

# Rational points on varieties and the Brauer-Manin obstruction

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My work appearing in this talk was predominantly completed on the lands of the Coast Salish, Duwamish, Stillaguamish, and Suquamish nations, & I am reporting on it from the lands of the East Shoshone and Ute nations.

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PCMI 2022

**Number theory informed by computation**

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If  $X(\mathbb{Q}) \hookrightarrow S$  and  $S = \emptyset$ , then  $X(\mathbb{Q}) = \emptyset$ .

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A class of varieties satisfies the  
**local-to-global principle**

if, for every such variety,  $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset \Rightarrow X(\mathbb{Q}) \neq \emptyset$ .

Examples: quadrics, Severi-Brauer varieties



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For “most” classes of varieties, LGP fails.

# **The Brauer-Manin obstruction to the existence of rational points**

Goal: Define an intermediate obstruction set

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$k$ : number field  $X(k) \subset X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)$

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Brauer-Manin set

If  $X(\mathbb{A}_k)^{\text{Br}} = \emptyset$  (and  $X(\mathbb{A}_k) \neq \emptyset$ ), then  
 **$X$  has a Brauer-Manin obstruction**  
to the existence of rational points.

# Leveraging Quadratic Reciprocity

**Assume** that for every  $F/k$ , we have  $X(F) \rightsquigarrow \{\text{conics}\}, x \mapsto C_x$

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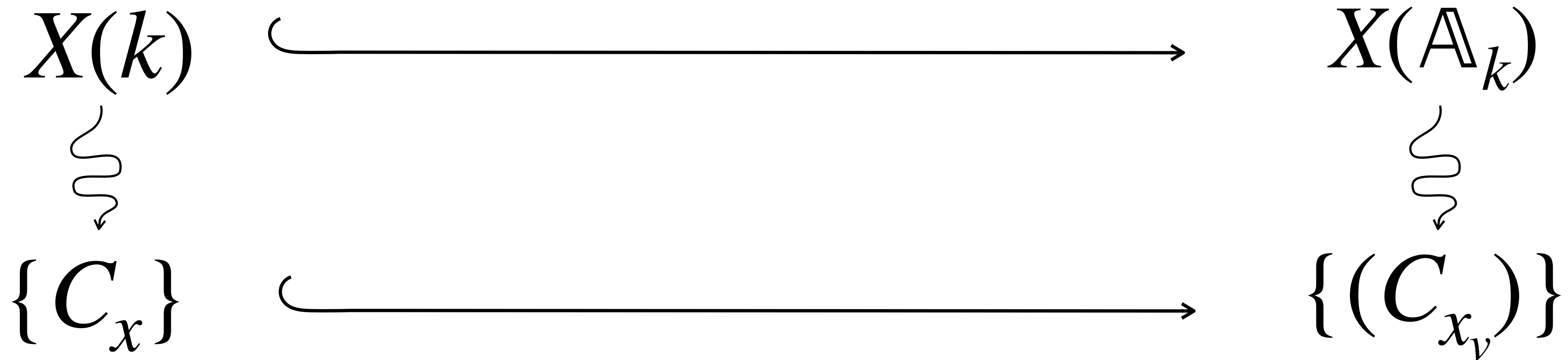
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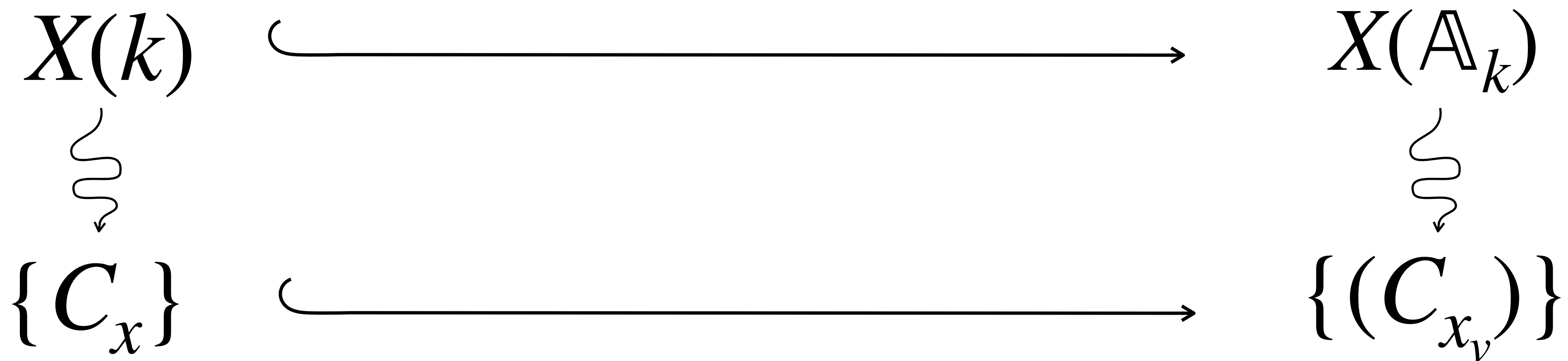
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 X(k) & \xrightarrow{\quad} & X(\mathbb{A}_k) & \xrightarrow{\varphi_\alpha} & & & \\
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$\varphi_\alpha^{-1}(0)$

