- Morel-Sawant cellular $\mathbb{A}^1$-homology:

  k perfect field

  Say X "cellular" when equipped with

$$\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \cdots \subseteq \Omega_d = X$$

$$\Omega_i - \Omega_{i-1} = \coprod^{b_i} \mathbb{A}^{d-i}$$

$C_*^{cell}(X)$

$$\cdots \to \tilde{H}_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \to \tilde{H}_{i-1}^{\mathbb{A}^1}(\Omega_{i-1}/\Omega_{i-2}) \to \cdots$$

$$\tilde{H}_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \cong \bigoplus^{b_i} \underline{K}_i^{MW}$$

$Ab(k)$ = Nisnevich sheaves abelian groups

$$Sm_k^{op} \longrightarrow Ab$$

$\cup$

HI = strictly $\mathbb{A}^1$ invariant sheaves, so

- $M(\mathbb{A}^1_U) \cong M(U)$ $\mathbb{A}^1$-invariant
- $H^i_{Nis}(U \times \mathbb{A}^1, M) \cong H^i_{Nis}(U, M)$ strictly $\mathbb{A}^1$-invariant

$\forall\, i, U$

$D(HI)$ Derived category: In chain complexes $Ch(HI)$ in HI,
a quasi-isomorphism is
$C_* \to C'_*$ inducing iso on all π_*
$D(HI)$ is formed by inverting
quasi-isomorphisms in $Ch(HI)$

$$\downarrow$$

$D_{\mathbb{A}^1}(k)$ $\mathbb{A}^1$-localized derived category:
$D_* \in Ch(Ab(k))$ is $\mathbb{A}^1$-local if

$$\mathrm{Hom}_{D(Ab(k))}(C_*, D_*) \xrightarrow{\sim} \mathrm{Hom}_{D(Ab(k))}(C_* \otimes \mathbb{Z}(\mathbb{A}^1), D_*)$$

$C_* \to C'_*$ is an $\mathbb{A}^1$-quasi-isomorphism if for all $\mathbb{A}^1$-local D_*

$$\mathrm{Hom}_{D(Ab(k))}(C'_*, D_*) \xrightarrow{\sim} \mathrm{Hom}_{D(Ab(k))}(C_*, D_*)$$

invert $\mathbb{A}^1$-quasi-isomorphisms in $D(Ab(k))$

reference: Morel "$\mathbb{A}^1$-algebraic topology over a field" 6.2

$$\begin{array}{ccc} L_{mot}\mathbb{Z}[X] & \longrightarrow & C^{cell}_*(X) \\ \cap & & \cap \\ D_{\mathbb{A}^1}(k) & & D(HI) \end{array}$$

sheaves

$$\mathrm{Hom}_{D(HI)}(C^{cell}_*, C'_*) \longrightarrow \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(L_{mot}\mathbb{Z}[X], C'_*)$$

$\Longrightarrow$ C^{cell}_* is functorial

joint in progress with T. Bachmann
on a generalization of C_*^{cell} to non-cellular varieties following Morel–Sawant, Bondarko

Heuristic: construct a left adjoint

$$D(HI) \subset \cancel{D_{\mathbb{A}^1}(k)}$$

$$SH(k) = Spc(k)_*[(\mathbb{P}^1)^{-1}]$$

Warning: Not same notation as in Ayoub's talk, where this is $\mathcal{H}(k)$

① Some Formalism

$\mathcal{C}^{\otimes}$ presentably symmetric monoidal stable ∞-cat with $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ a compatible t-structure

$$\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$$

$Ch(\mathcal{C}^{\heartsuit})^{\otimes}$ = ∞-category of chain complexes with chain homotopy equivalences

1a) Construct K such that

$$\begin{tikzcd}
\mathcal{C}^{\heartsuit} \arrow[r] \arrow[d] & \mathcal{C} \\
Ch(\mathcal{C}^{\heartsuit}) \arrow[ur, dashed]
\end{tikzcd}$$

K lax-$\otimes$ ⇜ "1-categorical heuristic: $K(X \otimes Y) \leftarrow K(X) \otimes K(Y)$"
preserves finite limits

$$\begin{tikzcd} \mathcal{C}^{\heartsuit} \arrow[r, shift right, "\text{right adjoint}"'] & \mathcal{C}_{\geq 0} \arrow[l, bend right, "\pi_0^{\otimes}"'] \end{tikzcd}$$

symmetric monoidal

$$\pi_0^{\otimes} \Rightarrow \mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0} \quad \text{lax } \otimes$$

$$\begin{tikzcd} \mathcal{C}^{\heartsuit} \arrow[r, "\text{lax } \otimes"] \arrow[d, "\otimes"] & \mathcal{C}_{\geq 0} \\ Ch^{b}_{\geq 0}(\mathcal{C}^{\heartsuit}) \arrow[ur, dashed] & \end{tikzcd}$$

