PCMI GRADUATE SUMMER SCHOOL - ASPECTS OF MOTIVIC COHOMOLOGY

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The overall goal of this lecture series is to present some aspects of motivic cohomology, K-theory, and their relation. All rings are assumed to be commutative.

1. Lecture 1

1.1. K_0. See [Wei13]. Let us begin by recalling a classical definition of Grothendieck from the 50's. Let R be a ring. Recall that an R-module P is called projective if there exists a module Q such that $P \oplus Q$ is free.

Definition 1.1.1. Let R be a ring, and let $\mathbf{P}(R)$ be the set of isomorphism classes of finitely generated projective R-modules. It is an abelian monoid with respect to the direct sum $\oplus$ and with identity 0. The Grothendieck group $K_0(R)$ is the group completion of $\mathbf{P}(R)$.

Equivalently, $K_0(R)$ can be described as the abelian group generated by symbols $[P]$, where P runs over (isomorphism classes) of finitely generated projective R-modules, modulo the relations

$$[P] = [P'] + [P'']$$

whenever there is a short exact sequence

$$0 \to P' \to P \to P'' \to 0$$

Example 1.1.2.
(1) Let F be a field. Then there is an isomorphism $K_0(F) \cong \mathbb{Z}$ given by $[P] \mapsto \dim(P)$. More generally, for every 0-dimensional ring R, we have $K_0(R) = H_0(R)$, where $H_0(R)$ denotes the ring of all continuous maps from $\mathrm{Spec}(R)$ to $\mathbb{Z}$ (if R is a domain, then $H_0(R) = \mathbb{Z}$).
(2) Let $\mathcal{O}$ be a Dedekind domain. Recall that the Class group of $\mathcal{O}$ is defined as

$$Cl(\mathcal{O}) := \{\text{isomorphism classes of non-zero ideals } I \text{ of } \mathcal{O}\}$$

which becomes an abelian group via multiplication $I \cdot J = IJ$ (the ideal generated by ab for $a \in I$ and $b \in J$). Since $\mathcal{O}$ is a Dedekind domain, every ideal I is a finitely generated projective $\mathcal{O}$-module, hence it has a class $[I]$ in $K_0(\mathcal{O})$. Then there is a short exact sequence

$$0 \to Cl(\mathcal{O}) \longrightarrow K_0(\mathcal{O}) \xrightarrow{\mathrm{rank}_{\mathcal{O}}(-)} \mathbb{Z} \to 0 \tag{1.1}$$

where the first map is given by $[I] \mapsto [I] - [\mathcal{O}]$.

The above definition generalizes to exact categories as follows.

Definition 1.1.3. Let $\mathcal{C}$ be a small exact category. The Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by symbols $[X]$ for each object X of $\mathcal{C}$, and relations $[X] = [Y] + [Z]$ for every exact sequence

$$0 \to Y \to X \to Z \to 0$$

in $\mathcal{C}$.

Example 1.1.4. (1) Let X be a scheme. The category $\mathbf{V}(X)$ of algebraic vector bundles on X (aka locally free $\mathcal{O}_X$-modules) is an exact category (it is an additive subcategory of the abelian category of $\mathcal{O}_X$-modules). We write $K_0(X)$ for $K_0(\mathbf{V}(X))$ and call it the Grothendieck group of X.

(2) For a Noetherian scheme X, we can also consider the exact category $\mathbf{M}(X)$ of all coherent $\mathcal{O}_X$-modules. Its Grothendieck group $K_0(\mathbf{M}(X))$ is usually denoted $G_0(X)$. The inclusion $\mathbf{V}(X) \subset \mathbf{M}(X)$ induces a natural map, called the Cartan map, $K_0(X) \to G_0(X)$.

(3) If X is separated regular Noetherian scheme, then the Resolution Theorem of [SGA71, II. 2.3.3] implies that every coherent $\mathcal{O}_X$-module $\mathcal{M}$ has a finite resolution

$$0 \to \mathcal{P}_d \to \ldots \to \mathcal{P}_0 \to \mathcal{M} \to 0$$

where each $\mathcal{P}_i$ is a locally free $\mathcal{O}_X$-module. In particular we can consider the class $\xi([\mathcal{M}]) = \sum_{i=0}^{d} [\mathcal{P}_i]$ in $K_0(X)$. This gives a morphism $\xi\colon G_0(X) \to K_0(X)$ that is actually an inverse to the Cartan map. Thus for such X we have an isomorphism $K_0(X) \cong G_0(X)$. Note that this is no longer true if any of the above assumptions on X are dropped.

