Sublinear expanders

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Introduction.

Expander graphs make one of the most widely useful class of graphs. They appear naturally across many different areas of mathematics and satisfy a variety of very useful properties.

There are many closely related definitions, and lots of their usefulness stems from translating between these definitions. At a rather high level one can think of them as "robustly well-connected" graphs. While many interpretations of both "robustly" and "well-connected" fit perfectly well here, for example one can think of well-connected as saying that there is a short path between any two vertices and robustness as saying that this property is preserved under minor modifications to the graph such as edge or vertex deletions.

With these properties in mind, and armed with the fact that one can find such graphs with very few edges one perhaps starts to see the utility. For example, both well-connectedness and robustness are very useful properties for a communication network, while having few edges translates to lower costs in building it.

Expander graphs have been extensively studied, both due to their applications, as well as the intrinsic interest. We point an interested reader to several wonderful surveys [4, 10, 13] on the topic. We make essentially no assumption on prior knowledge on the topic.

In this course we will focus on a certain weak notion of expansion, called sublinear expansion, which has found some spectacular applications in recent years. Our goal is to help develop some intuition of when this type of expanders are useful and how to use them. Due to this we will not include a comprehensive overview of the wide variety of applications and modifications that arose over the years but rather point an interested reader to an excellent recent survey by Letzter [11].

1 Linear expanders

Let us start with a definition of what is usually referred to as a vertex expander.

Definition 1.1. For any $\varepsilon > 0$, an n-vertex non-empty graph G is an ε-expander if for all $U \subseteq V(G)$ such that $|U| \leq n/2$ we have:

$$|N(U)| > \varepsilon |U|.$$

So a graph is an expander if by taking one step out from any set of vertices, which is not too large, we reach many new vertices. We note that here and throughout these notes $N(U) := \{v \in V(G) \setminus U \mid \exists u \in U : vu \in E(G)\}$ denotes the external neighbourhood of the set of vertices U, namely we do *not* include vertices already in U.[1]

We note also that when working with usual expanders one often restricts the set we are allowed to expand to be of size at most $\frac{n}{2(1+\varepsilon)}$ since expanding a set of this size one reaches $n/2$ vertices, so for larger sets one can simply expand a subset of this size. The benefit of this is that one can have a C-expander with $C > 1$ unlike in our definition, where a set of size $n/2$ can only ever expand by a factor of at most 1. Since our focus is on expanders with small expansion factors the above definition is a bit cleaner for us.

[1] In the context of usual, strong expanders, this point makes little difference since whether one has C or $C - 1$ factor expansion is rarely relevant, but our expanders are going to have ε very small and the distinction is important.

While there are by now plenty of interesting examples, arguably the simplest is simply K_n, the clique on n vertices. According to our definition it is easy to see it is a 1-expander. Perhaps the most important class of examples are random graphs. They also serve as an excellent source of intuition for a variety of arguments.

Exercise 1. How good of an expander is the binomial random graph $G(n, p)$ as p varies?

Let us now prove that our definition actually implies the well-connectedness property we mentioned earlier, namely any two vertices are not too far apart. Here, the diameter of a graph G, denoted $\operatorname{diam}(G)$, is the maximum distance[2] between two vertices in G.‘

Lemma 1.2 (Diameter bound)**.** *Let $\varepsilon > 0$ and G be an n-vertex ε-expander graph. Then,*

$$\operatorname{diam} G \leq 2\left\lceil \log_{1+\varepsilon} \frac{n}{2} \right\rceil.$$

Proof. Let $\ell = \left\lceil \log_{1+\varepsilon} \frac{n}{2} \right\rceil$. Our goal is to show that any pair of vertices v, u is joined by a walk of length at most 2ℓ. Let $U_0 = \{u\}$, and $U_{i+1} = U_i \cup N(U_i)$, for each $0 \leq i < \ell$. Then, we have $|U_i| > n/2$ or $|U_{i+1}| > (1+\varepsilon)|U_i|$. Iterating this we get,

$$|U_{i+1}| \geq \min\{(1+\varepsilon)^{i+1}, n/2\}.$$

By our choice of ℓ we have $|U_\ell| > n/2$. Similarly, if we define $V_0 = \{v\}$ and $V_{i+1} = V_i \cup N(V_i)$, then $|V_\ell| > n/2$.

