\quad S^2 \qquad S^1 \times S^2$

have same classical but not same $\mathbb{A}^1$ logarithmic zeta functions

Lifting $X/\mathbb{F}_q$ to characteristic 0:

$$\begin{tikzcd}
X_F \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "sm, pr"] & X \arrow[l, hook'] \arrow[d, "sm, pr"] \\
\mathrm{Spec}\, F \arrow[r] & \mathrm{Spec}\, A & \mathrm{Spec}\, \mathbb{F}_q \arrow[l, hook']
\end{tikzcd}$$

F field char 0

Rmk: • Don't have computation $\mathrm{End}_{SH(A)}(1) \cong GW(A)$ for A not a field

Thm (Calmès, Harpaz, Nardin) Let B be a regular Noetherian scheme. Then symmetric Grothendieck-Witt theory is represented by GW^s_B in $SH(B)$.

$B \mapsto GW^s_B$ is compatible with pull back.

<u>Thm</u> (Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle I/I)

D Dedekind domain, $\pi_0(GW_D^s)_0 \cong GW(D)$

<u>Thm</u> (ref. Milnor–Husemuller Ch IV "Symmetric bilinear forms") Let D be a Dedekind domain with fraction field F. Then

$$0 \to GW(D) \to GW(F) \xrightarrow{\oplus \partial_p} \bigoplus_{\text{primes } p} GW(D/p)$$

$$\partial_p : GW(F) \xrightarrow[\text{pullback}]{} GW(F_p^\wedge) \to GW(D/p)$$

$F_p^\wedge = \lim\limits_{\leftarrow i} D/p^i$ is a discrete valuation ring

uniformizer π

$$GW(F_p^\wedge) \to GW(D/p)$$
$$\langle \pi^i u \rangle \mapsto \langle u \rangle$$

$u \in D^\times$

$i \equiv 1 \ (2)$

Ex $\mathbb{Z}[\langle -1\rangle, \langle p\rangle \mid p \mid d] \twoheadrightarrow GW(\mathbb{Z}[\frac{1}{d}])$
p prime

Ex: $GW(\mathbb{Z}) \cong \mathbb{Z}[\langle -1\rangle] \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle -1\rangle$

Thm: $X/\mathbb{F}_q$ sm, proper. A lift of Fröbenius to $\mathcal{O}_K$
K global field determines a lift of $d\log Z_X^{\mathbb{A}^1}(t)$
to $GW(\mathcal{O}_K)[\![t]\!]$. If $\mathcal{O}_K$ has a real place,

$$\operatorname{Sign} d\log Z_X^{\mathbb{A}^1}(t) = \frac{d}{dt} \log \prod_{i=0}^{d} \det(1 - tF|_{H^i(X(\mathbb{R});\mathbb{Q})})^{(-1)^{i+1}}$$

Ex (ii): Toric varieties $/\mathbb{F}_q$. Fröbenius lifts to $\mathbb{Z}$

Pf: X dualizable in $SH(\mathcal{O}_K)$ Dubouloz-Déglise-Østvaer
$GW^s_{\mathcal{O}_K} \in SH(\mathcal{O}_K)$ Calmès-Harpaz-Nardin, #9
$\pi_0 GW^{gs} \xrightarrow{\cong} GW(\mathcal{O}_K)$ #9 III

$$\mathrm{Lift} := \sum_{m\geq 0} \left(\mathrm{Tr}\, f^m \wedge GW^s_{\mathcal{O}_K}\right) t^{m-1} \in GW(\mathcal{O}_K)[[t]]$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\sum_{m\geq 0} \left(\mathrm{Tr}\, f^m \wedge GW^s_{\mathbb{F}_q}\right) t^{m-1} \in GW(\mathbb{F}_q)[[t]]$$

$$\parallel$$

$$\sum_{m\geq 0} (\mathrm{tr}\, f^m)\, t^{m-1}$$

$\square$

<u>Cor</u> (Criterion for $\mathbb{R}$-point) Let $X/\mathbb{F}_q$ admit a lift of Fröbenius to $\mathbb{Z}$. S'pose $q \equiv 3 \bmod 4$

$\mathcal{X}(\mathbb{R}) = \emptyset \Rightarrow$

$\forall m$ $\quad \mathrm{disc} \sum_{d|m} \alpha(d)\, \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1\rangle \in (\mathbb{R}^*)^2 \Leftrightarrow |X(\mathbb{F}_{q^m})| \equiv 0 \pmod 4$

$(\star\star)$

$\mathrm{disc} \sum_{d|m} \alpha(d)\, \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1\rangle \notin (\mathbb{R}^*)^2 \Leftrightarrow |X(\mathbb{F}_{q^m})| \equiv 2 \pmod 4$

$\alpha(d) = \#$ points p of X with $k(p) \cong \mathbb{F}_{q^d}$

<u>Rmk</u>: $(\star\star)$ is congruence conditions on $|X(\mathbb{F}_{q^m})|$

$\mathbb{P}^2$ does not satisfy $(\star\star)$

<u>pf of cor</u>:

$\mathrm{Sign}\left(d\log Z^{\mathbb{A}^1}_{\mathcal{X}}\right) = 0$ because $\mathcal{X}(\mathbb{R}) = \emptyset$

$$\Rightarrow \operatorname{Sign} \operatorname{Tr}^{\mathbb{A}'}(F^m) = 0$$

$$\Rightarrow \operatorname{Tr}^{\mathbb{A}'}(F^m) = (\langle 1\rangle + \langle -1\rangle)\frac{|X(\mathbb{F}_{q^m})|}{2}$$

$\Rightarrow$ $|X(\mathbb{F}_{q^m})|$ even and

$$\operatorname{disc} \operatorname{Tr}^{\mathbb{A}'}(F^n) = (-1)^{\frac{|X(\mathbb{F}_{q^m})|}{2}} \in (\mathbb{R}^*)^2$$

$$\Leftrightarrow |X(\mathbb{F}_{q^m})| \equiv 0 \ (4)$$

$$\operatorname{disc} \operatorname{Tr}^{\mathbb{A}'}(F^m) \overset{\text{Hoyois}}{=} \operatorname{disc}\left(\sum_{d|m} \alpha(d) \operatorname{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1\rangle\right) \quad \square$$

<u>Application</u> Lower bounds on $b_i(X(\mathbb{R}))$:

Let $X/\mathbb{F}_q$ have a lift to $\mathbb{Z}$ with a lift of Fröbenius

<u>ex</u>: $X = \operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ $\quad u \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$ non-square

$$X = \{5x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{P}^3_{\mathbb{F}_7}$$

$$d\log Z_X^{\mathbb{A}'}(T) = \frac{d}{dt}\log\frac{1}{(1-T)(1-q_\varepsilon^2 T)}$$

$$+ \langle -u \rangle \frac{d}{dt} \log \frac{1}{1 - q_\varepsilon \langle u \rangle t} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{1 + q_\varepsilon t}$$

Sign $d\log Z_x^{\mathbb{A}^1}(T) = \dfrac{d}{dt} \log \dfrac{1}{(1-T)^2}$

thm

$\Rightarrow b_0(\mathbb{R}) + b_2(\mathbb{R}) \geq 2$