Building on the last example, let us consider the case where X is a quasi-projective regular integral scheme defined over a field k (for short, a k-variety). Given any irreducible subscheme $i\colon Z \subset X$, the structure sheaf $i_*(\mathcal{O}_Z)$ is a coherent $\mathcal{O}_X$-module, and thanks to the Resolution Theorem it admits a finite resolution

$$0 \to \mathcal{P}_d \to \ldots \to \mathcal{P}_0 \to i_*(\mathcal{O}_Z) \to 0.$$

We write $cyc([Z])$ for the class $\xi([i_*(\mathcal{O}_Z)])$ in $K_0(X)$, and call it the cycle class of $[Z]$. We can use this to define the coniveau filtration on $K_0(X)$

$$F^j K_0(X) = \langle cyc([Z]) \,|\, Z \subset X \text{ irreducible closed subscheme of codimension } \geq j \rangle.$$

Let $Z^j(X)$ be the free abelian group generated by the set of integral closed subschemes of codimension j on X (this is called the group of codimension j cycles on X). Every $\alpha \in Z^j(X)$ can be written as $\alpha = \sum_i n_i [Z_i]$ for integers $n_i \in \mathbb{Z}$. Extending by linearity the cycle class map considered above yields a group homomorphism

$$cyc(-)\colon Z^j(X) \to F^j K_0(X) \subset K_0(X)$$

and we can consider the composition, still denoted $cyc(-)$, given by $Z^j(X) \to F^j K_0(X) \to F^j K_0(X)/F^{j+1}K_0(X)$. Let $R^j(X)$ be the subgroup of cycles rationally equivalent to 0, and let $\mathrm{CH}^j(X) = Z^j(X)/R^j(X)$. (Part of) the Theorem of Grothendieck-Riemann-Roch states that the map $cyc(-)$ factors kills the subgroup of cycles rationally equivalent to zero, giving a surjective group homomorphism

$$\mathrm{CH}^j(X) \longrightarrow F^j K_0(X) \to F^j K_0(X)/F^{j+1}K_0(X), \quad [Z] \mapsto cyc([Z])$$

whose kernel is killed by $(j-1)!$

1.2. K_1. Next we review some classical aspects of K_1, introduced by Bass in the late 50's.

Definition 1.2.1. For a ring R, let $GL(R) = \bigcup_n GL_n(R)$ be the infinite general linear group (where the union is in fact a colimit over the inclusions $GL_n \subset GL_{n+1}$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$). Let $E(R) \subset GL(R)$ be the subgroup generated by elementary matrices. We define $K_1(R)$ as the quotient $GL(R)/E(R)$.

Example 1.2.2. (1) There is a well-defined map (the determinant)

$$det\colon K_1(R) \to R^\times$$

with right inverse $R^\times \hookrightarrow K_1(R)$ given by sending $a \in R^\times$ to the (infinite) diagonal matrix having a in position $(1,1)$ and 1 everywhere else along the diagonal.

(2) If F is a field, then the Gaussian elimination algorithm implies that in fact there is an isomorphism $K_1(F) \cong F^\times$. The same in fact holds for any local ring R.

(3) Let $\mathcal{O}_F$ be the ring of integers of a number field F. Then again we have $det\colon K_1(\mathcal{O}) \xrightarrow{\simeq} \mathcal{O}_F^\times$, but this is now a deep theorem related to the "congruence subgroup problem". Note that the classical Dirichlet's unit theorem in algebraic number theory implies that $K_1(\mathcal{O})$ is a finitely generated abelian group.

(4) Part (3) is not a general fact about Dedekind domains. For example, let us consider $A = \mathbb{R}[x,y]/(x^2+y^2-1)$. This is also a Dedekind domain (it is a regular Noetherian 1-dimensional ring), but it is known that the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ defines a non-zero element in $\ker(det)$.

1.3. **Higher algebraic K-groups.** Here in the notes we are not going to formally define Quillen's higher algebraic K-groups; we will say a few works in the actual lecture. The reader who is unfamiliar with them may want to consult Quillen's ICM talk (1974, volume 1, available at `www.mathunion.org/icm/proceedings`). For now, let just say that for any ring R (actually, for any scheme X) there are abelian groups $K_n(R)$, $n \geq 0$, defined as the homotopy groups of a certain topological space built out of $GL(R)$. For $n = 0$ and 1 they agree with the previous definitions.

Example 1.3.1. (1) One of the most fundamental calculation is due to Quillen, who computed in 1973 the higher K-groups of finite fields.

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/(q^j - 1) & \text{if } n = 2j-1, j > 0 \\ 0 & \text{else.} \end{cases}$$

(2) The situation is dramatically different for a ring like $\mathbb{Z}$, since its higher K-groups remain mysterious. For example, the fact that $K_n(\mathbb{Z}) = 0$ for $n = 4m$ is equivalent to the Kummer–Vandiver conjecture, which states that prime p does not divide the class number h_K of the maximal real subfield K of the p-th cyclotomic field.

The following extremely deep conjecture is due to Bass

Conjecture 1.3.2. If R is a regular ring of finite type over $\mathbb{Z}$, then $K_n(R)$ is finitely generated for all $n \geq 0$.