$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0) \rightsquigarrow \left\{ ([C_{x_v}]) : \sum_v \text{inv}_v [C_{x_v}] = 0 \right\}$$



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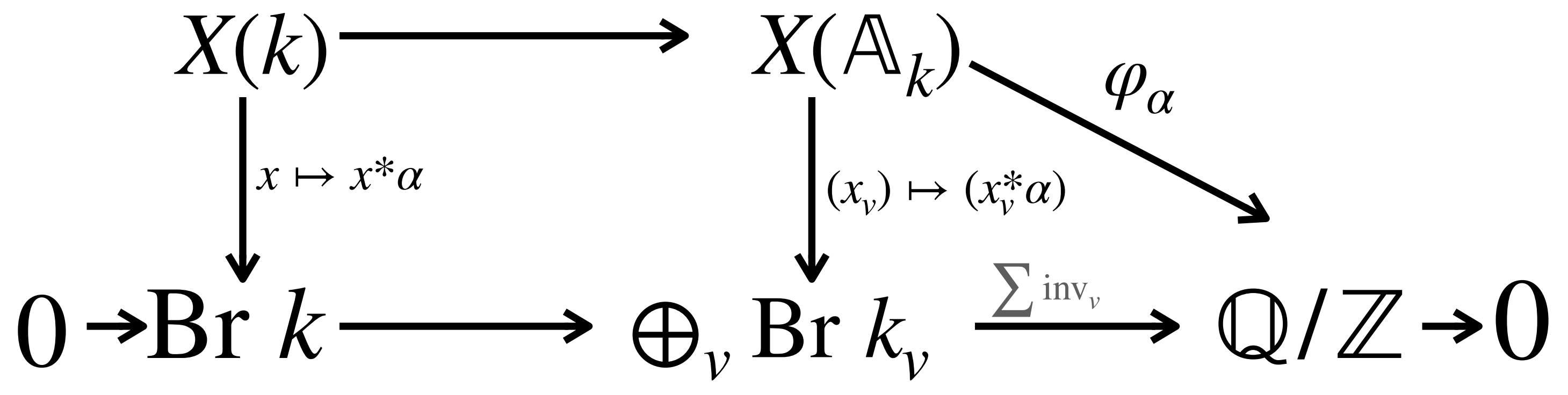
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$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0)$$