bounded

LKE = left Kan extension

has canonical lax $\otimes$ structure

ref: Nikolaus "stable ∞-operads and the multiplicative Yoneda embedding" Cor 3.8

$$Ch^{b}_{\geq 0}(\mathcal{C}^{\heartsuit}) \simeq P_{\Sigma, f}(\mathcal{C}) \subset P_{\Sigma}(\mathcal{C})$$

smallest prestable containing representables

$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spaces})$ preserving finite products

reference: Prop 7.4.5 and proof Bunke-Cisinski-Kasprowski-Winges

and $\mathcal{C}^{\heartsuit} \longrightarrow Ch^{b}_{\geq 0}(\mathcal{C}^{\heartsuit})$ is Yoneda embedding

$\rightsquigarrow$ LKE preserves finite (co)limits

1b) Stabilize LKE to produce K:

$\mathcal{C}$ pointed ∞-category which admits finite limits

Spectrum objects

$$\mathrm{Sp}(\mathcal{C}) = \lim_{\longleftarrow} \cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$$

Spanier-Whitehead

$$\mathrm{SW}(\mathcal{C}) = \operatorname{colim}_{\longrightarrow} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots$$

Nikolaus "stable ∞-operads and the multiplicative Yoneda embedding"

Preserve finite limits

$$(\star)\quad \mathrm{Fun}^{\mathrm{lex}}_{\mathrm{lax}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{lex}}_{\mathrm{lax}}(\mathcal{D}, \mathcal{C})$$

$$\mathrm{Ch}^b(\mathcal{C}^\heartsuit) \simeq \mathrm{SW}(\mathrm{Ch}^b_{\geq 0}(\mathcal{C}^\heartsuit))$$

ref: Lurie Higher Algebra Rmk C.1.1.6

$$\mathrm{SW}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathrm{SW}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{Ind}(\mathcal{C}))$$

$$\begin{array}{c} \cancel{\mathrm{Ind}}(\mathrm{Ch}^b_{\geq 0}(\mathcal{C}^\heartsuit)) \\ \uparrow \Omega \quad \text{lax}\otimes \\ \mathrm{Sp}(\mathrm{Ind}(\mathrm{Ch}^b_{\geq 0}(\mathcal{C}^\heartsuit))) \\ \uparrow \otimes \\ \mathrm{Ch}^b(\mathcal{C}^\heartsuit) \end{array}$$

lax $\otimes$, preserves finite limits

By $(\star)$ have

$$\begin{tikzcd} \mathrm{Ch}^b_{\geq 0}(\mathcal{C}^\heartsuit) \arrow[r, "\mathrm{LKE}"] & \mathcal{C}_{\geq 0} \\ \mathrm{Ch}^b(\mathcal{C}^\heartsuit) \arrow[u, "\Omega"'] \arrow[r, dashed, "K"'] & \mathrm{Sp}(\mathcal{C}_{\geq 0}) \simeq \mathcal{C} \arrow[u] \end{tikzcd}$$

lax $\otimes$, preserves finite limits

1c) $K^{\mathrm{Pro}} : \mathrm{Pro}(\mathrm{Ch}^b(\mathcal{C}^\heartsuit)) \longrightarrow \mathrm{Pro}(\mathcal{C})$

lax $\otimes$
preserves limits

Adjoint functor theorem $\Rightarrow$ K^{Pro} admits left adjoint

$C_*^{\mathrm{Pro}} : \mathrm{Pro}(\mathcal{C}) \longrightarrow \mathrm{Pro}(\mathrm{Ch}^b(\mathcal{C}^\heartsuit))$

oplax $\otimes$

Pro chains

heuristic: $C_*^{\mathrm{Pro}}(X) \otimes C_*^{\mathrm{Pro}}(Y) \leftarrow C_*^{\mathrm{Pro}}(X \otimes Y)$

2) Homotopy t-structure (Morel):

Rmk: homotopy t-structure does not require k perfect

$$SH(k)_{\geq 0} = \{ E \mid \underline{\pi}_i^{\mathbb{A}^1}(E)_j = 0 \quad \forall\, i<0,\, j\in\mathbb{Z} \}$$

$$\underline{\pi}_i^{\mathbb{A}^1}(E)_j = a_{Nis}\left(U \mapsto [U_+ \wedge S^i, E \wedge \mathbb{G}_m^{\wedge j}]_{SH(k)} \right)$$