We have already seen in Krashen' lecture that if F is a field of characteristic prime to $m \in \mathbb{Z}$, then the Tate symbol map defines an isomorphism

$$K_2(F)/m \xrightarrow{\cong} H^2_{\mathrm{Gal}}(F, \mu_m^{\otimes 2})$$

We will return to this later when we discuss more generally the Bloch-Kato conjecture (now known as the Norm residue theorem). Note that the isomorphism above is compatible with Quillen's computation when F is a finite field.

1.4. **Motivic cohomology.** The above examples show that the K-groups tend to "decompose" into pieces which can be described by more classical invariants. Motivated by such decompositions and by some analogous results in topological K-theory, the following detailed conjecture was proposed in the 80's by Beilinson and Lichtenbaum.

Conjecture 1.4.1. Let X be a regular variety over a field k. Then there exist a complex of abelian groups $\mathbb{Z}_{\mathrm{mot}}(j)(X)$, $j \geq 0$ satisfying the following properties:

(1) There is an "Atiyah-Hirzebruch" spectral sequence

$$E_2^{i,j} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Rightarrow K_{-i-j}(X)$$

where $H^*_{\mathrm{mot}}(X, \mathbb{Z}(j)) := H^*(\mathbb{Z}_{\mathrm{mot}}(j)(X))$ is the cohomology of the complex.

(2) $X \mapsto \mathbb{Z}_{\mathrm{mot}}(j)(X)$ satisfies Zariski descent; from a more classical point of view, this means we can view $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ as the hypercohomology of a complex of Zariski sheaves on X, typically denoted by $\mathbb{Z}_{\mathrm{mot}}(j)_X$.

(3) In low weights, we have

$$\mathbb{Z}_{\mathrm{mot}}(0)(X) \cong \mathbb{Z}^{\pi_0(X)}[0]$$

$$\mathbb{Z}_{\mathrm{mot}}(1)(X) \cong R\Gamma_{\mathrm{Zar}}(X, \mathcal{O}_X^\times)[-1]$$

(4) $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ should be supported in degrees $0, \ldots, 2j$ (the support in ≥ 0 is known as the Beilinson-Soulé vanishing conjecture; the support in the range $\leq 2j$ is essentially the Gersten conjecture).

(5) (Relation to algebraic cycles) One has $H^{2j}_{\mathrm{mot}}(X, \mathbb{Z}(j)) \cong \mathrm{CH}^j(X)$

(6) (Relation to étale cohomology – Beilinson-Lichtenbaum conjecture) One has

$$H^i_{\mathrm{mot}}(X, \mathbb{Z}/m(j)) \cong H^i_{\mathrm{et}}(X, \mu_m^{\otimes j})$$

whenever $i \leq j$ and m is prime to the characteristic of the ground field k.

This is now (almost) completely settled.

Theorem 1.4.2 (Suslin, Voevodsky, Friedlander, Bloch, Levine, Rost, ...)**.** The above conjecture is true (except for Beilinson–Soulé vanishing, which remains unknown).

Several definitions of motivic cohomology for smooth varieties have been proposed over the years: it is now known that they are all equivalent (but the proofs of the equivalences are all highly non-trivial). Historically, the first attempt is due to Bloch. We recall here its definition.

Let $\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(1 - \sum_{j=0}^n t_j))$ be the algebraic n-simplex. It is (non-canonically) isomorphic to affine space $\mathbb{A}^n_k$, but has distinguished subschemes of codimension s given by the vanishing locus $(t_{i_1} = \ldots = t_{i_s} = 0)$, for every $0 \leq i_1 < \ldots < i_s \leq n$. We call these subvarieties faces, and they are (canonically) isomorphic to Δ^{n-s}. The assignment $[n] \mapsto \Delta^n$ defines then a cosimplicial scheme $\Delta^\bullet$.

Definition 1.4.3. Let X be a regular k-scheme. We denote by $z^j(X, n)$ the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^n$ of codimension j, such that for every face F of codimension s on Δ^n, every irreducible component of the intersection $Z \cap X \times F$ has codimension (at least) s in $X \times F$ (we say that such Z is in good position).

Taking alternating sums over the intersections with codimension 1 faces gives a complex $z^j(X, *)$

$$\ldots \to z^j(X, n+1) \xrightarrow{d} z^j(X, n) \xrightarrow{d} z^j(X, n-1) \to \ldots$$

We denote its homology $H_n(z^j(X, *))$ by $\mathrm{CH}^j(X, n)$. These groups are called Bloch's higher Chow groups.

Example 1.4.4. For $n = 1$, note that the only faces of $\Delta^1 \cong \mathbb{A}^1$ are given by the closed points $t = 0$ and $t = 1$. A cycle $Z \subset X \times A^1$ is in good position if and only if none of its irreducible components are completely contained in $X \times \{0\}$ or in $X \times \{1\}$. Then the group $\mathrm{CH}^j(X, 0)$ is the cokernel

$$z^j(X,1) \xrightarrow{i_0^* - i_1^*} z^j(X,0) = Z^j(X)$$

where i_ϵ^* denotes the intersection $Z \cap X \times \{\epsilon\}$ for $\epsilon = 0, 1$. In particular it is easy to see that $\mathrm{CH}^j(X, 0) = \mathrm{CH}^j(X)$ agrees with the classical Chow group, whence the name.