So $|U_\ell| + |V_\ell| > n$, which implies that there exists a vertex $w \in U_\ell \cap V_\ell$. Since $w \in U_\ell$, there exists a uw walk of length at most ℓ, and since $w \in V_\ell$, there exists a vw walk of length at most ℓ. Concatenating these two walks gives a vu walk of length at most 2ℓ, as desired. □

Exercise 2. Prove that in an n-vertex ε-expander between any two sets of size $u \leq n/2$ there is a path of length at most

$$2\left\lceil \log_{1+\varepsilon} \frac{n}{2u} \right\rceil.$$

2 Sublinear expanders.

Note that in Definition 1.1 nothing prevents us from letting ε depend on n. Informally, a *sublinear* expander is an ε-expander where $\varepsilon \to 0$ as $n \to \infty$. Thinking of $\varepsilon = \frac{1}{\log n}$ is a good concrete choice to keep in mind.

Exercise 3. Prove that the $N = 2^n$ vertex hypercube graph Q_n with vertex set $\{0,1\}^n$ and two vertices adjacent iff they differ in exactly one coordinate is a $\Omega\left(\frac{1}{\sqrt{\log N}}\right)$-expander.

Hint. You may wish to use Harper's vertex isoperimetric inequality for the hypercube.

Now, a natural question is why would we consider a clearly weaker notion? The answer is that while usual expanders appear in many places, sublinear expanders appear essentially everywhere. We will see that every graph contains a sublinear expander with "essentially" the same degree, as well as that any graph can be decomposed into sublinear expanders (in both cases with expansion at least

[2]Defined as the minimum length of a path between the two vertices.

$1/\operatorname{polylog} n$). This means that for a wide variety of problems one can get a lot of mileage from understanding the problem well for sublinear expanders. On the other hand, they retain many of the useful properties of their stronger siblings (albeit usually slightly weakened), so it is often much more feasible to attack the problem on a sublinear expander than a completely arbitrary one.

For example, Lemma 1.2 applies equally as well for sublinear expanders, and for example, for $\frac{1}{\log n}$-expanders the diameter is guaranteed to be at most

$$2\left\lceil \log_{1+1/\log n} \frac{n}{2} \right\rceil = 2\left\lceil \frac{n/2}{\log(1+1/\log n)} \right\rceil \leq O(\log^2 n),$$

so still very small.

Finding long paths or cycles in graphs is a very classical problem in extremal combinatorics. Dirac's theorem is one of the most famous examples, which gives a criterion for a graph to contain the longest possible cycle (called Hamilton cycle), so one of length n. Namely, if a graph has minimum degree at least $\lfloor n/2 \rfloor$, and this is known to be tight. This, while tight is unfortunately quite dense. On the other hand, in a random graph $G(n,p)$ much less density is needed to guarantee a Hamilton cycle w.h.p., so most graphs do have one. This motivated a long line of work, with plenty of tantalizing open problems, of finding which conditions on a graph which guarantee Hamilton cycles (or at least very long ones). There has been very exciting recent progress in case of strong expander graphs [3], as well as for regular sublinear expanders [12]. We here will prove a simple, weaker lemma, which is nonetheless often very useful (as we will see later in the course). It also allows us to showcase the use of the so called DFS method of Ben-Eliezer, Krivelevich, and Sudakov [1], which has proven itself to be very useful in a variety of settings. See [10, Theorem 7.4] for a version with arbitrary expansion parameter and tighter bound for the cycle case.

Lemma 2.1. *Any n-vertex $\frac{1}{\log n}$-expander G, contains a path of length $\Omega(n/\log n)$ and a cycle of length at least $\Omega\big(n/\log^2 n\big)$.*

Proof. Observe first that the expansion condition guarantees G is connected, since otherwise we could find a connected component of G of size at most $\frac{|G|}{2}$, whose vertex set does not expand at all.

