Lecture 4.1

Proof of Crucial Ratio Lemma by iterating twice through recursive relations.

Remarks

- With more care one can run the same argument for $\mathcal{D}_\Delta := \{ \underline{d} \in \mathbb{Z}^n_{\geq 0} : \sum_{i=1}^n d_i \text{ is even}, \ 1 \leq d_i \leq \Delta \text{ for all } i \}$ provided that $\Delta = o(n^6)$.

  → Approximation for ratios for very irregular & sparse sequences.

- Do 1 more iteration to get accuracy $(1 + O(\frac{d^3}{n^3}))$ which would be good enough for $d = o(\frac{n^{4/7}}{(\log n)^{1/7}})$.

  → Possible to do on paper, but tedious.

What remains to be done to prove Conjecture for $d = o(\frac{n^{2/5}}{\log n})$?

Find $\mathcal{D}$ $(= \mathcal{D}_m{}')$ and formulae $Y_{\underline{d}}$, for each $\underline{d} \in \mathcal{D}_m$, such that

$$(1) \qquad \frac{Y_{\underline{d}-e_a}}{Y_{\underline{d}-e_b}} = \frac{d_a}{d_b}\left(1 + \frac{d_a - d_b}{dn}\left(1 + \frac{M_2(\underline{d})}{dn}\right) + o\left(\frac{1}{\operatorname{diam}(S)}\right)\right)$$

and
$$\sum_{\underline{d} \in \mathcal{D}_m} Y_{\underline{d}} = 1 + o(1). \qquad (2)$$

Conjecture says
$$Y_{\underline{d}} = \Pr(\mathcal{B}_m(n) = \underline{d}) \cdot \underbrace{\exp\left[\frac{1}{4} - \frac{\gamma_2^2(\underline{d})}{4\mu^2(1-\mu)^2}\right]}_{=: H(\underline{d})}$$
$$= \frac{\prod_{i=1}^n \binom{n-1}{d_i}}{\binom{n(n-1)}{2m}} \cdot H(\underline{d})$$

(1) is a calculation (sorry!)

$$\frac{\Pr(\mathcal{B}_m(n) = \underline{d} - e_a)}{\Pr(\mathcal{B}_m(n) = \underline{d} - e_b)} = \frac{d_a}{d_b}\,\frac{n - d_b}{n - d_a} = \frac{d_a}{d_b}\left(1 + \frac{d_a - d_b}{n} + O\left(\frac{d^2}{n^2}\right)\right)$$

(check!)

and $$\frac{H(\underline{d}-e_a)}{H(\underline{d}-e_b)} = \exp\left[\frac{(d_a-d_b)\, M_2}{(dn)^2} + \frac{d_a-d_b}{dn} - \frac{(d_a-d_b)}{n} + O\left(\frac{\log^2 n\, d^2}{n^2}\right)\right]$$

Exercise. Hint: Use $\gamma_2(\underline{d}-e_a) = \gamma_2(\underline{d}) - \frac{2d+1}{(n-1)^2} - \frac{2d_b}{(n-1)^2}$.

For (2): $$\sum_{\underline{d}\in \mathcal{D}_m} \Pr(\mathcal{B}_m(n)=\underline{d}) \cdot H(\underline{d}) = \sum_{\underline{d}:\ \sum_i d_i = 2m} \Pr(\mathcal{B}_m(n)=\underline{d})\cdot H(\underline{d}) + O(n^{-23})$$ (concentration lemma)

$$= \mathbb{E}_{\mathcal{B}_m} H(\underline{d}) + o(1)$$

$$= \mathbb{E}_{\mathcal{B}_m} \exp\left[\frac{1}{4} - \frac{\gamma_2^2(\underline{d})}{4\mu^2(1-\mu)^2}\right] \; o(1)$$

$\mu = \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)}$ const on $\mathcal{D}_m$.
& $\gamma_2(\underline{d}) = \frac{1}{(n-1)^2}\sum_i (d_i - d)^2$

<u>Lemma (Concentration)</u>

With probability $1-o(1)$, $f(\underline{d}) = \sum_{i=1}^n (d_i-d)^2$ satisfies
$f(\underline{d}) = \mathbb{E} f(\underline{d})\,(1+o(1))$. in $\mathcal{B}_m(n)$.

