# Homomorphism densities from edge-coloured trees and alternating paths into edge-coloured graphs

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# Preliminaries

# Homomorphism density

For two edge-coloured graphs $H$ and $G$, a *homomorphism* from $H$ to $G$ is a graph homomorphism that also preserves edge colours. We let $\mathrm{Hom}(H, G)$ denote the set of all such homomorphisms and write $\mathrm{hom}(H, G) = |\mathrm{Hom}(H, G)|$.

**Definition**

The **homomorphism density** of $H$ in $G$ is

$$t(H, G) := \frac{\mathrm{hom}(H, G)}{v(G)^{v(H)}},$$

where $v(F)$ denotes the number of vertices of $F$.

This can also be interpreted as the probability of a uniformly sampled function $\phi : V(H) \to V(G)$ being a colour-preserving homomorphism.

# Previous results

Let $P_l^A$ denote a 2-edge-coloured path with $l$ edges such that no two incident edges have the same colour.

**Theorem (Basit et al.)**

$$t(P_{2k}^A, G) \leq \left(\frac{1}{2}\right)^{2k}$$

**Theorem (Chen et al.)**

$$t(P_{2k+1}^A, G) \leq \left(\frac{k}{2k+1}\right)^{k} \left(\frac{k+1}{2k+1}\right)^{k+1}$$

# Even covering lemma

**Lemma (Chen et al.)**

Let T be any edge-coloured tree. If $H$ is a non-empty edge-coloured forest such that there exists a homomorphism $\varphi$ from $H$ to $T$ which covers every edge and vertex of $T$ exactly $e(H)/e(T)$ times, then

$$t(T, G)^{1/e(T)} \leq t(H, G)^{1/e(H)}$$

for every edge-coloured graph G.

# 3-edge-coloured alternating paths

# 3-edge-coloured alternating paths

Let $P_l^{(3)}$ denote any 3-edge-coloured path with $l$ edges such that no two incident edges have the same colour. Let $c_1$ and $c_l$ be the "starting" and "ending" colours.

**Theorem (3-edge-coloured alternating paths)**

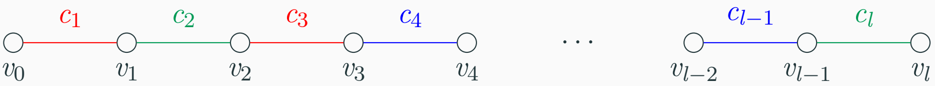
$$t(P_l^{(3)}, G) \leq \begin{cases} \frac{1}{2^l}\left(1 - \frac{1}{n}\right)^l & c_1 \neq c_l \\ \frac{1}{2^{l-1}}\left(1 - \frac{1}{n}\right)^l & c_1 = c_l \end{cases}$$

for any $n$-vertex edge-coloured $G$.

# 3-edge-coloured alternating paths

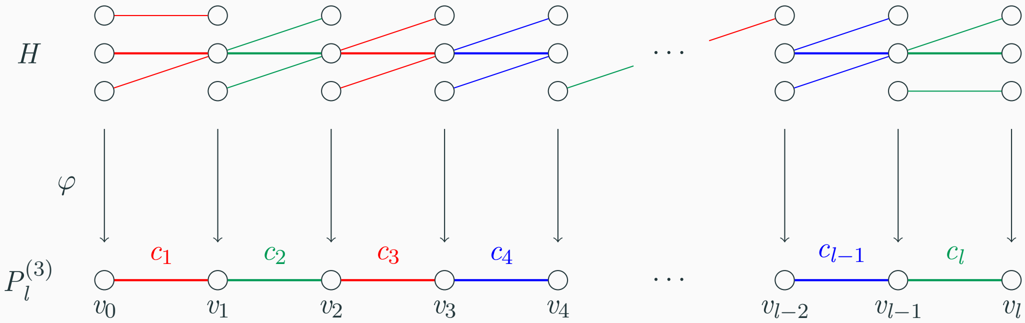
**Proof:**

Label the vertices and colours of $P_l^{(3)}$.



# 3-edge-coloured alternating paths

Construct $H$ from $P_l^{(3)}$ as follows, such that there is a homomorphism $\varphi : H \to P_l^{(3)}$ that covers each edge and vertex exactly 3 times.



