Weak solutions to Ideal MHD

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Ideal Magnetohydrodynamics

The equations in the Torus.

$$\partial_t u + \operatorname{div}(u \otimes u - B \otimes B) + \nabla \Pi = 0,$$
$$\partial_t B + \nabla \times B \times u = 0,$$
$$\operatorname{div}\ u = \operatorname{div}\ B = 0,$$
$$\int_{\mathbb{T}^3} u(x,t)\,dx = \int_{\mathbb{T}^3} b(x,t)\,dx = 0 \qquad \text{for almost every } t \in [0,T[,$$

- u is the velocity field, B the magnetic field, $\Pi = p + \frac{1}{2}|B|^2$ the Total presure.
- Ampere law $J = \nabla \times B$. Ohm law $E = \frac{1}{\sigma} J + u \times B$
- The evolution of B is given by Faraday law of induction in combination with Ohm (and Ampere if there is resistivity).
- The first equation is Euler (N-S) with the Lorentz force as external force. $\mathcal{F}_L = J \times B = (\nabla \times B) \times B = B \cdot \nabla B + \nabla \frac{1}{2}|B|^2$

Three preserved integral cuantities in the smooth regime

Vector Potential of b

$\nabla \times \Psi = b$ and $\int_{\mathbb{T}^3} \Psi(x,t)\,dx = 0$ for every $t \in]0,T[$

We define three classically conserved quantities of ideal 3D MHD on the torus $\mathbb{T}^3$; Previous results allow certain Besov regularity in the spirit of Onsager conjecture.

Total Energy $\frac{1}{2}\int_{\mathbb{T}^3}(|u(x,t)|^2 + |b(x,t)|^2)\,dx$,

Cross Helicity $\int_{\mathbb{T}^3} u(x,t)\cdot b(x,t).\,dx.$

Magnetic Helicity $\mathcal{H}(t) = \int_{\mathbb{T}^3} \Psi(x,t)\cdot b(x,t)\,dx$,

Conserved for $u, b \in L^3$ (Kang and Lee, Aluie-Eyink)

Exercise: if we solve the Faraday system.

$$\partial_t \mathcal{H}(t) = -2\int E\cdot B dx$$

Corollary: If $E = B\times u$, magnetic helicity is constant. Notice that $E \in L^{\frac{3}{2}}$.

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More Results

- Bronzi, Lopes, Lopes. In $3D$ There exists compactly supported in time solutions of MHD with non trivial b.

  $$u(x_1, x_2, x_3, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0),$$
  $$b(x_1, x_2, x_3, t) = (0, 0, b_3(x_1, x_2, t)),$$

  Such u, b solve MHD equivalent to solve passive tracer (b) equation in $2D$
- F-Lindberg. Magnetic Helicity is preserved in the vanishing resistivity limit of Leray-Hopff solutions.
- Beekie-Buckmaster-Vicol. There exists a $\beta > 0$ and a weak solutions in $C[(0, T), H^\beta]$ which do not preserve magnetic helicity. Extended by Lie, Zeng and Zhang to vanishing resistivity limits.
- Convex integration for Hall MHD (M.Dai), EMHD (M.Dai and Han Liu)

Theorems F-Lindberg-Székelyhidi Jr

- Theorem 1. There exists solutions $u, B \in L^\infty[(0,T), L^{3,\infty} \times L^{3,\infty})]$ which do not preserve magnetic helicity, nor energy nor cross helicity.
- Theorem 2. There exists bounded solutions which do not preserve energy, nor cross helicity but whose helicity (constant a forteriori) is an arbitrary constant h.

Faraday system in terms of forms

The compensated compactness of the Faraday system (from Luc Tartar notes)
The Faraday 2 form: $\omega \in \Lambda^2(\mathbb{R}^3_x \times \mathbb{R}_t)$

$$\omega = \sum \epsilon_{ijk} B_i dx^j \wedge dx^k + E_i dx^i \wedge dt$$

The Faraday system is equivalent to ω being closed. $d\omega = 0$.
$\alpha = \psi_i dx^i + \varphi dt$

$$\omega = d\alpha, B = \nabla \times \psi, E = \nabla_x \varphi - \partial_t \psi$$

$$\omega \wedge \omega = E \cdot B dV$$

But $\omega \wedge \omega = d(\alpha d\alpha)$ which implies is a compensated compactness (weakly continuous) Quantity!