Part of the previous Theorem is to show that the complex $z^j(X, *)[-2j]$ (seen as a complex of sheaves on the small Zariski site of X) has all the conjectural properties. This goes through a comparison between Bloch's higher Chow groups and Voevodsky's definition of motivic cohomology (we refer to Déglise's talks for that).

1.5. **Milnor K-theory and the Bloch-Kato conjecture.** A major consequence of the theory of motivic cohomology is a proof of the so-called Bloch-Kato conjecture, relating Milnor K-theory and étale cohomology. Let us recall the definition.

Definition 1.5.1. Let R be a ring and $n \geq 0$. The j-th Milnor K-group of R is defined as

$$K_j^M(R) = \frac{R^\times \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} R^\times}{\langle a_1 \otimes \ldots \otimes a_j \,|\, a_m + a_n = 1 \text{ for some } m \neq n \rangle}$$

here the tensor product is as abelian groups, n-times.

Remark 1.5.2. There is a map $K_j^M(R) \to K_j(R)$, with kernel killed by $(j-1)!$

We write $\{a_1, \ldots, a_j\}$ for a typical element of $K_j^M(R)$. Recall from Krashen' lecture that for any field F and any m prime to the characteristic of F, we have the Tate symbol map

$$K_j^M(F)/m \longrightarrow H^j_{\mathrm{Gal}}(F, \mu_m^{\otimes j}) \tag{1.2}$$

induced by sending a symbol $\{a_1, \ldots, a_j\}$ to the cup product $a_1 \cap \ldots \cap a_j$ where $a_i \in K_1^M(F)/m = F^\times/(F^\times)^m \cong H^1_{\mathrm{Gal}}(F, \mu_m)$ by Kummer theory.

Theorem 1.5.3. (Bloch-Kato conjecture, Theorem of Voevodsky and Rost). For all F and m as above, the Tate symbol map is an isomorphism.

The relevance of motivic cohomology to this problem is the surprising fact that Milnor K-theory turns out to be motivic in nature.

Theorem 1.5.4. For any field F, there are natural isomorphisms

$$K_j^M(F) \cong H^j_{\mathrm{mot}}(F, \mathbb{Z}(j)) \tag{1.3}$$

for all $j \geq 0$

There are two approaches to this problem. Using the definition of motivic cohomology in terms of Bloch's higher Chow groups, it is a theorem of Nesterenko-Suslin and Totaro that $\mathrm{CH}^j(F, j) \cong K_j^M(F)$ for every $j \geq 0$ (this was later generalized by Kerz for regular local rings S containing a field). This can be seen explicitly by associating to every symbol a special "graph" cycle. For another approach, see §2.4 of Déglise' lecture notes.

If we take mod m coefficients in (1.3), we get

$$K_j^M(F)/m \cong H^j_{\mathrm{mot}}(F, \mathbb{Z}/m(j))$$

and the Beilinson-Lichtenbaum conjecture predicts exactly that $H^j_{\mathrm{mot}}(F, \mathbb{Z}/m(j)) \cong H^j_{\mathrm{Gal}}(F, \mu_n^{\otimes j})$.

Remark 1.5.5. Conversely, to see that the Bloch-Kato conjecture essentially implies the Beilinson-Lichtenbaum conjecture, see Chapter 2 of [HW19].

2. Lecture 2

2.1. **Mod p K-theory in characteristic p.** Given a field F and $m \geq 1$ prime to the characteristic of F, the Beilinson–Lichtenbaum conjectures say that the mod m motivic cohomology is given by $\mathbb{Z}_{\mathrm{mot}}(j)(F)/m \simeq \tau^{\leq m} R\Gamma_{\mathrm{Gal}}(F, \mu_m^{\otimes j})$; so the Atiyah–Hirzebruch spectral sequence with mod m coefficients looks like

$$E_2^{ij} = H^{i-j}_{\mathrm{mot}}(F, \mathbb{Z}/m(j)) = \begin{cases} H^{i-j}_{\mathrm{Gal}}(F, \mu_m^{\otimes j}) & i \leq 0 \\ 0 & \text{else} \end{cases} \implies K_{-i-j}(F, \mathbb{Z}/m)$$

Thus we describe both motivic cohomology and K-theory in terms of a known invariant, namely étale cohomology. Moreover, the j-axis of the spectral sequence is given by mod m Milnor K-theory thanks to the Bloch–Kato isomorphisms $H^j_{\mathrm{Gal}}(F, \mu_m^{\otimes j}) \cong K_j^M(F)/m)$.

In this section we explore similar results in which m is no longer prime to the characteristic. For simplicity we focus on the case that $m = p$ is equal to the characteristic.

Theorem 2.1.1 (Bloch–Kato–Gabber 1986, Geisser–Levine 2000)**.** Let F be a field of characteristic $p > 0$. Then there are natural isomorphisms

$$K_j^M(F)/p \cong K_j^M(F)/p \cong \Omega^j_{F,\log}$$

for all $j \geq 0$.