# 3-edge-coloured alternating paths

Fix an $n$-vertex edge-coloured graph $G$.

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We can view $P_l^{(3)}$ as a subgraph of $H$. Then for any $\tilde{f} \in \mathrm{Hom}(H, G)$, we have $\tilde{f}|_{P_l^{(3)}} = f \in \mathrm{Hom}(P_l^{(3)}, G)$.

# 3-edge-coloured alternating paths

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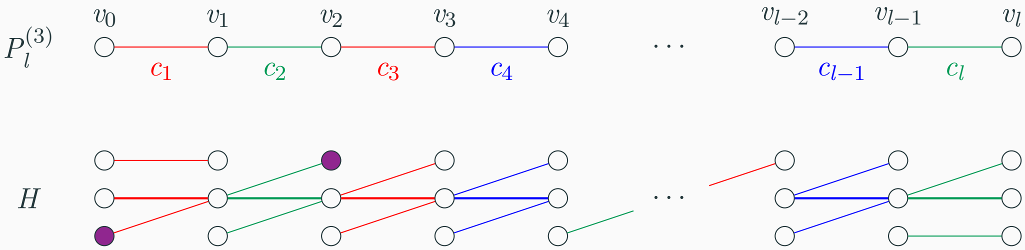
Thus we can count $\hom(H, G)$ by picking a $f \in \mathrm{Hom}(P_l^{(3)}, G)$ and then deciding where the duplicate vertices map to.

$$\hom(H, G) = \sum_{f \in \mathrm{Hom}(P_l^{(3)}, G)} \hom(H, G; f)$$

# 3-edge-coloured alternating paths

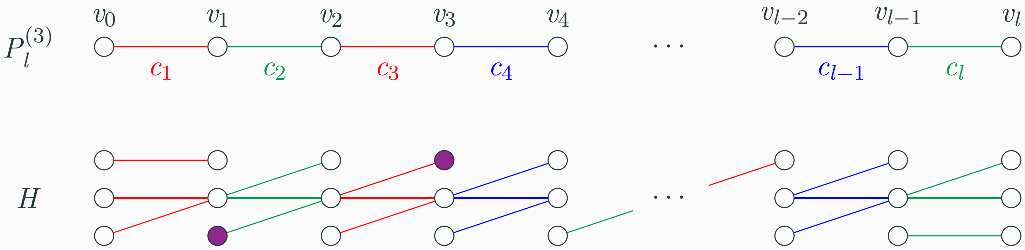



$$\hom(H, G) = \sum_{f \in \mathrm{Hom}(P_l^{(3)}, G)} \hom(H, G; f)$$

$$= \sum_{f \in \mathrm{Hom}(P_l^{(3)}, G)} \underline{d_{c_1}(f(v_1)) d_{c_2}(f(v_1))} \cdot d_{c_2}(f(v_2)) d_{c_3}(f(v_2)) \cdot$$

$$\cdots \cdot d_{c_{l-1}}(f(v_{l-1})) d_{c_l}(f(v_{l-1})) \cdot 2e_{c_1}(G) \cdot 2e_{c_l}(G)$$

# 3-edge-coloured alternating paths




$$\hom(H,G) = \sum_{f \in \mathrm{Hom}(P_l^{(3)}, G)} \hom(H, G; f)$$

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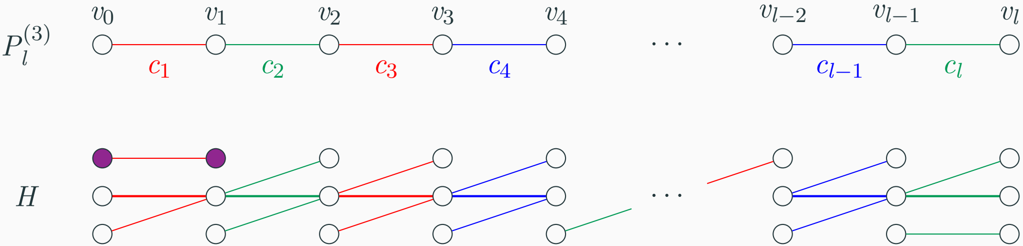
# 3-edge-coloured alternating paths

