~~Lecture 4.3~~ Lecture 5.1

Recursive relations

(1) $R_{(a,b)}(\underline{d}) = \frac{d_a}{d_b} \cdot \frac{1 - \mathrm{Bad}_{(a,b)}(\underline{d} - e_b)}{1 - \mathrm{Bad}_{(b,a)}(\underline{d} - e_a)}$ for odd $\underline{d}$

(1.5) $\mathrm{Bad}_{a,b}(\underline{d}) = \frac{1}{d_a}\left(P_{ab}(\underline{d}) + \sum_{v \neq a,b} Y_{a,v,b}(\underline{d})\right)$. even $\underline{d}$

(2) $P_{av}(\underline{d}) = d_v \left(\sum_{b \neq v} R_{(b,a)}(\underline{d} - e_v) \frac{1 - P_{bv}(\underline{d} - e_b - e_v)}{1 - P_{av}(d - e_a - e_v)} \right)^{-1}$

(3) $Y_{avb}(\underline{d}) = \frac{P_{av}(\underline{d})(P_{bv}(d - e_a - e_v) - Y_{avb}(\underline{d} - e_a - e_v))}{1 - P_{av}(d - e_a - e_v)}$

Assuming things are well-defined (e.g. realisable in $P_{av}(\underline{d})$).

Start with $P_{av}(d) = O(\frac{d}{n})$ ~~(= o(1))~~ $\ll 1$.

$\longrightarrow$ iterate applying (3), (1.5), (1); (2)

& get better & better approximations of $R_{ab}(\underline{d})$.

What's the improvement after 1 round of iterations?

after 1st round $R_{ab}(\underline{d}) = \frac{d_a}{d_b}(1 + O(\frac{d}{n}))$

2nd $R_{ab}(\underline{d}) = \frac{d_a}{d_b}(1 + \text{sth.} + O(\frac{d^2}{n^2}))$

3rd round $\frac{d_a}{d_b}(1 + \text{sth. more complicated} + O(\frac{d^3}{n^3}))$

?

k rounds $R_{ab}(\underline{d}) = \text{Formula}\,(1 + O((\frac{d}{n})^k))$

For which d is $(\frac{d}{n})^k$ precise enough? $(\frac{d}{n})^k = o\left(\frac{1}{n\sqrt{d\log n}}\right)$

$\Longleftrightarrow d^{\frac{2k+1}{2}} \ll n^{k-1} / (\log n)^{1/2}$

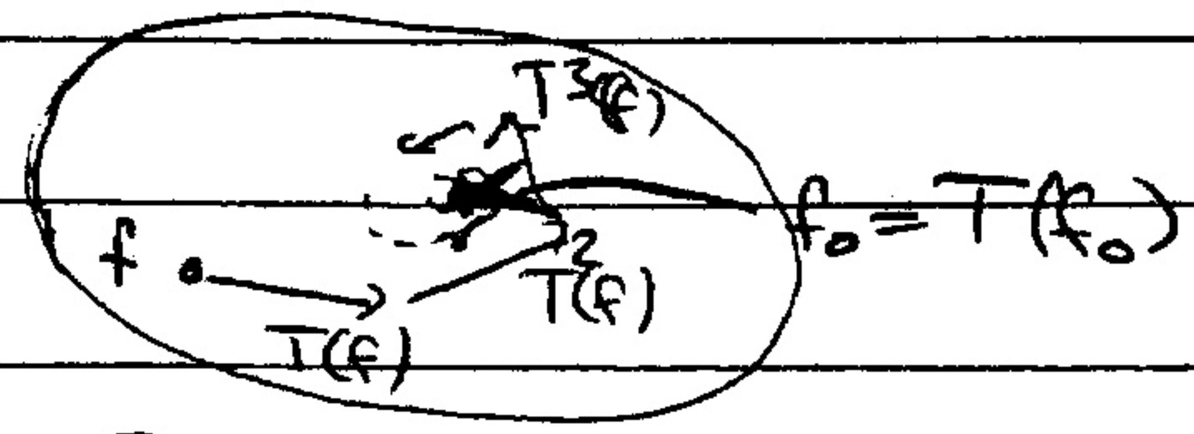
$\Longleftrightarrow d \ll n^{1-\frac{2}{2k+1}} / (\log n)^{\frac{1}{2k+1}}$

In theory: $\forall \varepsilon > 0$, $d = n^{1-\varepsilon}$, need $O(1)$ iterations to obtain desired accuracy.

Problem 1: formulae get very complicated very quickly (try to do 3^{rd} iteration on paper).

Problem 2: that does not get us up to $d = \Theta(\frac{n}{\log n})$ or even $d = \varepsilon n$.