E.g.: via Chebyshev's Inequality
or sharp concentration inequality (McDiarmid, bounded differences, Azuma)

$$\mathbb{E}_{\mathcal{B}_m} f(\underline{d}) = \mathbb{E}_{\mathcal{B}_m} \sum_i (d_i-d)^2 = \sum_i \mathbb{E}_{\mathcal{B}_m} (d_i-d)^2 = n\cdot \mathrm{Var}(d_1)$$

$\uparrow$ Hypergeom$\left(\binom{n}{2}, n(n-1), n-1, 2m\right)$

$$= n\cdot\left(\frac{2m}{n}\left(1-\frac{1}{n}\right)\left(1-\frac{2m}{n(n-1)-1}\right)\right)$$
$$\approx dn\left(1-\frac{d}{n}\right) + O(d)$$

$\Rightarrow$ with prob $1-o(1)$ $\quad \gamma_2(\underline{d}) = \frac{d}{n}\left(1-\frac{d}{n}\right) \cancel{+O(\ldots)} (1+o(1))$

$\Rightarrow$ $\quad$ – " – $\quad H(\underline{d}) = \exp\left[\frac{1}{4} - \frac{\mu^2(1-\mu)^2}{4\mu(1-\mu)^2}(1+o(1))\right]$
$= 1+o(1)$

$\Rightarrow$ $\mathbb{E}_{\mathcal{B}_m} H(\underline{d}) = 1+o(1) + o(1)\cdot \underbrace{\max H(\underline{d})}_{=O(1)} = 1+o(1).$

Lecture 4.2

Summary: we proved

Thm If $\log^2 n \ll d \ll n^{2/5}/\log n$ then

$$|\mathcal{G}(n,d)| = \binom{\binom{n}{2}}{m} \Pr(\mathcal{B}_m(n) = \underline{d}) \cdot \exp\left[\frac{1}{4} + o(1)\right],$$

where $m = dn/2$.

Moreover, for all $\underline{d} \in \mathcal{D}_m$,

$$\Pr(D(G_{n,m}) = \underline{d}) = \Pr(\mathcal{B}_m(n) = \underline{d}) \cdot \underbrace{\exp\left[\frac{1}{4} - \frac{\gamma_2^2(\underline{d})}{4\mu^2(1-\mu)^2} + o(1)\right]}_{1+o(1) \text{ w.h.p. in } \mathcal{B}_m(n)}$$

Pf summary

- define local change graph S, $\mathrm{diam}(S) \leq 10\sqrt{d\log n} \cdot n$.
- CRL: $\dfrac{\Pr(D(G_{n,m}) = \underline{d} - e_a)}{\Pr(D(G_{n,m}) = \underline{d} - e_b)} = [\text{formula}]\left(1 + o\left(\frac{1}{\mathrm{diam}(S)}\right)\right)$
- Calculation: $\dfrac{Y_{\underline{d}-e_a}}{Y_{\underline{d}-e_b}} = [\text{SAME formula}]\left(1 + o\left(\frac{1}{\mathrm{diam}(S)}\right)\right)$
- (Sharp) concentration: $\displaystyle\sum_{\underline{d} \in \mathcal{D}_m} Y_{\underline{d}} = \mathbb{E}_{\mathcal{B}_m} H(\underline{d}) + O(n^{-23})$
$= 1 + o(1)$ □

Larger d?

1^{st}, 3^{rd} & 4^{th} point: standard toolbox

CRL → need to get finer approximation for $R_{(a,b)}(\underline{d})$.

