Ordering the topological order in the FQHE

Nathan Seiberg

IAS

Meng Cheng, Seth Musser, Amir Raz, NS, and T. Senthil,
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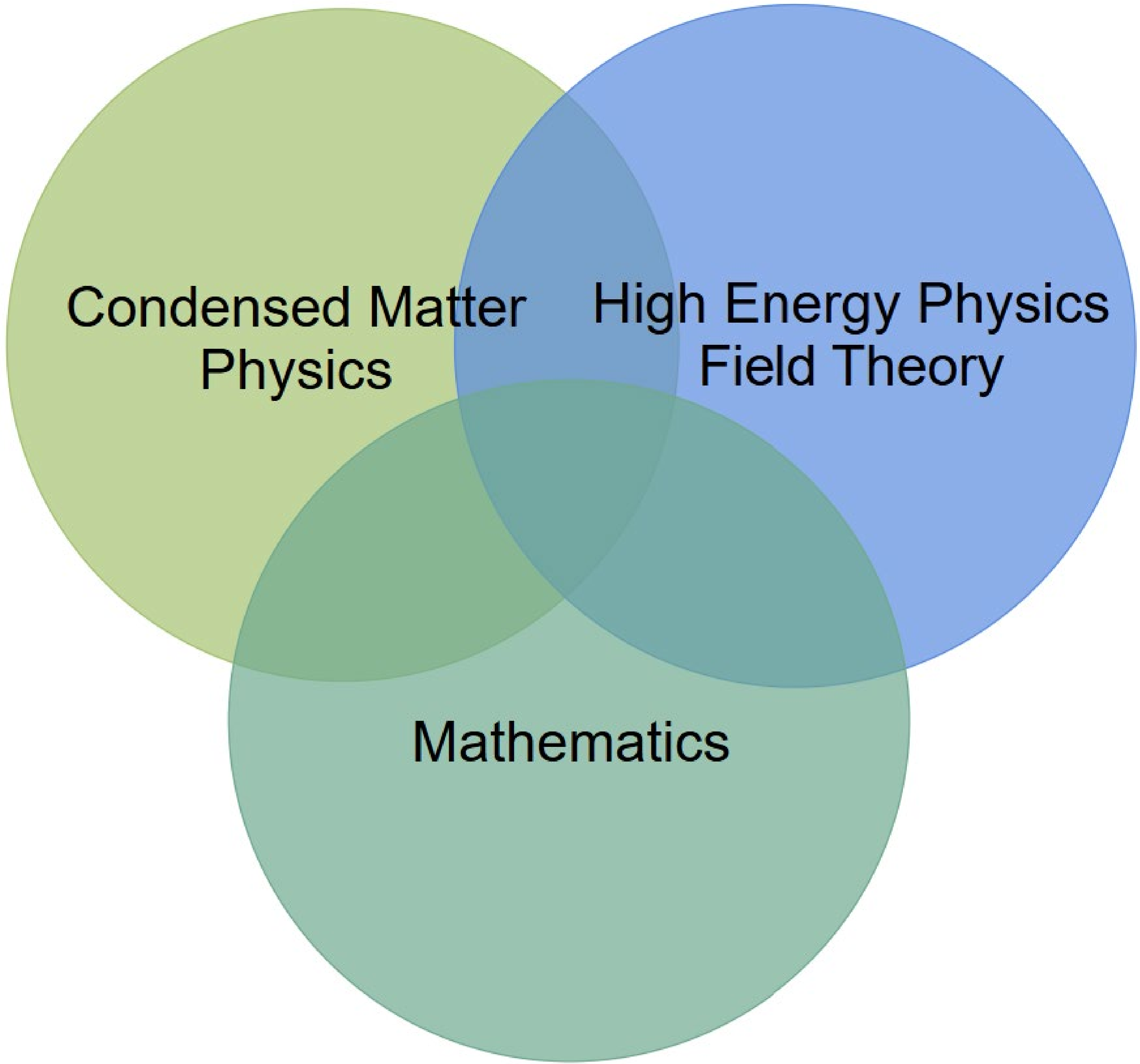