To explain the theorem further, we introduce:

Definition 2.1.2. For any ring R, there is a group homomorphism

$$\mathrm{dlog} : K_j^M(R) \longrightarrow \Omega^j_R, \quad a_1 \otimes \cdots \otimes a_j \mapsto \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_j}{a_j}$$

(**Exercise**: check that this is well-defined), and when R is local we define $\Omega^j_{R,\log} \subseteq \Omega^j_R$ to be its image.

The content of the Bloch–Kato–Gabber theorem is that, for a field F of characteristic p, the dlog map induces an injection $\mathrm{dlog} : K_j^M(F)/p \hookrightarrow \Omega^j_F$, thus an isomorphism $K_j^M(F)/p \xrightarrow{\sim} \Omega^j_{F,\log}$. The proof is essentially an explicit calculation. We mention also Izhbolidin's 1990 theorem, stating that $K_j^M(F)$ has no p-torsion.

Meanwhile, Geisser–Levine's contribution is really a theorem about the motivic cohomology; they showed that

$$\mathbb{Z}_{\mathrm{mot}}(j)(F)/p \simeq \Omega^j_{F,\log}[-j],$$

i.e., the motivic complex mod p of F is supported in the single cohomological degree j, where it is given by $\Omega^j_{F,\log}$. They prove this using the Frobenius action on Bloch's cycle complex. Therefore the Atiyah–Hirzebruch spectral sequence with mod p coefficients collapses to

$$E_2^{ij} = \begin{cases} \Omega^j_{F,\log} & i = 1 \\ 0 & \text{else} \end{cases} \implies K_{-i-j}(F; \mathbb{Z}/p),$$

giving the desired isomorphisms $\Omega^j_{F,\log} \cong K_j(F; \mathbb{Z}/p)$.

Remark 2.1.3. From a K-theoretic perspective, the remarkable part of the Geisser–Levine theorem is that the canonical map $K_j^M(F) \to K_j(F)$ is surjective mod p. For the moment there is no known "direct" proof of this bypassing motivic cohomology.

We also note that there is a conjecture of Beilinson predicting that $K_j^M(F) \to K_j(F)$ should be an isomorphism rationally (for any field F of characteristic p); this remains wide open.

2.2. **Variants of motivic cohomology and K-theory.** The point of view we have hoped to adopt so far is that motivic cohomology is a refinement of algebraic K-theory which breaks it into "simpler" pieces, related to cycles, étale cohomology, and other existing invariants. This philosophy continues to apply in other related contexts; the ultimate degree of generality is not yet clear.

2.2.1. *Bloch's cycle complex as Borel–Moore homology.* Bloch's higher cycle complex $z^j(X,\bullet)$ makes sense for any scheme of finite type over a field (perhaps quasi-projective, but not necessarily smooth), and in that generality is Borel–Moore motivic *homology.* In fact, $X \mapsto z^j(X,\bullet)$ is covariant (as one expects of a homology theory), contravariant if we restrict to smooth X, and always fits into an Atiyah–Hirzebruch spectral sequence

$$E_2^{ij} = CH^{-j}(X, -i-j) \implies G_{-i-j}(X),$$

where the target denotes the G-groups of X defined in terms of coherent sheaves rather than locally free sheaves.

2.2.2. $\mathbb{A}^1$*-invariant theories.* There has been considerable work on defining theories of homotopy invariant motivic cohomology for Noetherian schemes X, i.e., complexes $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ for $j \geq 0$ such that the following (and more) hold:

(1) the motivic cohomology refines Weibel's homotopy invariant K-theory $\mathrm{KH}(X)$ in the sense that there exists an Atiyah–Hirzebruch spectral sequence $E_2^{ij} = H^{i-j}(\mathbb{Z}_{\mathrm{mot}}(-j)(X)) \implies \mathrm{KH}_{-i-j}(X)$.
(2) $X \mapsto \mathbb{Z}_{\mathrm{mot}}(j)(X)$ satisfies cdh-descent (like $X \mapsto \mathrm{KH}(X)$).
(3) When X is smooth over a field, $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ agrees with the existing theory presented above.

See the book of Cisinski–Déglise for work in this direction.

2.2.3. *Motivic cohomology with modulus.* Another context which has received a lot of attention is that of motivic cohomology with modulus, which detected ramification phenomena along a divisor. For example, let X be a smooth algebraic variety and $D \hookrightarrow X$ an effective Cartier divisor (not necessarily reduced). This theory introduces complexes $z^j(X|D,\bullet)$ of higher cycles on X which are "in good position with respect to X"; see in particular the work of Binda–Saito. It is hoped that these will refine the relative algebraic K-theory $K(X,D)$ via an Atiyah–Hirzebruch spectral sequence $E_2^{ij} = CH^{-j}(X|D, -i-j) \implies K_{-i-j}(X,D)$.