$E \cdot B = 0$ in the relaxation of MHD

Relaxed MHD=weak solutions, Subsolutions, coarse-grained solutions.

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Projection from the Faraday system to Relaxed MHD

Theorem 3. Let $\bar{\omega} = (\overline{B}, \overline{E}) \in L^\infty(\mathbb{T}^3 \times [0, T])$ be a p.c solution to $d\bar{\omega} = 0$ in the sense of distributions, and let p, p' Hölder-dual exponents with $3/2 < p < \infty$.
Then there exist p.c $\omega = (B, E) \in L^1(\mathbb{T}^3 \times [0, T])$ solving $d\omega = 0$

- $B \in L^\infty(0, T; L^{p,\infty}(\mathbb{T}^3)), \quad E \in L^\infty(0, T; L^{p',\infty}(\mathbb{T}^3))$
- $B \cdot E = 0$
- $\mathcal{H}(B)(t) = \mathcal{H}(\overline{B})(t)$

The proof is based on the Tartar framework adapted to the Faraday system, and a convex integration type iteration for unbounded sets (Staircase laminates).

Projection from Relaxed MHD to MHD

Theorem 4. Let $\bar{\omega} = (\bar{B}, \bar{E})$ piecewise constant (p.c) with $d\bar{\omega} = 0$ $\omega \wedge \omega = \bar{B} \cdot \bar{E} = 0$. Then there exists a constant such that if ξ_+, ξ_- regular enough (and positive) , and

$$|\bar{B}|^2 + |\bar{E}| \leq M_0 min|\xi_+|^2, \xi_-|^2$$

Then there exists u, B solving MHD and such that

- $|u + B| = \xi_+$
- $|u - B| = \xi_-$
- $MH(\bar{B}) = MH(B)$

The proof is based on the Tartar framework for the full MHD. Apart from the fact that the sets live in R^{15}, The K^{Λ} has non empty interior.

Basic Construction

The Faraday wave cone. $\Lambda^F = \{w = (\bar{B}, \bar{E}) : \bar{B} \cdot \bar{E} = 0\}$. Indeed $\bar{E} = |\bar{E}|\xi, \bar{B} = |B|\eta \times \xi$. Let $\omega_1, \omega_2 \in \mathbb{R}^3$ with $\omega_1 - \omega_2 \in \Lambda^F$ and $\lambda_1, \lambda_2 \in (0, 1)$ with $\lambda_1 + \lambda_2 = 1$. For any open bounded domain $Q \subset \mathbb{R}^4$ with $|\partial Q| = 0$ and any $r, \epsilon > 0$ there exist p.c $\omega \in L^\infty(Q; \mathbb{R}^3)$ satisfying $d\omega = 0$ with "boundary conditions" given by $\omega_0 = \lambda_1\omega_1 + \lambda_2\omega_2$

- $Q = Q^{(1)} \cup Q^{(2)} \cup Q^{(error)} \cup \mathcal{N}$ where $\mathcal{N}$ a nullset, $Q^{(1)}, Q^{(2)}$ and $Q^{(error)}$ are open sets where ω is locally constant, and such that $\omega = \omega_i$ in Q_i, $i = 1, 2$ and $|\omega - \omega_0| < r$ in $Q^{(error)}$.
- For $i = 1, 2$ and any $t \in \mathbb{R}$

$$|Q^{(error)}(t)| + \frac{1}{\lambda_i}|Q^{(i)}(t)| \leq (1 + \epsilon)|Q(t)| \qquad (1)$$

  and $|Q^{(error)}| \leq \epsilon|Q|$.
- If $\eta \cdot B_0 = 0$ and $\tilde{\omega}$ is a p.c with $\omega\chi_Q = \omega_0$.

$$\mathcal{H}(\omega + \tilde{\omega}) = \mathcal{H}(\omega)$$

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Laminates.Definition

The above lemma implies that the measure $\lambda\delta_{\omega_1} + \lambda_2\delta_{\omega_2}$ is well approximated with the distribution of a p.c solution to Faraday.
The set of *laminates* (with respect to Λ), denoted $\mathcal{L}(Y)$, is the smallest class of atomic probability measures supported on Y with the following properties:

(i) $\mathcal{L}(Y)$ contains all the Dirac masses with support in Y.

(ii) $\mathcal{L}(Y)$ is closed under splitting along Λ-segments inside Y.

