Rational points on varieties and the Brauer-Manin obstruction

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My work appearing in this talk was predominantly completed on the lands of the Coast Salish, Duwamish, Stillaguamish, and Suquamish nations, & I am reporting on it from the lands of the East Shoshone and Ute nations.

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PCMI 2022

Number theory informed by computation

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Examples: $S = X(\mathbb{Q}_p), X(\mathbb{R}), X(\mathbb{A}_\mathbb{Q})$

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Examples: $S = X(\mathbb{Q}_p), X(\mathbb{R}), X(\mathbb{A}_\mathbb{Q})$

If $X(\mathbb{Q}) \hookrightarrow S$ and $S = \varnothing$, then $X(\mathbb{Q}) = \varnothing$.

Approach: Embed $X(\mathbb{Q})$ into another set S that is more understandable/computable

Examples: $S = X(\mathbb{Q}_p), X(\mathbb{R}), X(\mathbb{A}_\mathbb{Q})$

A class of varieties satisfies the **local-to-global principle** if, for every such variety, $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset \Rightarrow X(\mathbb{Q}) \neq \emptyset$.

Examples: quadrics, Severi-Brauer varieties

Approach: Embed $X(\mathbb{Q})$ into another set S that is more understandable/computable

Examples: $S = X(\mathbb{Q}_p), X(\mathbb{R}), X(\mathbb{A}_{\mathbb{Q}})$

A class of varieties satisfies the **local-to-global principle** if, for every such variety, $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset \Rightarrow X(\mathbb{Q}) \neq \emptyset$.

For "most" classes of varieties, LGP **fails.**

The Brauer-Manin obstruction to the existence of rational points

Goal: Define an intermediate obstruction set

$$X(\mathbb{Q}) \subset X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \subset X(\mathbb{A}_{\mathbb{Q}})$$

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Brauer-Manin set

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k: number field $\quad X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$

Brauer-Manin set

Goal: Define an intermediate obstruction set

k: number field

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$$

Brauer-Manin set

If $X(\mathbb{A}_k)^{\mathrm{Br}} = \varnothing$ (and $X(\mathbb{A}_k) \neq \varnothing$), then
***X* has a Brauer-Manin obstruction**
to the existence of rational points.

Leveraging Quadratic Reciprocity

Assume that for every F/k, we have $X(F) \rightsquigarrow \{\text{conics}\}, x \mapsto C_x$

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$$\rightsquigarrow$$

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$$\#\{v : C_x(k_v) = \varnothing\} \equiv 0 \pmod{2}$$

QR

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Assume that for every F/k, we have $X(F) \rightsquigarrow \{\text{conics}\}, x \mapsto C_x$

$$X(k) \qquad\qquad\qquad\qquad X(\mathbb{A}_k)$$

$$\{C_x\} \subset \left\{ (C_{x_v}) : \begin{matrix} \#\{v : C_{x_v}(k_v) = \varnothing\} \\ \text{is even} \end{matrix} \right\} \subset \{(C_{x_v})\}$$

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Assume that for every F/k, we have $X(F) \rightsquigarrow \mathrm{Br}\, F, \quad x \mapsto [C_x]$

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$$0 \to \operatorname{Br} k \to \oplus_v \operatorname{Br} k_v \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

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The Brauer-Manin set

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The Brauer-Manin set

Given $\alpha \in \mathrm{Br}\ X$ and $\mathrm{Spec}\ F \xrightarrow{x} X$, $x^*\alpha \in \mathrm{Br}\ F$.

$$X(k) \rightsquigarrow \{C_x\} \subset \left\{([C_{x_v}]) : \sum_v \mathrm{inv}_v[C_{x_v}] = 0\right\} \subset \{(C_{x_v})\} \leftsquigarrow X(\mathbb{A}_k)$$

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$$\downarrow \qquad\qquad \rightsquigarrow$$

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\end{array}
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$$\left\{([C_{x_v}]) : \sum_v \operatorname{inv}_v [C_{x_v}] = 0\right\}$$

The Brauer-Manin set

Given $\alpha \in \operatorname{Br} X$ and $\operatorname{Spec} F \xrightarrow{x} X$, $x^*\alpha \in \operatorname{Br} F$.

$$\varphi_\alpha^{-1}(0)$$

$$\begin{array}{ccccccccc}
 & & X(k) & \longrightarrow & X(\mathbb{A}_k) & \xrightarrow{\varphi_\alpha} & & & \\
 & & \downarrow & & \downarrow & & \searrow & & \\
0 & \longrightarrow & \operatorname{Br} k & \longrightarrow & \bigoplus_v \operatorname{Br} k_v & \xrightarrow{\sum \operatorname{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array}$$

$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0) \rightsquigarrow \left\{([C_{x_v}]) : \sum_v \operatorname{inv}_v [C_{x_v}] = 0\right\}$$

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\end{array}$$

$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0) \qquad X(\mathbb{A}_k)^{\operatorname{Br}} := \bigcap_\alpha X(\mathbb{A}_k)^\alpha$$

$\alpha \in \operatorname{Br} X$

$$
\begin{array}{ccccccc}
 & & X(k) & \longrightarrow & X(\mathbb{A}_k) & \overset{\varphi_\alpha}{\searrow} & \\
 & & \downarrow{\scriptstyle x \mapsto x^*\alpha} & & \downarrow{\scriptstyle (x_v) \mapsto (x_v^*\alpha)} & & \\
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$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0)$$

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Approach: Embed $X(\mathbb{Q})$ into another set S that is more understandable/computable

$\alpha \in \mathrm{Br}\, X$

$$
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Approach: Embed $X(\mathbb{Q})$ into another set S that is more understandable/computable

Can we understand/compute $X(\mathbb{A}_k)^{\mathrm{Br}}$?