We run the DFS algorithm on G as follows. At any point during the process we have the set of unexplored vertices U, the path P with active endvertex $t(P)$ (called a stack), and the set of processed vertices R. Picking an arbitrary vertex $r \in V(G)$, We start with $U = V(G) \setminus \{r\}, R = \emptyset$ and with P being the path with vertex set $\{r\}$, and set $t(P) = r$. At each step, if there is a neighbor v of $t(P)$ in U we add it to P with the edge $t(P)v$ and let $t(P) = v$. Otherwise, we move $t(P)$ from P to R and set its neighbor in P as the new $t(P)$.

In the above process, in each step we either move precisely one vertex from U to P or precisely one vertex from P to R. Note also that at any point in the process there are no edges between U and R since a vertex is only moved to R once it has no neighbors in U, and U only ever has vertices removed from it. Finally, as G is connected, note that the process finishes with all the vertices being in R and P, and U being empty.

Thus, we start with $|U| = n-1$ and $|R| = 0$ and finish with $|U| = 0$ and $|R| = n$, at each step reducing $|U|$ by one or increasing $|R|$ by one. Therefore, at some point in the process we must have $|U| = |R|$. Since there are no edges between U and R we know that all the neighbors of U must belong to P, which therefore has size $|P| = n - 2|U| \geq |N(U)| \geq |U|/\log n$, where the last inequality follows by the expansion property applied to U, which we can do since $|U| = (n - |P|)/2 \le n/2$. This in turn implies that $|P| \ge \frac{n}{3\log n}$ since either $|U| \ge \frac{n}{3}$ or $|P| = n - 2|U| \ge \frac{n}{3}$. This shows a path of desired length exists.

Let us turn to showing there is also a long cycle. let X, Y, Z be sets of consecutive vertices of P in that order which partition $V(P)$ so that $|X|, |Z| \ge \frac{|P|}{3}$ and $\frac{n}{6\log^2 n} \le |Y| < \frac{n}{3\log^2 n}$. If X and Z are connected by some path in $G \setminus Y$, then take a shortest path, Q say, between X and Z in $G \setminus Y$ and note that, combined with the segment of P between the endvertices of Q, this gives a cycle containing each vertex in Y, which thus has size $\Omega\left(\frac{n}{\log^2 n}\right)$. If X and Z are not connected by a path in $G \setminus Y$, then we can take a partition $V(G) \setminus Y = X' \cup Z'$ with no edges between X' and Z' in G, and $X \subseteq X'$ and $Z \subseteq Z'$. Without loss of generality, suppose that $|X'| \le \frac{n}{2}$. By the expansion condition we have $|N_G(X')| \ge \frac{|X|}{\log n} \ge \frac{|P|}{3\log n} \ge \frac{n}{3\log^2 n}$, yet we also have $|N_G(X')| \le |Y| < \frac{n}{3\log^2 n}$, a contradiction. □

We note that one can in fact find a cycle of length $\Omega(n/\log n)$ and that this lemma is tight up to a constant factor as can be seen by taking a complete bipartite graph with one side of size $n/\log n$. We opted to show this argument here since we will repeatedly use this type of "robustness" arguments (which say that if you remove some set of vertices from your expander, then the expansion properties of slightly larger sets remain essentially intact).

Exercise 4. Show that any n-vertex $\frac{1}{\log n}$-expander G, contains a cycle of length $\Omega(n/\log n)$.

2.1 Pass to expander lemma

In this section we will show how to find a sublinear expander inside an arbitrary graph while "essentially" preserving the average degree. We first introduce a definition of a more robust version of sublinear expanders which often give one a crucial extra bit of power when working with them.