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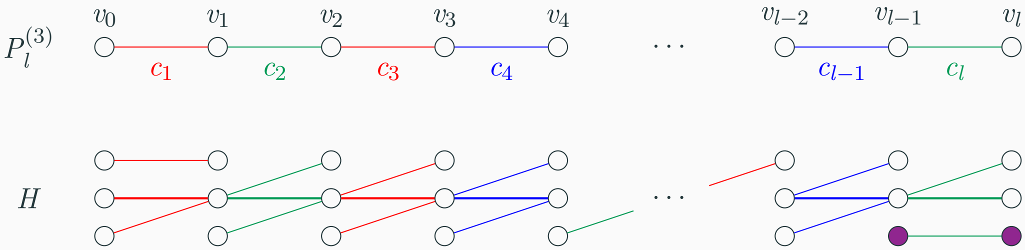


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# 3-edge-coloured alternating paths

$$\hom(H, G) = \sum_{f \in \mathrm{Hom}(P_l^{(3)}, G)} \left[ \prod_{j=1}^{l-1} d_{c_j}(f(v_j)) d_{c_{j+1}}(f(v_j)) \right] 4 e_{c_1}(G) e_{c_l}(G)$$

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$$\leq \sum_{f \in \mathrm{Hom}(P_l^{(3)},G)} \left[ \prod_{j=1}^{l-1} \left( \frac{d_{c_j}(f(v_j)) + d_{c_{j+1}}(f(v_j))}{2} \right)^2 \right] 4 e_{c_1}(G) e_{c_l}(G) \quad \text{AM-GM}$$

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$$= \hom(P_l^{(3)},G)\cdot\left(\frac{n-1}{2}\right)^{2l-2}\cdot 4e_{c_1}(G)e_{c_l}(G)$$

# 3-edge-coloured alternating paths

Dividing by $n^{v(H)} = n^{v(P_l^{(3)})+2l+2}$ yields:

$$t(H, G) \leq \frac{1}{2^{2l-4}} \left(1 - \frac{1}{n}\right)^{2l-2} \frac{e_{c_1}(G) e_{c_l}(G)}{n^4} t(P_l^{(3)}, G) \tag{1}$$

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Meanwhile, since $\varphi$ is an even 3-covering, we have

$$t(P_l^{(3)},G)^{1/l} \leq t(H,G)^{1/3l} \tag{2}$$

from the even covering lemma.

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from the even covering lemma.

Combining (1) and (2) yields:

$$t(P_l^{(3)},G) \leq \frac{1}{2^{l-2}}\left(1-\frac{1}{n}\right)^{l-1}\frac{\sqrt{e_{c_1}(G)e_{c_l}(G)}}{n^2}$$

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When $c_1 \neq c_l$: $\sqrt{e_{c_1}(G) e_{c_l}(G)} \leq \frac{e_{c_1}(G) + e_{c_l}(G)}{2} \leq \frac{e(G)}{2} \leq \frac{1}{2}\binom{n}{2}$.

$$t(P_l^{(3)}, G) \le \frac{1}{2^{l-2}} \left(1 - \frac{1}{n}\right)^{l-1} \frac{\sqrt{e_{c_1}(G) e_{c_l}(G)}}{n^2}$$

When $c_1 \neq c_l$: $\sqrt{e_{c_1}(G)e_{c_l}(G)} \le \frac{e_{c_1}(G)+e_{c_l}(G)}{2} \le \frac{e(G)}{2} \le \frac{1}{2}\binom{n}{2}$.

When $c_1 = c_l$: $\sqrt{e_{c_1}(G)e_{c_l}(G)} = e_{c_1}(G) \le \binom{n}{2}$.