Nisnevich sheafification

$$SH(k)^{\heartsuit} = HI_* = \text{homotopy modules}$$

$$\{ \mathcal{F}_i \in HI : i\in\mathbb{Z} \quad \underset{\substack{\| \\ \mathrm{map}_*(\mathbb{G}_m, \mathcal{F}_i)}}{(\mathcal{F}_i)_{-1}} \cong \mathcal{F}_{i-1} \}$$

By 1) have

$$C_*^{Pro} : Pro(SH(k)) \longrightarrow Pro\, Ch^b(HI_*)$$

commutes colimits because oplax left adjoint

In fact, commutes limits (omitted)

<u>Claim</u>: C_*^{Pro} is $\otimes$ on dualizable objects of $SH(k)$

Reductions
- dualizable objects are compact (see problems)
- Compact objects of $SH(k)$ are generated under finite limits, retracts, and bigraded shifts by Sm_k

- Sm_k is in the subcategory of $Pro(SH(k))$ generated under limits by affine essentially sm, semilocal schemes

Thus it suffices to show $C_*^{Pro}(X) \otimes C_*^{Pro}(Y) \simeq C_*^{Pro}(X \otimes Y)$ for X, Y affine, essentially sm, semilocal

Bondarko Prop 1.4.8 "Gersten weight structures for motivic homotopy categories; retracts of coh of function... "

plus Elmanto-Hoyois-Khan-Sosnilo-Yakerson "Motivic infinite loop spaces" Prop B.2.1 to remove hypothesis that k infinite

X affine, essentially sm, semilocal $\Rightarrow H^n_{Nis}(X, \mathcal{F}) = 0$ for all $n > 0$ and $\mathcal{F} \in HI_*$

(Gabber's lemma)

Lemma: $C_*^{Pro}(X) = \underline{\pi}_0^{\mathbb{A}^1}(X)$ for X affine, essentially sm, semilocal

pf: $X \to \underline{\pi}_0^{\mathbb{A}^1}(X)$

$C_*^{Pro}(X) \to C_*^{Pro}(\underline{\pi}_0^{\mathbb{A}^1}(X))$

Since $\underline{\pi}_0^{\mathbb{A}^1}(X) \in SH^{\heartsuit}$, $K^{Pro}(\underline{\pi}_0^{\mathbb{A}^1}(X)) = \underline{\pi}_0^{\mathbb{A}^1}(X)$

$$\underbrace{C_*^{\mathrm{Pro}}}_{\text{left adjoint}} \dashv K^{\mathrm{Pro}} \quad \Rightarrow \quad \pi_0^{\mathbb{A}^1}(X) \longleftarrow C_*^{\mathrm{Pro}}(\underline{\pi}_0^{\mathbb{A}^1}(X))$$

Thus have $C_*^{\mathrm{Pro}}(X) \to \pi_0^{\mathbb{A}^1}(X)$

To show equivalence, can show

$$[\pi_0^{\mathbb{A}^1}(X), C'_*]_{\mathrm{Pro}\,Ch^b(HI_*)} \xrightarrow{\cong} [C_*^{\mathrm{Pro}}(X), C'_*]_{\mathrm{Pro}\,Ch^b(HI_*)}$$

Since HI_* generates $\mathrm{Pro}\,Ch^b(HI_*)$ under limits, may assume $C'_* \simeq \Sigma^j \mathcal{F}$ $\quad \mathcal{F} \in HI_*$

$$[C_*^{\mathrm{Pro}}(X), \Sigma^j\mathcal{F}]_{\mathrm{Pro}\,Ch^b(HI_*)} \underset{\text{adjunction}}{=} [X, \underbrace{\mathrm{Pro}K(\Sigma^j\mathcal{F})}_{\Sigma^j\mathcal{F}}] = H^j(X, \mathcal{F}_0) = [\pi_0^{\mathbb{A}^1}(X), C'_*]_{\mathrm{Pro}\,Ch^b(HI_*)}$$

□

Since $\underline{\pi}_0^{\mathbb{A}^1}$ is symmetric monoidal, this proves the claim □

Thus have $C_*^{\mathrm{Pro}} : \underbrace{SH(k)^{\omega}}_{\text{dualizable objects}} \xrightarrow{\otimes} \mathrm{Pro}\,Ch^b(HI_*)$

- Since C_*^{Pro} is $\otimes$ it takes dualizable objects to dualizable objects

- dualizable objects of Pro $Ch^b(HI_*)$ are in $Ch^b(HI_*)$

$\rightsquigarrow$

<u>Thm</u> (Bachmann-W)

There is a symmetric monoidal

$$C_* : SH(k)^\omega \longrightarrow Ch^b(HI_*)$$

defining a homology theory

For X cellular

$$C_*(X) = \left(\bigoplus^{b_i} \underline{K}^{MW}_{*+i} , \partial_n \right)$$