Definition 2.2. A graph G on $n \ge 1$ vertices is called a *robust sublinear expander* if

- for every $0 \le \varepsilon \le 1$ and every non-empty subset $U \subseteq V(G)$ of size $|U| \le n^{1-\varepsilon}$, and
- for every subset $F \subseteq E(G)$ of $|F| \le \frac{\varepsilon}{4} \cdot d(G) \cdot |U|$ edges, we have

$$|N_{G-F}(U)| \ge \frac{\varepsilon}{4} \cdot |U|.$$

To build intuition let us for the moment ignore the second point. In this case, the definition becomes the usual vertex expansion one, albeit with a different level of expansion for sets of different sizes (the size is controlled by the ε parameter). For example if one picks $\varepsilon = 1/\log n$ the required condition simply becomes $\frac{1}{3\log n}$-expansion (since the upper bound on $|U|$ becomes $n/2$). On the other hand, if ε is an absolute constant then we require that sets of size up to $n^{1-\varepsilon}$ expand by an $\varepsilon/3$ factor. As ε transitions from an absolute constant to $1/\log n$ the size of the sets we require expand grows while the required expansion factor decreases smoothly. For comparison to simply pure $\frac{1}{\log n}$-expanders which we saw have diameter upper bounded by $O(\log^2 n)$ robust sublinear expanders have quite a bit better guarantee and come closer to the usual constant factor expander guarantee of $O(\log n)$.

Exercise 5. Show that an n-vertex robust sublinear expander H has $\text{diam}(H) \le O(\log n \log\log n)$.

Turning back to the second condition, it in addition allows us to remove a substantial number of edges from the graph and still preserve the expansion guarantees we discussed above. In most cases, it is used in order to be able to forbid usage of a number of "inconvenient" edges. We stress that the choice of F can be made after the subset U is specified, and its allowed size is controlled by the size of the set we are expanding.

The following lemma shows we can find a robust sublinear expander inside any graph while preserving a certain relative notion of the degree. We note that there are several different, pass to a sublinear expander lemmas in use depending on how large the average degree of the initial graph is. The following particular lemma is designed for graphs with average degree at least a large constant times $\log n$. The proofs of all these lemmas essentially follow the same strategy.

Lemma 2.3. *For every non-empty graph G contains a subgraph H which is a robust sublinear expander and has average degree*

$$d(H) \geq \frac{\log |V(H)|}{\log |V(G)|} \cdot d(G).$$

Proof. Among all subgraphs H of G with at least two vertices, let us choose a subgraph H maximizing the expression

$$\frac{d(H)}{\log |V(H)|}.$$

Then, we get the desired inequality since $H = G$ is itself a viable choice, namely

$$\frac{d(H)}{\log |V(H)|} \geq \frac{d(G)}{\log |V(G)|}.$$

It remains to show that H is a robust sublinear expander. Let $n = |V(H)| \geq 2$ and $d = d(H) > 0$. Consider $0 \leq \varepsilon \leq 1$, and let $U \subseteq V(H)$ be a non-empty subset of size $|U| = u \leq n^{1-\varepsilon}$. Furthermore, let $F \subseteq E(H)$ be a subset of $|F| \leq \frac{\varepsilon}{4} \cdot du$ edges. Our goal is to show that $|N_{H-F}(U)| \geq \frac{\varepsilon}{4} \cdot u$. Suppose towards a contradiction that $|N_{H-F}(U)| < \frac{\varepsilon}{4} \cdot u$.

Let $H_1 = H[U \cup N_{G-F}(U)]$ and let $H_2 = H[V(H) \setminus U]$, and denote by $n_1 = |H| \leq (1 + \varepsilon/4)u$ and $n_2 = |H_2| = n - u$. Then, every edge of H belongs to H_1, H_2, or F (with some edges within $H[N_{G-F}(U)]$ counted twice). This implies

$$e(H) \leq e(H_1) + e(H_2) + |F|.$$

On the other hand, the maximality[3] of H implies $d(H_i) \leq d \cdot \frac{\log n_i}{\log n}$ for both i. This, plugged into the above inequality implies

$$\begin{aligned} n &\leq \frac{2e(H_1) + 2e(H_2) + 2|F|}{d} \leq n_1 \cdot \frac{\log n_1}{\log n} + n_2 \cdot \frac{\log n_2}{\log n} + \frac{\varepsilon}{2} \cdot u \\ &\leq \left(1 + \frac{\varepsilon}{4}\right) u \cdot \frac{(1-\varepsilon)\log n + \log(1 + \varepsilon/4)}{\log n} + n - u + \frac{\varepsilon}{2} \cdot u \\ &\leq \left(1 + \frac{\varepsilon}{4}\right) u \cdot \left(1 - \frac{3\varepsilon}{4}\right) + n - u + \frac{\varepsilon}{2} \cdot u < n, \end{aligned}$$