(iii) $\mathcal{L}(Y)$ is weakly closed.

Condition (ii) means that if $\nu = \sum_{i=1}^{M} \nu_i \delta_{V_i} \in \mathcal{L}(Y)$ and $V_M \in [Z_1, Z_2] \subset Y$ with $Z_1 - Z_2 \in \Lambda$, then

$$\sum_{i=1}^{M-1} \nu_i \delta_{V_i} + \nu_M(\lambda\delta_{Z_1} + (1-\lambda)\delta_{Z_2}) \in \mathcal{L}(Y),$$

where $\lambda \in [0,1]$ such that $V_M = \lambda Z_1 + (1-\lambda)Z_2$.

Staircase Laminates

For any β, ω_0 there exist a (Faraday) laminate such that

- $\int \lambda d\nu = \omega_0$
- ν is supported in $|E||B| = 0$
- $\nu(\{|B|^p + |E|^{p'} \geq t\}) \leq \frac{\beta^p}{t}$

Building block for staircase laminates

Everything as in the building block for laminates but now:

- $Q = Q^{(good)} \cup Q^{(error)} \cup \mathcal{N}$ $|B||E| = 0$ in $Q^{(good)}$.
- For all t and all $s > 1$ we have

$$\mathcal{I}(s,t) \leq \beta^{2(p+1)} |Q(t)| \min(|B_0|^p + |E_0|^{p'}, s), \tag{2}$$

  where $\mathcal{I}(s,t)$ is

$$\int_{Q^{(error)}(t)} \min\{|B|^p + |E|^{p'}, s\} dx + s \left| \{x \in Q^{(good)}(t) : |B|^p + |E|^{p'} > s\} \right|$$

  and p' is the Hölder dual of p.
- $\int \int_{Q^{(error)}} |B|^p + |E|^{p'} \, dxdt \leq \epsilon$.
- Magnetic helicity is conserved.

Theorem

Let ω_0, $1 < p < \infty$ *and* $Q \subset \mathbb{R}^4$ *an open bounded domain with* $|\partial Q| = 0$. *there exist piecewise constant vector fields* ω *with* $d\omega = 0$ *and the boundary condition* ω_0

- $|B||E| = 0$ *for a.e.* $(x, t) \in Q$;
- *For all* t *and any* $s > 1$

$$\left|\left\{x \in Q(t) : |B|^p + |E|^{p'} > s\right\}\right| \leq \frac{2}{s}|Q(t)| \min(|B_0|^p + |E|_0|^{p'}, s), \tag{3}$$

  so that, in particular, $B \in L_t^\infty L_x^{p,\infty}$ *and* $E \in L_t^\infty L_x^{p',\infty}$.
- *There exists a vector potential* $\tilde{A} \in Lip_0(Q)$ *which guarantees preservation of magnetic helicity.*
  $\int_{\mathbb{R}^3} [(A_0 + \tilde{A}) \cdot B - A_0 \cdot B_0] \, dx = 0 \;\; a.e. \;\; t \in \mathbb{R}$.

Projecting to MHD

We are given now $\omega = \sum \omega_i \chi_{\Omega_i(x,t)}$ such that $\omega_i \wedge \omega_i = B_i \cdot E_i = 0$. The plan is to apply convex integration at each ω_i adding a compactly supported perturbation u_i, B_i. Now however the linear system is more complicated and there is a nonlinear constraint.

- The linear system.

$$\nabla \cdot u = \nabla \cdot B = 0, \tag{4}$$

$$\partial_t u + \nabla \cdot S = 0, \tag{5}$$

$$\partial_t B + \nabla \times E = 0 \tag{6}$$

$S \in \mathbb{R}^{3\times 3}_{\text{sym}}$

- The constitutive relations.
$K := \{(u, S, B, E) \colon S = u \otimes u - B \otimes B + \Pi I,\ \Pi \in \mathbb{R},\ E = B \times u\}$.
Note that if (u, S, B, E) satisfies (**??**)–(**??**) and takes values in K a.e. (x, t), then (u, B, Π) satisfies the MHD equations

The Λ geometry

- The Lambda cone (up to corner cases).

$$\Lambda^{MHD} = \{(u, S, B, E) : S(B \times u) + (E \cdot u)u = 0 \quad ; E \cdot B = 0\}$$

- $E \cdot B$ is still Λ affine (Compensated compactness quantity)

$$K^{\Lambda} \subset \mathcal{M} := \{(u, S, B, E) : B \cdot E = 0\}$$

Problem: When we approximate a laminate by an actual solution to the linear system, the error falls off the manifold $\mathcal{M}$. K^{Λ} has non empty interior.