Therefore,

$$t(P_l^{(3)}, G) \le \begin{cases} \frac{1}{2^l}\left(1 - \frac{1}{n}\right)^l & c_1 \neq c_l \\ \frac{1}{2^{l-1}}\left(1 - \frac{1}{n}\right)^l & c_1 = c_l \end{cases}$$

□

# 3-edge-coloured perfect trees

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**Definition**

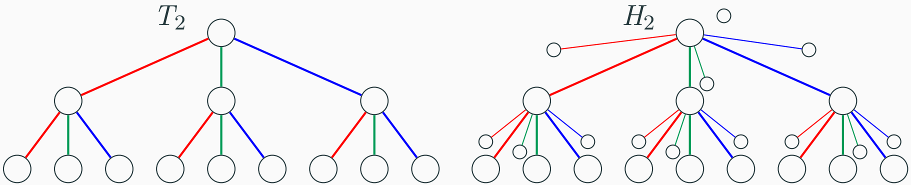
A **perfect** 3-edge-coloured tree is a tree which is symmetric about one of its vertices (except the colours) and where each non-leaf vertex has 3 ”children”, one of each colour.

Let $T_k$ denote the 3-edge-coloured perfect tree of depth $k$.

**Theorem**

For any $k \geq 1$, any 3-edge-coloured $G$, we have $t(T_k, G) \leq \left(\frac{1}{3}\right)^{3e(T_k)}$.

Proof:




$$\hom(H_k, G) = \sum_{f \in \mathrm{Hom}(T_k, G)} \hom(H_k, G; f)$$

# 3-edge-coloured perfect trees

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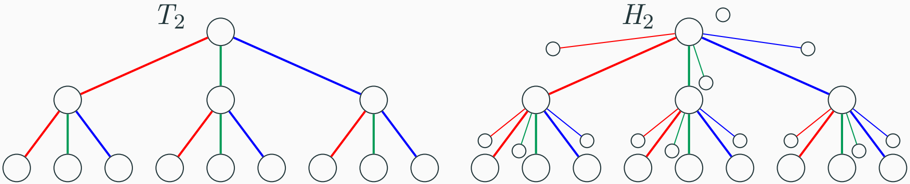


$$\hom(H_k, G) = \sum_{f \in \mathrm{Hom}(T_k, G)} \hom(H_k, G; f)$$

$$= \sum_{f \in \mathrm{Hom}(T_k, G)} v(G) \prod_{v \in V(T_{k-1})} d_R(f(v))\, d_G(f(v))\, d_B(f(v))$$

# 3-edge-coloured perfect trees

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$$\begin{aligned}
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&\leq \sum_{f \in \mathrm{Hom}(T_k, G)} v(G) \prod_{v \in V(T_{k-1})} \left( \frac{n-1}{3} \right)^3 && \text{max deg}
\end{aligned}$$

# 3-edge-coloured perfect trees

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&\leq \sum_{f \in \mathrm{Hom}(T_k, G)} v(G) \prod_{v \in V(T_{k-1})} \left( \frac{n-1}{3} \right)^3 && \text{max deg} \\
&\leq v(G)^{3v(T_{k-1})+1} \left( \frac{1}{3} \right)^{3v(T_{k-1})} \hom(T_k, G)
\end{aligned}$$

# 3-edge-coloured perfect trees

Now, dividing both sides by $v(G)^{v(H_k)} = v(G)^{v(T_k)+3v(T_{k-1})+1}$ (which can be verified by a simple counting argument) gives the following:

$$t(H_k, G) \leq \left(\frac{1}{3}\right)^{3(e(H_k)-e(T_k))} t(T_k, G) \tag{3}$$

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Also, by the even covering lemma, we have that

$$t(T_k, G)^{1/e(T_k)} \leq t(H_k, G)^{1/e(H_k)} \tag{4}$$

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Combining (3) and (4) gives the desired result:

$$t(T_k, G) \leq \left(\frac{1}{3}\right)^{3e(T_k)}.$$

□

# An imperfect 3-edge-coloured tree

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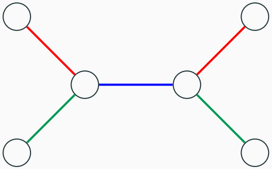
- What about imperfect trees?

In this direction, we will only consider one case, which is illustrative of what one might expect to see in a more general case.

# An imperfect 3-edge-coloured tree

Let $T$ be the graph on the picture to the right.