Lecture 4.3

Recursive relations

(1) $$R_{(a,b)}(\underline{d}) = \frac{d_a}{d_b}\,\frac{1-\mathrm{Bad}_{(a,b)}(\underline{d}-e_b)}{1-\mathrm{Bad}_{(b,a)}(\underline{d}-e_a)} \quad \text{for odd } \underline{d}$$

(1.5) $$\mathrm{Bad}_{a,b}(\underline{d}) = \frac{1}{d_a}\Big(P_{ab}(\underline{d}) + \sum_{v\neq a,b} Y_{a,v,b}(\underline{d})\Big). \quad \text{even } \underline{d}$$

(2) $$P_{av}(\underline{d}) = d_v\left(\sum_{b\neq v} R_{(b,a)}(\underline{d}-e_v)\,\frac{1-P_{bv}(\underline{d}-e_b-e_v)}{1-P_{av}(\underline{d}-e_a-e_v)}\right)^{-1}$$

(3) $$Y_{avb}(\underline{d}) = \frac{P_{av}(\underline{d})\big(P_{bv}(\underline{d}-e_a-e_v)-Y_{avb}(\underline{d}-e_a-e_v)\big)}{1-P_{av}(\underline{d}-e_a-e_v)}.$$

Assuming things are well-defined (e.g. realisable in $P_{av}(\underline{d})$).

Start with $P_{av}(d) = O(\frac{d}{n})$ ~~$(=o(1))$~~ $\ll 1$.

$\rightarrow$ iterate applying (3), (1.5), (1), (2)

& get better & better approximations of $R_{a,b}(\underline{d})$.

What's the improvement after 1 round of iterations?

after 1st round $R_{ab}(\underline{d}) = \frac{d_a}{d_b}(1+O(\frac{d}{n}))$

2nd $R_{a,b}(\underline{d}) = \frac{d_a}{d_b}(1+ \text{sth.} + O(\frac{d^2}{n^2}))$

3rd round $\frac{d_a}{d_b}(1+ \text{sth. more complicated} + O(\frac{d^3}{n^3}))$

⋮

k rounds $R_{a,b}(\underline{d}) = \text{Formula}\,(1+O((\frac{d}{n})^k))$

For which d is $(\frac{d}{n})^k$ precise enough? $(\frac{d}{n})^k = o\left(\frac{1}{n\sqrt{d\log n}}\right)$

$$\Leftrightarrow d^{\frac{2k+1}{2}} \ll n^{k-1}/(\log n)^{1/2}$$

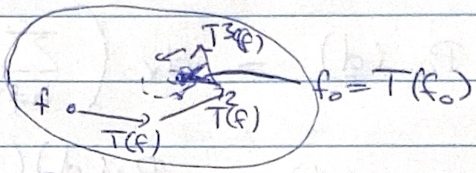
$$\Leftrightarrow d \ll n^{1-\frac{2}{2k+1}}\Big/(\log n)^{\frac{1}{2k+1}}$$

In theory: $\forall \varepsilon>0$, $d=n^{1-\varepsilon}$, need $O(1)$ iterations to obtain desired accuracy.

Problem 1: formulae get very complicated very quickly (try to do 3^{rd} iteration on paper).

Problem 2: that does not get us up to $d = \Theta(\frac{n}{\log n})$ or even $d = \varepsilon n$.

Operators $T(f)$ on functions $p = p(a,b,\underline{d})$



$p: \binom{n}{2} \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad p_{av}(\underline{d})$

$y: \binom{n}{3} \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad y(a,v,b,\underline{d})$ $\quad y_{avb}(\underline{d})$

$r: [n]^2 \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad r(a,b;\underline{d})$. $\quad r_{a,b}(\underline{d})$

(1) $$\mathcal{R}(p,y)(a,b,\underline{d}) = \frac{d_a}{d_b} \cdot \frac{1 - bad(p,y)(a,b,\underline{d}-e_b)}{1 - bad(p,y)(a,b,\underline{d}-e_a)},$$