[3]Technically if $n_i = 1$ these inequalities do not follow from the maximality assumption but are immediate.

where in the penultimate inequality we used $\log(1+\varepsilon/4) < \varepsilon/4 \leq \frac{\varepsilon}{4} \cdot \log n$. This is a contradiction completing the proof. $\square$

As we already mentioned, the above lemma is only useful if $d(G) \gg \log n$ since, e.g. if $d(G) \leq \log n$ then the density guarantee we get on H becomes vacuous and H being a single edge is a valid (albeit useless) output. When we have $d(G) \geq C \log n$ on the other hand we get the same guarantee of $d(H) \geq C \log |H|$ and can at least guarantee $|H| \geq C$. This in fact, is one of the main downsides of essentially all pass to expander lemmas, namely while one preserves the "relative" guarantee on the average degree, one might lose a lot in terms of the size of the graph. This however, can't be avoided, our original graph G might simply be a vertex disjoint union of cliques of size $d(G)+1$, so it is not possible to find a very large expander subgraph.

Note that the definition of a robust sublinear expander already guarantees that its minimum degree $\delta(G) \geq d(G)/4$, by simply applying the expansion property to the sets of size one. With this in mind one can view the pass to expander lemma as a significantly stronger version of the easy classical result that any graph of average degree d contains a subgraph with minimum degree at least $d/2$.

Exercise 6. Show that the above proof in fact ensures $\delta(H) \geq d(H)/2$.

One can obtain pass to expander lemmas even for graphs with only a large constant degree, and in fact these are precisely the first ones ever used due to Komlós and Szemerédi [9]. They are in a certain sense weaker than the robust sublinear expanders we introduced above, and more cumbersome, but more versatile and are behind an impressive number of recent results using the theory. We point an interested reader to [11] for a gentle description of their properties.

On the other hand, if one has at least polynomial average degree $d(H) \gg n^{\alpha}$ for some absolute constant $\alpha > 0$, then one can find stronger expander subgraphs by repeating the above proof with the potential function $\frac{d(H)}{n^{\alpha}}$. Note that as above, the lower bound on $d(H)$ is simply required to guarantee the expander we find is not too small.

Exercise 7. Show that

a) For any $1 \geq \alpha > 0$, every non-empty graph G contains an $\frac{\alpha}{1+\alpha}$-expander subgraph H with

$$d(H) \geq \frac{|V(H)|^{\alpha}}{|V(G)|^{\alpha}} \cdot d(G).$$

b) One can in fact get (significantly) better expansion for smaller sets.

c) One can sacrifice a tiny bit of vertex expansion to ensure robustness, namely that the expansion is guaranteed even upon deletion of a substantial number of edges.

3 Applications of pass to expander lemmas.

In this section we will illustrate the "pass to expander" method with some examples. We note that the examples are picked to be simple and instructive rather than as flashy as possible, we direct the reader to [11] for a collection of some of the most spectacular examples.

3.1 Finding topological cliques

How many edges does one need in a graph in order to guarantee K_k as a subgraph? This is precisely answered by the classical Turán's theorem and is one of the cornerstone results of extremal graph theory. Unfortunately, from the view of using this result, the answer is rather large, even for a triangle we require a quadratic number of edges. In some applications, it turns out a slightly relaxed notion of a clique, where we replace every edge by a path in such a way that the paths are internally vertex disjoint, suffices.

Definition 3.1. A *topological clique* is a clique in which we replaced every edge with a path in such a way that all the paths are internally vertex disjoint. The original vertices are called the *anchor points* and their number is the *order* of the topological clique.