Condensed Matter
Physics
High Energy Physics
Field Theory
Mathematics

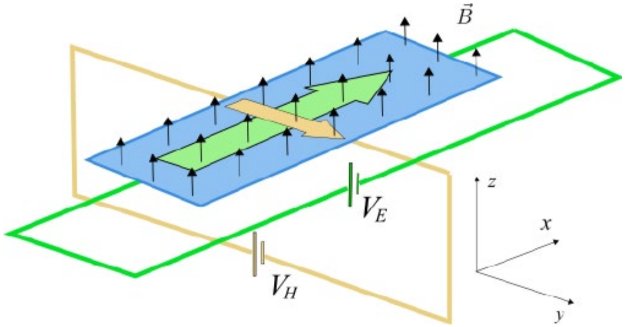
QFT coupled to background metric and $U(1)_A$

- Bosons are described by a bosonic theory.
- Fermions (and also bosons) are described by a fermionic/spin theory. Can be defined only on spin manifolds and can depend on the choice of spin structure.
- Bosons with even charges and fermions with odd charges are described by a $spin^c$ theory (electronic theory). Need a $spin^c$ structure. The flux of the background $U(1)_A$ connection is correlated with the obstruction to spin structure $\left(2\oint\frac{dA}{2\pi}\right) mod\ 2 = w_2$.

The fermionic theories are the largest class. Some of them are also bosonic and some of them are also $spin^c$.

For condensed matter applications, mostly the $spin^c$ theories.

Classical Hall Effect



Characterized by a response theory including

$$\frac{\sigma_H}{4\pi}\int_{M^{(3)}} AdA$$

(We suppress various important terms including gravitational terms.)

Differentiating with respect to A_i gives the current

$$J^i = \frac{\sigma_H}{2\pi}\epsilon^{ij}E_j$$

Differentiating with respect to A_t gives the charge density

$$\rho = \frac{\sigma_H}{2\pi}B$$

Integer Quantum Hall Effect

Set $\hbar = 1$.

The response theory (Chern-Simons term)

$$\exp\left(i \frac{\sigma_H}{4\pi} \int_{M^{(3)}} AdA \right)$$

is well-defined when:

- $\sigma_H \in 2\mathbb{Z}$ for bosons
- $\sigma_H \in \mathbb{Z}$ for fermions or electrons

(No such issue in the classical theory.)

Then, $\frac{\sigma_H}{4\pi} \int_{M^{(3)}} AdA$ is the action of a 2+1d invertible field theory.

Physically, the system can be trivially gapped, but even then, it is nontrivial, as it can have edge modes.

Fractional Quantum Hall Effect

We will be interested in fractional

$$\sigma_H = \frac{p}{q} \quad , \qquad \gcd(p,q) = 1$$

Then, the response theory (Chern-Simons term)

$$\exp\left(i \frac{\sigma_H}{4\pi} \int_{M^{(3)}} A dA\right)$$

is not well-defined.

This reflects the fact that the low-energy theory cannot be trivially gapped.

If the system is gapped, but not trivially gapped, it is (expected to be) described by a nontrivial TQFT.

Adding an integer σ_H corresponds to adding a local counterterm (stacking an IQH state).

Response theory $\frac{\sigma_H}{4\pi} AdA$

$$\sigma_H = \frac{p}{q} \quad , \qquad \gcd(p,q) = 1$$

Such a fractional Chern-Simons coefficient is a useful characteristic of a nontrivial quantum field theory. (It was used in the study of dualities between supersymmetric gapless theories [Closset, Dumitrescu, Festuccia, Komargodski, NS] .)

We will limit ourselves to gapped systems, which are described by a TQFT.

Given $\sigma_H = \frac{p}{q}$, what can we say about the TQFT?

Response theory $\frac{\sigma_H}{4\pi} AdA$

$$\sigma_H = \frac{p}{q} \quad , \qquad \gcd(p,q) = 1$$

Condensed Matter perspective	Field Theory perspective
“Flux threading” interpolates $A_x = 0 \rightarrow \frac{2\pi}{L_x}$. They should be the same, but they differ by moving an anyon, a vison v.	This expression is not well-defined, reflecting the imprecise integration out of the TQFT modes.
Its $U(1)$ charge and spin are $Q(v) = \frac{p}{q}$, $h(v) = \frac{p}{2q}$ [Laughlin; Halperin; Arovas, Schrieffer, Wilczek; Zhang, Hansson, Kivelson; Lopez Fradkin;…] $v^s = v^{\ell q} = 1$	It reflects a $\mathbb{Z}_s$ one-form symmetry with anomaly r, such that $s = \ell q,\ r = \ell p$. Its generator determines the $U(1)$ charges of all anyons.

