Sunflowers in semi-linear set systems

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The Sunflower Problem

Let $\mathcal{H} \subseteq \mathcal{P}(X)$ be a set-system on a ground set X.

Definition

A **sunflower** is a set-system $\mathcal{F}$ such that $A \cap B$ is the same for all pairs of distinct sets $A, B \in \mathcal{F}$. An **r-sunflower** implies $|\mathcal{F}| = r$.

A set-system $\mathcal{H}$ is ℓ-bounded if $|S| \leq \ell$ for each $S \in \mathcal{H}$.
Denote by $f_r(\ell)$ the maximum possible size of $\mathcal{H}$ such that it contains no sunflower of size r.

Lemma (Erdős - Rado '60)

$$(r-1)^\ell \leq f_r(\ell) \leq (r-1)^\ell \ell! \leq O(r\ell)^\ell$$

We write $f_r(\ell) \leq C(r,\ell)^\ell$. Is $C(r,\ell)$ independent of ℓ?

Improved bounds on $f_r(\ell)$ and VC-dimension

- $C(r,\ell) \leq O(r\ell)$ (Erdős–Rado '60)
- $C(r,\ell) \leq O(r^3 \log \ell \log\log \ell)$ (Alweiss–Lovett–Wu–Zhang '21)
- $C(r,\ell) \leq O(r \log \ell)$ (Bell–Chueluecha–Warnke '21)

Given $\mathcal{H} \subseteq \mathcal{P}(X)$, we say that $\mathcal{H}$ *shatters* a finite set $T \subseteq X$ if

$$\{T \cap S : S \in \mathcal{H}\} = \mathcal{P}(T).$$

Definition (Vapnik - Chervonenkis, 1971)

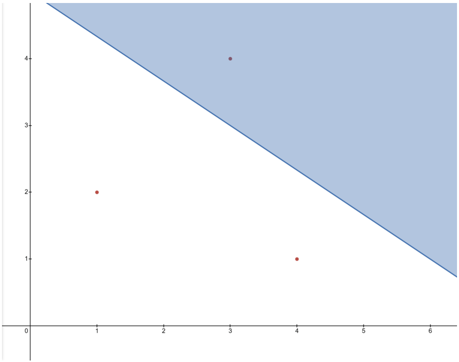
The *VC-dimension* of $\mathcal{H}$ is defined as

$$\mathrm{VC}(\mathcal{H}) := \sup\{|T| : T \subseteq X,\ \mathcal{H} \text{ shatters } T\} \in \mathbb{N} \cup \{\infty\}.$$

VC Dimension

VC Dimension: $X = \mathbb{R}^2$, and $\mathcal{H} =$ the set of half-planes in $\mathbb{R}^2$

Claim: $VC(\mathcal{H}) = 3$:



Motivation

Assume the ℓ-bounded set-system $\mathcal{H} \subseteq \mathcal{P}(X)$ has VC-dimension d, then we get improved bounds on $f_{r,d}(l)$.

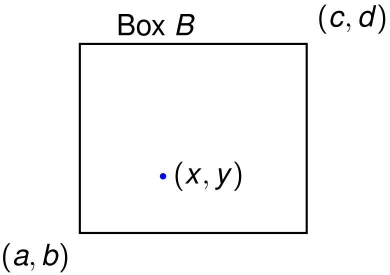
- $C_d(r,\ell) \leq 2^{10(dr)^{\log^* \ell}}$ (Fox–Pach–Suk '23)
- $C_d(r,\ell) \leq O(r(\log d + \log^* \ell))$ (Balogh et al. '24+)
- $C_1(r,\ell) \leq r - 1$ (Balogh et al. '24+)

Fox et al. '23 asked whether we get an independent of ℓ bound on $C(r,\ell)$ for set-systems $\mathcal{H}$ arising from the intersection of ground set X with semi-algebraic sets of bounded description complexity. These families have bounded VC-dimension.

Sunflower Problem in semi-linear set systems

Definition

Given ground set $X \in \mathbb{R}^{d_1}$ and a defining set $Y \in \mathbb{R}^{d_2}$, $x \sim y$ for $x \in X, y \in Y$ if they satisfy a system (boolean combination) of linear inequalities $\phi(x, y)$. Let $N(y) := \{x \in X : x \sim y\}$ and $\mathcal{H} := \{N(y) : y \in Y\}$.



$$(x, y) \in B \iff \{a < x < c, b < y < d\}$$

Our progress

Theorem

Given ℓ-bounded set system $\mathcal{H} \subseteq \mathcal{P}(X)$ defined by semi-linear sets with bounded description complexity with $|\mathcal{H}| > r\ell^c$, then $\mathcal{H}$ contains an r-sunflower. The constant $c \geq 1$ depends on the description complexity of the set system.

For $r \geq 3$ and $\ell \geq 2$, observe $r\ell^c \leq (r\ell)^c \leq r^{c\ell} = C(r)^\ell$.

Corollary

The bound works for families of sets arising from the intersection of $X \subset \mathbb{R}^d$ with polytopes generated by arbitrary translates of finitely many half-spaces in $\mathbb{R}^d$.

Proof idea

1. Induction on the complexity of the semi-linear set system
2. Rewrite a linear inequality $f(x, y) < 0$ as $f_1(x) + f_2(y) < 0$ for f_1, f_2 linear. For $y_1, y_2 \in Y$ with $f_2(y_1) < f_2(y_2)$, we have $N(y_2) \subseteq N(y_1)$. Thus, the set of neighbourhoods form a chain w.r.t $\subseteq$. If a chain of r-many sets contain the same elements, then they form an r-sunflower. Thus, chain with length $> r\ell$ contains an r-sunflower.
3. If we find k-sunflowers each w.r.t $f(x, y) < 0$ and w.r.t $g(x, y) < 0$ on same set of elements $\{y_1, y_2 \cdots y_r\}$, then we have a k-sunflower w.r.t the boolean combinations of f and g.

Thank You!