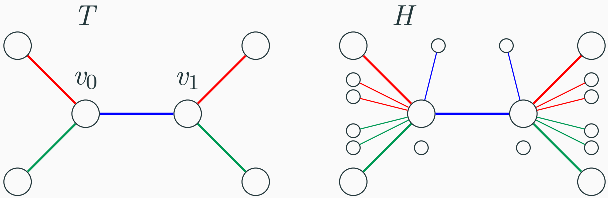


**Theorem**

For any 3-edge-coloured $G$, we have $t(T,G) \leq \left(\frac{1}{5}\right)\left(\frac{2}{5}\right)^2\left(\frac{2}{5}\right)^2.$

# An imperfect 3-edge-coloured tree

**Proof:** We introduce $H$ as per the picture below. Note that such a forest covers every edge and vertex of $T$ exactly $e(H)/e(T) = 3$ times.

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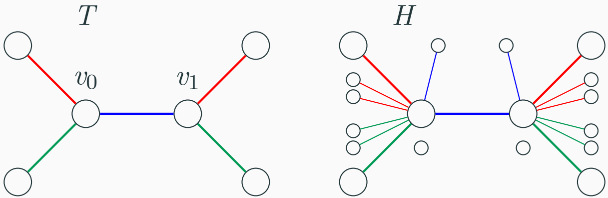


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$$= \sum_{f \in \mathrm{Hom}(T,G)} v(G)^2 \prod_{v \in \{v_0, v_1\}} d_R(f(v))^2 d_G(f(v))^2 d_B(f(v))$$

# An imperfect 3-edge-coloured tree

**Lemma**

If $a, b, c, m > 0$ and $x, y, z \geq 0$ are such that $x + y + z \leq m$, then

$$x^a y^b z^c \leq m^{a+b+c} \left(\frac{a}{a+b+c}\right)^a \left(\frac{b}{a+b+c}\right)^b \left(\frac{c}{a+b+c}\right)^c.$$

$$\begin{aligned}\hom(H, G) &= \sum_{f \in \mathrm{Hom}(T,G)} v(G)^2 \prod_{v \in \{v_0, v_1\}} d_R(f(v))^2 d_G(f(v))^2 d_B(f(v)) \\ &\leq v(G)^2 \left[ v(G)^5 \left(\frac{2}{5}\right)^2 \left(\frac{2}{5}\right)^2 \left(\frac{1}{5}\right) \right]^2 \text{hom}(T, G).\end{aligned}$$

# An imperfect 3-edge-coloured tree

Now, dividing both sides of the inequality by $v(G)^{v(H)} = v(G)^{v(T)+12}$, we get

$$t(H, G) \leq \left(\frac{2}{5}\right)^4 \left(\frac{2}{5}\right)^4 \left(\frac{1}{5}\right)^2 t(T, G)$$

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Combining (5) and (6) gives the desired result:

$$t(T, G) \le \left(\frac{1}{5}\right) \left(\frac{2}{5}\right)^2 \left(\frac{2}{5}\right)^2.$$

□

# Distributions of $t(P_l^A, G)$ for randomly 2-coloured $G \sim G(n, 1/2)$

# Randomly 2-coloured $G(n, 1/2)$

How would $t(P_l^A, G)$ be distributed if $G$ is a random graph?

# Randomly 2-coloured $G(n, 1/2)$

How would $t(P_l^A, G)$ be distributed if $G$ is a random graph?

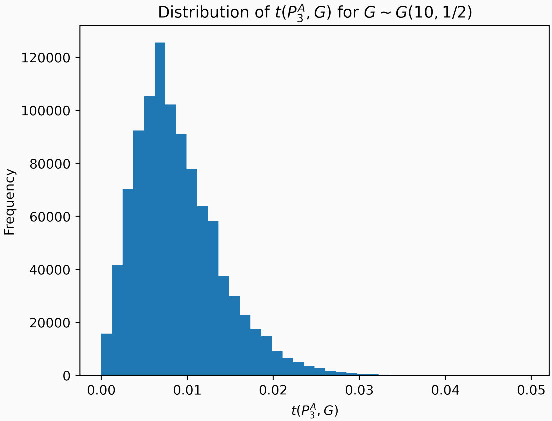
**Definition (Erdös-Rényi random graph with probability $1/2$)**

$G \sim G(n, 1/2)$ samples a random graph on $n$ vertices such that any two vertices are connected with probability $1/2$, independently from any other pair.
In other words, a graph is chosen uniformly from the set of all graphs on $n$ vertices.