$$X(\mathbb{A}_k)^{\text{Br}} := \bigcap_{\alpha} X(\mathbb{A}_k)^\alpha$$

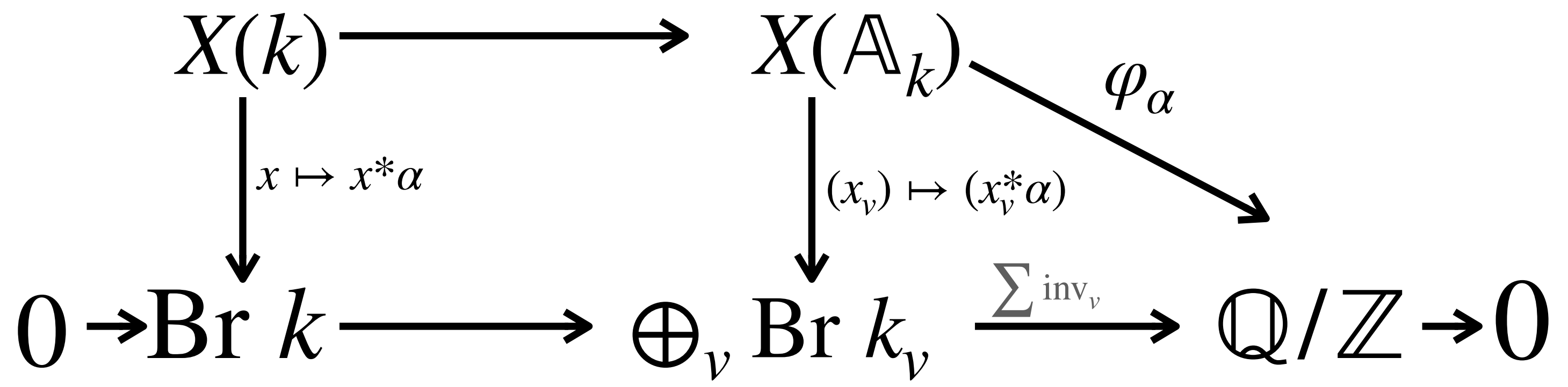
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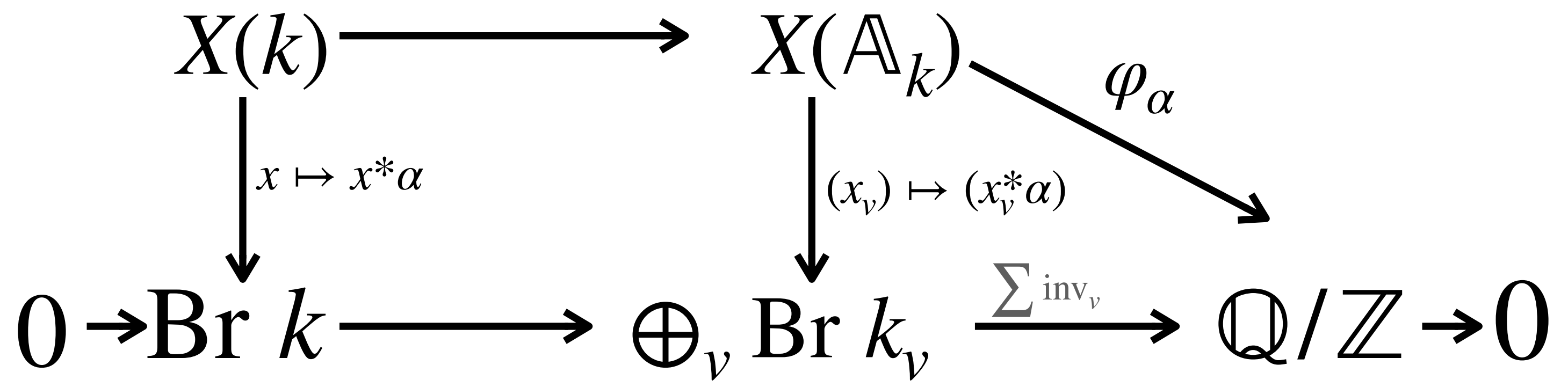
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Can we understand/compute  $X(\mathbb{A}_k)^{\text{Br}}$ ?