2.2.4. *Étale motivic cohomology.* K-theory is quite simple étale locally, at least away from the characteristic, thanks to the following theorems:

Theorem 2.2.5 (Gabber, Suslin)**.** Let A be a strictly Henselian local ring, and $m \geq 1$ prime to the characteristic of its residue field k. Then there are natural isomorphisms

$$K_j(A;\mathbb{Z}/m) \cong K_j(k;\mathbb{Z}/m) \cong \begin{cases} \mu_m(k)^{\otimes j/2} & j \geq 0 \text{ even} \\ 0 & j \geq 0 \text{ odd} \end{cases}$$

Unfortunately, K-theory is not determined étale locally; more precisely it does not satisfy descent for the étale topology, the problem being that if $A \subseteq B$ is a finite étale extension with Galois group G, then $K(A) \to K(B)^{hG}$ is not an equivalence in general. To get around this, various modifications of K-theory have been introduced where this failure is eliminated (Soulé, Friedlander, Thomason, Clausen-Mathew,...). Very roughly, to produce the universal such modification, we let $K^{\text{ét}} : \mathrm{op}\mathit{Schemes} \to \mathrm{op}\mathit{Spectra}$ be the étale sheafification (in the ∞-categorical) sense of the connective K-theory functor $K : \mathrm{op}\mathit{Schemes} \to \mathrm{op}\mathit{Spectra}$.

Up to subtle convergence issues (see Thomason and Clausen–Mathew), which work out if the scheme X has sufficiently bounded étale cohomological dimension, there will then be a Atiyah–Hirzebruch spectral sequence

$$E_2^{ij} = H^{i-j}_{\text{ét}}(X, \mu_m^{\otimes -j}) \implies K^{\text{ét}}_{-i-j}(X; \mathbb{Z}/m)$$

for any $m \geq 1$ which is prime to all residue characteristics of X. Note that, unlike usual K-theory and motivic cohomology, the étale cohomology is not being truncated à la Beilinson–Lichtenbaum – we are seeing all degrees and all Tate twists.

2.3. **Recent progress: étale motivic cohomology at the characteristic.** The remainder of the notes is devoted to describing the aforementioned étale K-theory in terms of a "motivic" invariant when we are no longer prime to the residue characteristics; so fix a prime number p. By Zariski gluing we focus on affine schemes $\operatorname{Spec} R$, and we even allow ourselves to replace R by its p-adic completion $\hat{R}$; we are allowed to do this thanks to the gluing square

$$\begin{array}{ccc} R & \longrightarrow & R[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \hat{R} & \longrightarrow & \hat{R}[\frac{1}{p}], \end{array}$$

since in principle we already understand the two terms on the right (note that p is prime to their residue characteristics!).

Theorem 2.3.1 (Bhatt–M.–Scholze, Antieau–Mathew–M.–Nikolaus, Lüders–M, Bhatt–Clausen–Mathew, Kelly–M.)**.** To any p-adically complete ring R (or more generally derived p-complete simplicial ring), we may naturally associate a family of derived p-complete complexes

$$\mathbb{Z}_p(j)(R), \quad j \geq 0$$

satisfying the following analogues of the Beilinson–Lichtenbaum and Bloch–Kato conjectures:

(1) Setting $H^*_{\text{ét-mot}}(R, \mathbb{Z}_p(j)) := H^*(\mathbb{Z}_p(j)(R))$, there is an Atiyah–Hirzebruch spectral sequence converging to the p-adic completion of the étale K-theory of R:
$$E_2^{ij} = H^{i-j}_{\text{ét-mot}}(R, \mathbb{Z}_p(-j)) \implies K^{\text{ét}}_{-i-j}(R; \mathbb{Z}_p)$$
(2) In low weights we have $\mathbb{Z}_p(0)(R) \simeq$ and $\mathbb{Z}_p(1)(R) \simeq R\Gamma_{\text{ét}}(\hat{R}, \mathbb{G}_m)$.
(3) Range of support: $\mathbb{Z}_p(j)(R)$ is supported in degrees $\leq j+1$ (and in degree $\leq$ if the R-modules $\Omega^1_{R/p}$ can be generated by $\leq$elements); it is supported in degrees ≥ 0 is R is quasisyntomic (maybe not enough time to discuss this notion, which is a mild smoothness condition on R).
(4) There is a Nesterensko–Suslin style isomorphism
$$K_j^{\hat{M}}(R) \cong H^j_{\text{ét-mot}}(R, \mathbb{Z}_p(j))$$
if R is local with infinite residue field.
(5) Comparison to Geisser–Levine: if R is smooth over a field of characteristic p (or more generally a regular Noetherian $\mathbb{F}_p$-algebra, or even more generally a "Cartier smooth" $\mathbb{F}_p$-algebra, then there are natural equivalences $\mathbb{Z}_p(j)(R)/p \simeq R\Gamma_{\text{ét}}(\operatorname{Spec} R, \Omega^j_{\log})[-j]$.