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Simple forms make life simpler

$\omega \wedge \omega = 0 \iff \omega = v \wedge w.$

Lemma

It turns out that $\xi = (\xi_x, \xi_t)$ *is a* Λ *direction for* (E, B) *if and only if* $\omega \wedge \xi = 0$. *Suppose that* ω_0 *and* $\omega \neq 0$ *are simple bivectors and that* $\omega \wedge \xi = 0$, *where* $\xi \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$. *The following conditions are equivalent:*

(i) $\omega_0 + t\omega$ *is simple for all* $t \in \mathbb{R}$.

(ii) $\omega_0 \wedge \omega = 0$.

(iii) *We can write* $\omega = v \wedge \xi$ *and either* $\omega_0 = v_0 \wedge \xi$ *or* $\omega_0 = v \wedge w_0$

In order to find potential, $\omega_0 = v_0 \wedge \xi$ is a bad case, and $\omega_0 = v \wedge w_0$ is a good case.

Good potentials

set

$$\alpha = \varphi \, d\psi, \omega = d\alpha = d\varphi \wedge d\psi;$$

Remark: It is not a Poincare lemma. Thus we need to find specific potentials for the planar waves This we can not do for all lines $\omega_0 + t\omega$. Namely if $\omega_0 = v_0 \wedge w_0$ and $\Phi_\ell(x,t) = x + \ell^{-1} h'(\ell x \cdot \xi) a$
$\Phi^*(v_0 \wedge w) = \Phi^* v_0 \wedge \Phi^* w_0 = d\psi \wedge d\varphi$

$$d\varphi_\ell(x,t) \wedge d\psi_\ell(x,t) = v_0 \wedge w_0 + \chi(x,t) h''(\ell(x,t) \cdot \xi) \underbrace{(c_2 v_0 - c_1 w_0)}_{=w} \wedge \xi + O\left(\frac{1}{\ell}\right),$$

Λ good segments

- Unfortunately this does not work for all Λ lines (not directions). If ξ is the direction of oscillation of ω and $\omega_0 = w \wedge \xi$,we can not construct potentials for the plane wave taking values in the line $\omega_0 + t\omega$
- Thus, segments (ω_0, ω) are divided in Λ_g (segments) and Λ_b (segments). Similarly we can speak of Λ_g laminates and of Λ_g lamination hull, where only good segments are involved.
- We can talk about good laminates and good Λ hulls and have good solutions to the linear system but..

A heart breaking discovery

K^{Λ_g} is rigid; $E = B \times u$

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The simple form formalism at work

Two good news.

- For open sets $\mathcal{O}^{lc,\Lambda} = \mathcal{O}^{\Lambda_g,lc}$
- If $\mathcal{O}$ is $\mathcal{M}$-open then $\mathcal{O}^{\Lambda,lc}$ is $\mathcal{M}$-open

Very difficult to prove if one does not work at the level of the factors of the simple two forms.

An strategy which do not require the full Λ hull

In order to prescribe the energy density and cross helicity densities we normlized K. Thus we declare

$$K_{r,s} = \{|u+B| = r, |u-B| = s,\} \qquad \mathcal{U}_{r,s} = \mathrm{int} K_{r,s}^{\Lambda,lc}$$

An interesting remark is that we do not compute the Λ hull of a constraint set $K_{r,s}$ but we show (lamination of order 12) such that for any $0 < \tau_0 < 1$, there exists $\epsilon_\tau > 0$,

$$B_\delta \subset \mathcal{U}_{r,s} = \cup_{1>\tau\geq\tau_0} \mathcal{O}_\tau^{\Lambda_g}$$

with $\mathcal{O}_\tau = B_{\mathcal{M}}(K_{\tau r, \tau s}, \epsilon_\tau), B_\delta = \{|B|^2 + |u|^2 + |S| + |E| \leq \delta \min r^2, s^2\}$ The construction is indeed closer in spirit very to the in-approximation approach but we argue by a Baire category argument in each subddomain and glue the solutions. It also helps that the potential $d\varphi \wedge d\psi$ are good for magnetic helicity concentration.

Thanks for the hospitality!