Response theory $\frac{p}{4\pi q} AdA$

Can define it by adding a "bulk" $M^{(4)}$ and writing

$$\frac{p}{4\pi q}\int_{M^{(4)}} dAdA \quad , \qquad M^{(3)} = \partial M^{(4)}$$

This expression is well-defined, but depends on more data.

How can the TQFT lead to such a term?

If the TQFT has a $\mathbb{Z}_s$ one-form symmetry, we can couple it to a two-form background field $\mathcal{B}$. Then, the 't Hooft anomaly in this symmetry is labeled by r and is characterized by the anomaly theory [Kapustin, NS; Gaiotto, Kapustin, NS, Willett; Hsin, Lam, NS]

$$\frac{2\pi r}{2s}\int_{M^{(4)}} \mathcal{P}(\mathcal{B})$$

$\mathcal{P}(\mathcal{B})$ is the Pontryagin square (roughly $\mathcal{B}^2$).

Response theory $\frac{p}{4\pi q} AdA$

Then, the background A couples to the TQFT through its one-form global symmetry, i.e., its coupling to $\mathcal{B}$.

Roughly,* $\mathcal{B} = \frac{dA}{2\pi}$

$$\frac{r}{4\pi s}\int_{M^{(4)}} \mathcal{P}(\mathcal{B}) \to \frac{r}{4\pi s}\int_{M^{(4)}} dAdA \to$$

$$\frac{p}{4\pi q}\int_{M^{(3)}=\partial M^{(4)}} AdA$$

$$s = \ell q \quad , \qquad r = \ell p$$

* "Roughly" because $\mathcal{B}$ is a discrete field (integer modulo s), rather than a differential form $\frac{dA}{2\pi}$ and $\mathcal{P}(\mathcal{B})$ is not exactly a square.

The symmetry lines [Hsin, Lam, NS]

First, ignore the $U(1)$ symmetry.

One-form $\mathbb{Z}_s$ symmetry generated by v, with anomaly r has s symmetry lines $\mathcal{V}^{s,r} = \{1, v, v^2 \dots v^{s-1}\}$

$$h(v^j) = \frac{rj^2}{2s}$$

v^s should be a transparent line (identity).

Therefore:

- Fermionic theory: $h(v^s) = 0 \, mod \, \frac{1}{2}$. Hence, $r, s \in \mathbb{Z}$, $r \sim r + s$.
- Bosonic theory: $h(v^s) = 0 \, mod \, 1$. Hence, $r, s \in \mathbb{Z}$, $rs \in 2\mathbb{Z}$, $r \sim r + 2s$.

The symmetry lines with $U(1)_A$

The $U(1)$ charge Q of every line in the TQFT is determined by braiding with v. In particular,

$$h(v^j) = \frac{rj^2}{2s} \quad , \qquad Q(v^j) = \frac{rj}{s}$$

- Fermionic theory: $h(v^s) = 0 \; mod \, \frac{1}{2}$, $Q(v^s) = 0 \; mod \; 1$.

$$r, s \in \mathbb{Z}, r \sim r + s$$

- Bosonic theory: Fermionic and also $h(v^s) = 0 \; mod \; 1$.

$$r, s \in \mathbb{Z} \quad , \quad rs \in 2\mathbb{Z} \quad , \quad r \sim r + 2s$$

- spin^c theory: Fermionic and also $h(v^s) + \frac{1}{2} Q(v^s) = 0 \; mod \; 1$.

$$r, s \in \mathbb{Z} \quad , \quad r(s+1) \in 2\mathbb{Z} \, , \quad r \sim r + s$$

The fermionic theory is the easiest to think about and the spin^c theory is the most interesting in CMP.

The symmetry lines for $\ell = 1$

$$s = \ell q \ , \qquad r = \ell p \ , \qquad \gcd(p,q) = 1$$

The symmetry lines $\mathcal{V}^{s,r} = \mathcal{V}^{q,p}$ realize $\mathbb{Z}_q$ faithfully.

Therefore, the TQFT factorizes [Hsin, Lam, NS]

$$\mathcal{T} = \mathcal{V}^{q,p} \otimes \mathcal{T}'$$

All the anyons in $\mathcal{T}'$ are $\mathbb{Z}_s$ invariant and $U(1)$ neutral.

That part of the theory is trivial from our perspective and we are going to ignore it.