It turns out that the top degree of this motivic cohomology is also understood: it is given by $\mathbb{H}^{j+1}_{\text{ét-mot}}(R, \mathbb{Z}_p(j))\operatorname{proj}\lim_{r\geq 0} \tilde{\nu}_r(j)(R)$ where $\tilde{\nu}_r(j)(R)$ is a certain "mod p^r, weight j, Artin–Schreier obstruction". For example,

$$\tilde{\nu}_1(j)(R) = \operatorname{coker}(1 - C^{-1} : \Omega^j_{R/pR} \to \Omega^j_{R/pR}/d\Omega^{j-1}_{R/pR}$$

where the map is $adb_1 \wedge \cdots \wedge db_j \mapsto (a - a^p b_1^{p-1} \cdots b_j^{p-1}) db_1 \wedge \cdots \wedge db_j$. This invariant is related to class field theory, the Tate conjecture, and the Kato conjectures.

For p-adic cohomologies of p-adic rings, the difference between Zariski and étale cohomology is much less than when working prime to the characteristic; this is essentially because the p-étale cohomological dimension of any ring of characteristic p is ≤ 1. Although the previous theorem concerns étale K-theory, it therefore still has consequences for usual K-theory:

Theorem 2.3.2 (continued from previous theorem)**.** If R is local (and still p-adically complete), then discarding the top degree of $\mathbb{Z}_p(j)(R)$ results in a motivic invariant describing the (connective) p-adic K-theory of R, i.e., an Atiyah–Hirzebruch spectral sequence

$$E_2^{ij} = \begin{cases} H^{i-j}_{\text{ét-mot}}(R, \mathbb{Z}_p(-j)) & i \leq 0 \\ 0 & i > 0 \end{cases} \Rightarrow K_{-i-j}(R; \mathbb{Z}_p)$$

Exercise 2.3.3. In the lecture we will examine this spectral sequence for truncated polynomial algebras over a field of characteristic p, and for $\mathbb{Z}/p^n\mathbb{Z}$.

3. Problems and Exercises

3.1. K-theory.

Exercise 3.1.1 (Milnor patching)**.** Let I be an ideal in a ring R and let $f\colon R \to S$ be a ring map such that I is mapped isomorphically onto an ideal of S (also denoted I). The square

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

is called a Milnor square.

1. Given an S-module M_1, R/I-module M_2, and an S/I-module isomorphism $g\colon M_2 \otimes_{R/I} S/I \xrightarrow{M}{}_1 /IM_1$, show that there exists a unique R-module M such that $M/IM = M_2$ and $M \otimes_R S = M_1$. We say that M is obtained by patching M_1 and M_2 along g.
2. Show that if P is obtained by patching the free modules S^n and R/I^n along $g \in GL_n(S/I)$, and Q is obtained by patching together the free modules S^n and R/I^n along g^{-1}, then $P \oplus Q \cong R^{2n}$ as R-modules. Hint: the patching is obtained along the matrix $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in GL_{2n}(S/I)$.
3. If P is obtained by patching together a finitely generated projective S-module P_1 and a finitely generated projective R/I-module P_2, show that P is a finitely generated projective R-module. Hint: use part (2).
4. Show that every finitely generated projective R-module is obtained by patching.

Exercise 3.1.2. Let R be a ring and I be an ideal contained in the Jacobson radical of R. Show that $K_0(R) \cong K_0(R/I)$. In particular, show that this implies that for any local ring R we have an isomorphism $K_0(R) \cong \mathbb{Z}$.

Exercise 3.1.3. Check that the sequence (1.1) is indeed exact. Why is the map $Cl(\mathcal{O}) \to K_0(\mathcal{O})$ a well defined group homomorphism?

Exercise 3.1.4. Check that for a ring R, the subgroup $E(R)$ of $GL(R)$ is normal and we have $E(R) = [GL(R), GL(R)]$.

Exercise 3.1.5. What does Bass' finite generation conjecture say about $K_0(\mathcal{O}_F)$ and $K_1(\mathcal{O}_F)$ for $\mathcal{O}_F$ the ring of integers of a number field F?

Exercise 3.1.6. (Harder) Let F be a number field and $m \geq 1$ odd. Use the motivic Atiyah-Hirzebruch spectral sequence with mod m coefficients together with the Beilinson-Lichtenbaum conjecture to express $K_n(F, \mathbb{Z}/m)$ in terms of étale cohomology. Note that the m-cohomological dimension of F is ≤ 2.

3.2. **Weight one motivic cohomology.** The goal of this series of exercises is to compute the motivic complexes $\mathbb{Z}(1)$ and $\mathbb{Z}(1)/\ell$, following Voevodsky [MVW06]. Let k be a field and let $\mathbf{Cor}(k)$ be the additive category of finite correspondences. For any regular k-scheme, we write $\mathbb{Z}_{tr}(X)$ for the presheaf of abelian groups $Y \mapsto \mathbf{Cor}_k(Y, X)$.