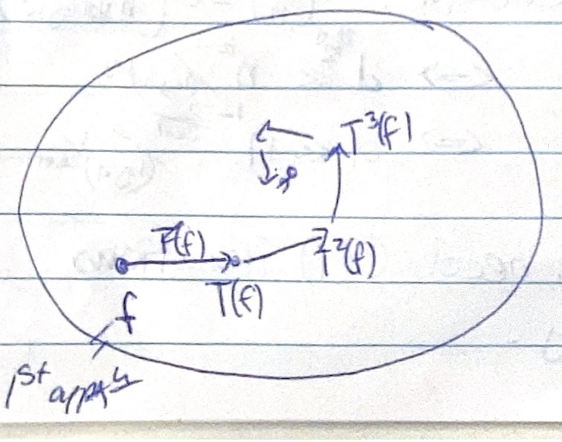
(R$) $$\mathbf{bad}(p,y)(a,b,\underline{d}) = \frac{1}{d_a}\left(p_{ab}(\underline{d}) + \sum_{v \neq a,b} y_{avb}(\underline{d})\right)$$

(2) $$\mathcal{P}(p,r)(a,v,\underline{d}) = d_v \left(\sum_{b \neq v} r_{b,a}(\underline{d}-e_v) \frac{1 - p_{bv}(\underline{d}-e_b-e_v)}{1 - p_{av}(\underline{d}-e_a-e_v)}\right)^{-1}$$

(3) $$\mathcal{Y}(p,y)(a,v,b,\underline{d}) = \frac{p_{av}(\underline{d})\left(p_{bv}(\underline{d}-e_a-e_v) - y_{avb}(\underline{d}-e_a-e_v)\right)}{1 - p_{av}(\underline{d}-e_a-e_v)}$$

Observation: "real" probabilities & ratios P_{av}, Y_{avb} & $R_{(a,b)}$ are "fixed points"

$$\mathcal{R}(P,Y) = R$$
$$\mathcal{P}(P,R) = P$$
$$\mathcal{Y}(P,Y) = Y.$$



Lecture 4 ②

Operators are "contractive" for probability-like p & y:

Call (p, y) μ-probability-like on $\mathcal{D}_0$ ← some set $\subseteq \mathbb{Z}_{\geq 0}^n$

(P1) $0 \leq p_{av}(d) \leq \mu \quad \forall d \in \mathcal{D}_0$

(P2) $\sum_{v \neq a,b} y_{avb}(d) \leq \mu \cdot d_a \quad \forall d \in \mathcal{D}_0, \; a,b \in [n]$ distinct.

(P3) $0 \leq y_{avb}(d) \leq \mu \cdot p_{bv}(d) \quad \forall d \in \mathcal{D}_0$ & $a,v,b \in [n]$ distinct.

Exercise: Show that real (P, Y) is μ-probability like on $\mathcal{B}_{16 \log n}(\mathcal{D}_m)$, assuming $n \log^3 n \leq m = o(n^2)$ where $\mu = 5 \cdot \frac{m}{\binom{n}{2}}$.

Hint: Use simple switching lemma & Recursion for Y_{avb}.

Contraction lemma (p, y), (p', y') are μ-probality like on $\mathcal{D}_0$, r, r' fct^{ns}.

(a) If $p = p'(1 \pm \xi)$ & $y = y'(1 \pm \xi)$

then $\mathcal{R}(p, y) = \mathcal{R}(p', y')(1 + O(\mu \cdot \xi))$

(b) If $p = p'(1 \pm \xi)$ & $r = r'(1 \pm \mu\xi)$

then $\mathcal{P}(p, r) = \mathcal{P}(p', r')(1 + O(\mu\xi))$

(c) If $p = p'(1 \pm \mu\xi)$ and $y = y'(1 \pm \xi)$

then $\mathcal{Y}(p, y) = \mathcal{Y}(p', y')(1 + O(\mu\xi))$

[w/o proof]

Take $(p, y, r) = (P, Y, R)$

and (p', y', r') are guessed formulae.