We end up with the minimal theory $\mathcal{V}^{q,p}$ [Moore, NS; Hsin, Lam, NS].

For $\ell \neq 1$, the symmetry lines $\mathcal{V}^{s,r}$ are not a complete theory. (It is not modular invariant – pre-modular.)

Reducing ℓ

$$s = \ell q \ , \qquad r = \ell p \ , \qquad \gcd(p, q) = 1$$

Can try to gauge $\mathbb{Z}_\ell \subset \mathbb{Z}_s$ to reduce $(s, r) \to (q, p)$.

This procedure is often described as "condensation."

I have always had issues with this terminology.

In a fermionic theory, $r, s \in \mathbb{Z}$ without a restriction. Therefore, this is always possible.

After doing that, $\mathcal{V}^{s,r} \to \mathcal{V}^{q,p}$, the TQFT factorizes, we can remove the decoupled neutral TQFT, and end up with the minimal theory $\mathcal{V}^{q,p}$.

Reducing ℓ

$$s = \ell q \;, \qquad r = \ell p \;, \qquad \gcd(p, q) = 1$$

Can try to gauge $\mathbb{Z}_\ell \subset \mathbb{Z}_s$ to reduce $(s, r) \to (q, p)$.

This procedure is often described as “condensation.”

I have always had issues with this terminology.

In a bosonic theory, $rs \in 2\mathbb{Z}$.

- if $pq \in 2\mathbb{Z}$, we can gauge $\mathbb{Z}_\ell$ and end up with the minimal theory $\mathcal{V}^{q,p}$.
- if $pq \in 2\mathbb{Z} + 1$, we can gauge only $\mathbb{Z}_{\ell/2}$ and end up with $\mathcal{V}^{2q,2p}$. This TQFT does not factorize.
  - For $\sigma_H = 1$, we have $p = q = 1$. The smallest symmetry is $\mathcal{V}^{2,2}$. It can be embedded in $Spin(n)_1$.

Reducing ℓ

$$s = \ell q \ , \qquad r = \ell p \ , \qquad \gcd(p,q) = 1$$

Can try to gauge $\mathbb{Z}_\ell \subset \mathbb{Z}_s$ to reduce $(s,r) \to (q,p)$.

This procedure is often described as "condensation."

I have always had issues with this terminology.

In a spin^c theory, $r(s+1) \in 2\mathbb{Z}$

- if $q \in 2\mathbb{Z}+1$, we can gauge $\mathbb{Z}_\ell$ and end up with the minimal theory $\mathcal{V}^{q,p}$.
- if $q \in 2\mathbb{Z}$, we can gauge only $\mathbb{Z}_{\ell/2}$ and end up with $\mathcal{V}^{2q,2p}$. This TQFT does not factorize.
  - For $\sigma_H = 1/2$, we have $p = 1$, $q = 2$. The smallest symmetry is $\mathcal{V}^{4,2}$. It can be embedded in $U(1)_8$ or in the Moore-Read theory.

Additional topics in the paper

- Where did the one-form symmetry come from?
  - For electrons in a constant magnetic field, the UV translation symmetry becomes the IR one-form symmetry [Jensen, Raz; NS, Seifnashri]. The UV translation symmetry is transmuted into the IR one-form symmetry.
  - This symmetry is robust because the low-energy theory is gapped. And our discussion applies more generally, e.g., when there is no translation symmetry in the UV.
- Extend to bi-layer systems
- A more detailed picture of how the $\mathbb{Z}_s$ symmetry generated by the vison v is embedded in a possibly larger one-form symmetry of the TQFT.
- New results about spin and spin^c TQFTs.

Conclusions

- One-form global symmetry and its anomaly are the organizing principle of Quantum Hall systems.
- There is a natural notion of a minimal TQFT for every $\sigma_H = \frac{p}{q}$.
- Bosonic systems
  - Even pq, a minimal Abelian TQFT with q anyons $\mathcal{V}^{q,p}$.
  - Odd pq, the minimal symmetry lines are $\mathcal{V}^{2q,2p}$ and the system is more complicated.
- spin^c (electronic) systems
  - Odd q, a minimal Abelian TQFT with q anyons $\mathcal{V}^{q,p}$
  - Even q, the minimal symmetry lines are $\mathcal{V}^{2q,2p}$ and the system is more complicated.
- Most, if not all, known examples are the minimal ones.

Thank you