Recall from the lectures that for any additive presheaf $F \in \mathrm{PSh}(\mathbf{Cor}(k), Ab)$ we have a complex of presheaves $X \mapsto C_*(F)(X)$,

$$C_*(F)(X) := \quad \ldots \to F(X \times \Delta^2) \xrightarrow{d^2} F(X \times \Delta^1) \xrightarrow{d^1} F(X)$$

where $\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_1 = 1))$.

Let $\mathbb{Z}(\mathbb{G}_m) = \mathrm{Coker}(\mathbb{Z} \to \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}))$ where the morphism is induced by the rational point $1 \in \mathbb{A}^1 - \{0\}(k)$. We write $\mathbb{Z}(1)$ for the complex $C_*(\mathbb{Z}(\mathbb{G}_m))[-1]$. We are going to show that $\mathbb{Z}(1) \xrightarrow{\simeq} \mathcal{O}^\times[-1]$, where $\mathcal{O}^\times$ is the sheaf of global units.

Exercise 3.2.1. Let $\mathcal{M}^*$ be the functor $\mathbf{Sm}(k) \to Ab$ sending a scheme X to the group of rational function $X \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ which are regular in a neighborhood of $X \times \{0, \infty\}$ and exactly equal to 1 on $X \times \{0, \infty\}$.

(1) Show that $\mathcal{M}^*$ is a sheaf for the Zariski topology.
(2) Given $X \in \mathbf{Sm}(k)$ and a section $f \in \mathcal{M}^*(X)$, let $D(f)$ denote the *Weil divisor* associated to f. Show that $D(f)$ is supported on $X \times \mathbb{A}^1 - \{0\}$, and each of its components is finite and surjective over X. In particular, it defines an element of $\mathbf{Cor}(X, \mathbb{A}^1 - \{0\})$. Hint: reduce to the case where $X = \mathrm{Spec}(A)$ is affine, and write $f = f_+/f_-$ with $f_+, f_- \in A[t]$. Note that $D(f) = D(f_+) - D(f_-)$.
(3) Deduce that we have an injective morphism of sheaves (of Abelian groups) $\mathcal{M}^* \to \mathbb{Z}_t r(\mathbb{A}^1 - \{0\})$.

Exercise 3.2.2. (1) Show that for any $Z \in \mathbf{Cor}(X, \mathbb{A}^1)$ there is a unique rational function f on $X \times \mathbb{P}^1$ such that $Z = D(f)$ and $f/t^n = 1$ on $X \times \{\infty\}$ for some $n \in \mathbb{Z}$.
(2) Using the previous point, show that there is a well-defined and surjective map of sheaves $\lambda \colon \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}) \to \mathbb{Z} \oplus \mathcal{O}^\times$.
(3) Recall that the Norm map induces a structure of (pre)sheaf with transfers on $\mathcal{O}^\times$. Show that λ is compatible with transfers, i.e. that it is a morphism in $\mathrm{PSh}(\mathbf{Cor}(k), Ab)$. Hint: reduce to check the compatibility with the transfer structure for sections on fields.
(4) Show that the kernel of λ is exactly the subsheaf $\mathcal{M}^*$. Note that this implies that $\mathcal{M}^* \in \mathrm{PSh}(\mathbf{Cor}(k), Ab)$.

Using the previous two exercises we have a short exact sequence of complexes

$$0 \to C_*(\mathcal{M}^\times) \to C_*(\mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\})) \to C_*(\mathbb{Z} \oplus \mathcal{O}^\times) \to 0$$

and thus a short exact sequence of complexes (splitting off a $\mathbb{Z}$ term)

$$0 \to C_*(\mathcal{M}^\times) \to \mathbb{Z}(1)[1] \to C_*(\mathcal{O}^\times) \to 0 \tag{3.1}$$

Exercise 3.2.3. (1) Show that for every n and X regular and irreducible, we have $\mathcal{O}^\times(X \times \Delta^n) = \mathcal{O}^\times(X)$.

(2) Show that $C_*(\mathcal{M}^\times)(X)$ is an acyclic complex of abelian groups for every regular k-scheme. Hint: let $f \in C_i(\mathcal{M}^*)(X) = \mathcal{M}^*(X \times \Delta^i)$ be an element in the kernel of the differential. Consider the function $h(f) = 1 - t(1 - f)$ and show that $h(f)$ can be used to get a contracting homotopy for f. Conclude.
(3) Use (1), (2) and (3.1) to complete the proof of the Theorem.

Exercise 3.2.4. Using the previous results and the universal coefficient theorem for étale cohomology, show that the étale sheafification of $\mathbb{Z}/\ell(1)$ is isomorphic to μ_ℓ.

Exercise 3.2.5. Assume that $\ell \in k^\times$ and that X is regular. Compute the following motivic cohomology groups for every $p \geq 0$:
(1) $H^p_{\mathrm{Zar}}(X, \mathbb{Z}(1))$;
(2) $H^p_{\mathrm{Zar}}(X, \mathbb{Z}/\ell(1))$.

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