We note that an alternative way of defining a topological clique is as a subdivision of a clique. For example, a topological clique of order three is simply a cycle. We know that it is much easier to guarantee a cycle in a graph (one needs average degree of at least 2) than it is to guarantee a triangle (one needs average degree at least $n/2$). A very natural question is what happens for larger orders. This is a classical question of Erdős and Hajnal, and Mader. It was resolved by Bollobás and Thomason [2] and independently Komlós and Szemerédi [9] who showed the following.

Theorem 3.2. *Any graph with average degree d contains a topological clique of order $\Omega(\sqrt{d})$.*

It is not hard to see that this theorem is tight.

Exercise 8. Show that for any integer d there exists a graph which does not contain a topological clique of order larger than $10\sqrt{d}$.

The proofs of Bollobás and Thomason and independently Komlós and Szemerédi are very different, and in fact the one from the latter paper is the origin of the theory of sublinear expanders[4].

We will prove a slightly weaker version of the result, with a significantly simpler proof.

Theorem 3.3. *Any n-vertex graph with average degree d contains a topological clique of order $\Omega(\sqrt{d}/\log^2 n)$.*

Proof. By Lemma 2.3 we can find a robust sublinear expander subgraph H with minimum degree

$$\delta(H) \geq d(H)/4 \geq \frac{\log |H|}{4 \log n} \cdot d \geq \frac{d}{4 \log n}.$$

Let $m = |H|$. Now fix an arbitrary set of distinct vertices $v_1, \ldots, v_t \in V(H)$ with $t = \sqrt{d}/(8 \log^2 n)$ to serve as anchor points.

Now take a maximal collection of internally vertex disjoint paths of length at most $4 \log^2 m$ between the anchor points, with at most one per pair. If each pair got a path we found the desired topological clique and are done so let us assume there are some $i < j$ such that we were not able to find a $v_i v_j$ path which would extend the collection.

[4] Technically Komlós and Szemerédi introduced the method in a paper [8] which preceded this one by three weeks and proved only a slightly weaker bound.

Notice that the current collection of paths uses at most $t + \binom{t}{2} \cdot 4\log^2 m \le 2t^2 \log^2 m \le d/(32\log^2 n)$ vertices. Call this set of vertices V_F.

Next observe that in $G \setminus V_F$ any set U of size $m/2 \ge |U| \ge 2|V_F| \log n$ still has

$$|N_{G\setminus F_v}(U)| \ge |N_G(U)| - |F_V| \ge |U|/\log n - |F_V| \ge \frac{|U|}{2\log n}.$$

Finally, notice that

$$|N_{G\setminus F_v}(v_i)| \ge \delta(H) - |F_V| \ge \frac{d}{4\log n} - |F_v| \ge \frac{d}{8\log n} \ge 2|V_F| \log n$$

So, for any $\ell \ge 1$ we have

$$|B^{\ell}_{G\setminus F_v}(v_i)| \ge \frac{d}{4\log n}(1 + 1/(2\log n))^{\ell-1}.$$

In particular, we get $|B^{2\log^2 m-2}_{G\setminus F_v}(v_i)| > m/2$. Repeating for v_j we get $|B^{2\log^2 m-2}_{G\setminus F_v}(v_j)| > m/2$ and conclude there is a path in $G \setminus F_v$ between v_i and v_j of length at most $4\log^2 m$, contradicting maximality. □

We note that we only used the vertex expansion property above to keep things simple. We were also rather loose with the logarithms.

Exercise 9.

a) Improve the power of the $\log n$ in the above result as much as you can.

b) Prove Theorem 3.2 under the assumption $d(G) \ge \varepsilon n$ for any $\varepsilon > 0$.

c)* Prove Theorem 3.2 in full.

For part b) it might help to work out Exercise 7. As a warning, part c) is still quite hard, there are several key ideas we did not touch upon. But part b) is useful and you want to develop a sparse variant of the pass to expander lemmas. If you need further hints, looking at section 2 of [11] contains a high level sketch which should help.

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