We first sample $G \sim G(n, 1/2)$, then uniformly and independently colour the edges of the sampled graph.

# Distribution of $t(P_3^A, G)$ for $G \sim G(10, 1/2)$

Distribution of $t(P_3^A, G)$ over 1000 samples of $G \sim G(10, 1/2)$, each coloured with 1000 random 2-colourings ($\hom(P_3^A, G)$ counted using depth-first search):



# Expectation of $t(P_3^A, G)$

**Proposition**

For $G \sim G(n, 1/2)$,

$$\mathbb{E}[t(P_3^A, G)] = \frac{1}{4^3}\frac{(n-1)(n-2)^2}{n^3}$$

For $n = 10$, this gives $\mathbb{E}[t(P_3^A, G)] = 0.009$, which seems to agree with the sampled data (mean of $t(P_3^A, G)$ was $0.008947 \pm 0.000068$ over 5 trials).

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Proof:


$n$ choices of starting point ($v_0$)

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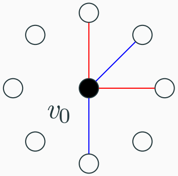


$n$ choices of starting point ($v_0$)

At $v_0$: $\frac{n-1}{2}$ expected edges, each coloured red with probability $1/2 \Rightarrow \frac{n-1}{4}$

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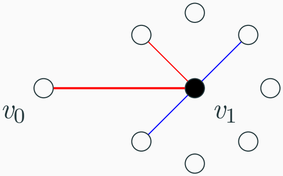


$n$ choices of starting point ($v_0$)
At $v_0$: $\frac{n-1}{4}$ expected red edges
At $v_1$: $\frac{n-2}{4}$ expected blue edges (excluding the previous edge)

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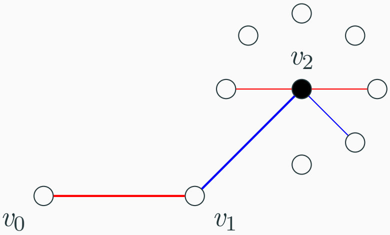


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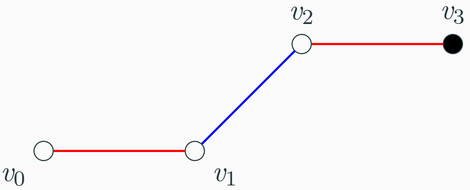


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**Proof:**

Connections and colours are all independent, so the expectation is

$$\mathbb{E}[t(P_3^A, G)] = \frac{n\left(\frac{n-1}{4}\right)\left(\frac{n-2}{4}\right)\left(\frac{n-2}{4}\right)}{n^4} = \frac{1}{4^3}\frac{(n-1)(n-2)^2}{n^3}$$

$\square$

# Expectation of $t(P_l^A, G)$ for $l \geq 4$

Does this work for $l \geq 4$?

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No!

It's possible to go through a vertex that was already used, in which case independence is not guaranteed.

# Recursive inequality for $a(n,l) = \mathbb{E}[\hom(P_l^A, G)]$, where $G \sim G(n, 1/2)$

**Proposition**

Let $a(n,l) = \mathbb{E}[\hom(P_l^A, G)]$ for $G \sim G(n, 1/2)$. Then we have:

$$a(n+1,l) \geq a(n,l) + \frac{1}{2}a(n,l-1) + \frac{n-1}{8n}\sum_{i=0}^{\lfloor \frac{l}{2} \rfloor - 1} a(n,i)a(n,l-2-i)$$

where $a(n,0) = n$.

**Proof:**

Use induction on $n$ by adding another vertex to $n$ vertices, assuming the new vertex is used only once. (If one considers all paths that visit the new vertex more than once, one could get an exact recursive formula involving all partitions of $l$.)

# References


[1] H. Chen, F. C. Clemen, and J. A. Noel. Maximizing Alternating Paths via Entropy. E-print arXiv: 2505.03903, 2025.

[2] A. Basit, B. Granet, D. Horsley, A. Kündgen, and K. Staden. The semi-inducibility problem. E-print arXiv: 2501.09842v2, 2025.


No Questions?