Operators on functions $T(f)$



$p = p(a,b,\underline{d})$

$p: \binom{n}{2} \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad p_{av}(\underline{d})$

$y: \binom{n}{3} \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad y(a,v,b,d)$ $\quad y_{avb}(d)$

$r: [n]^2 \times \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\quad r(a,b,\underline{d})$. $\quad r_{ab}(\underline{d})$

(1) $$R(p,y)(a,b,\underline{d}) = \frac{d_a}{d_b} \cdot \frac{1 - \text{bad}(p,y)(a,b,d-e_b)}{1 - \text{bad}(p,y)(a,b,d-e_a)},$$

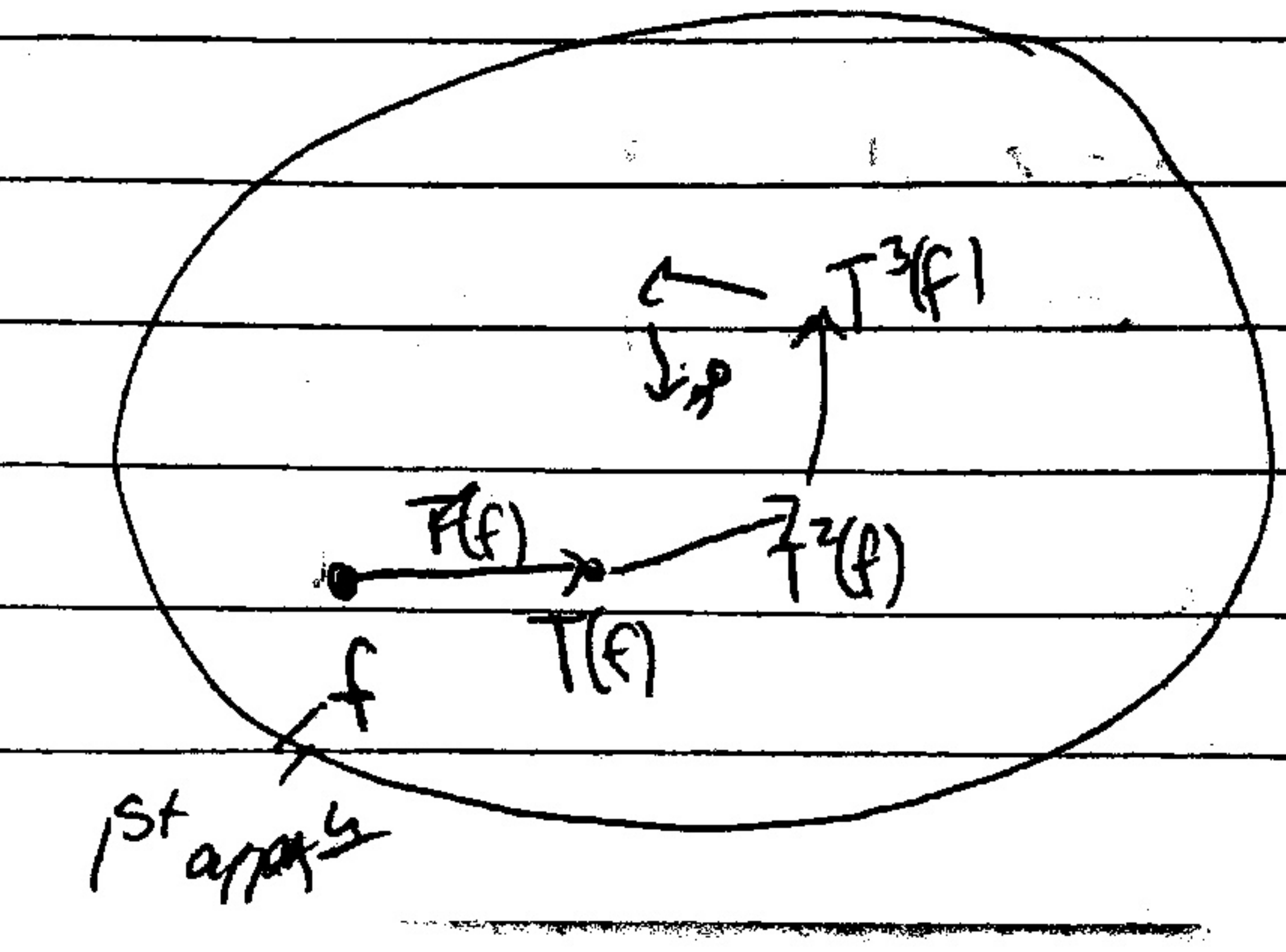
(2\$) $$\text{bad}(p,y)(a,b,d) = \frac{1}{d_a}\Big(p_{ab}(\underline{d}) + \sum_{v \neq a,b} y_{avb}(\underline{d}) \Big)$$

(2) $$P(p,r)(a,v,d) = d_v \Big(\sum_{b \neq v} r_{b,a}(\underline{d} - e_v) \frac{1 - p_{bv}(\underline{d} - e_b - e_v)}{1 - p_{av}(\underline{d} - e_a - e_v)} \Big)^{-1}$$

(3) $$Y(p,y)(a,v,b,\underline{d}) = \frac{p_{av}(\underline{d})(p_{bv}(\underline{d} - e_a - e_v) - y_{avb}(d - e_a - e_v))}{1 - p_{av}(d - e_a - e_v)}$$

Observation: "real" probabilities & ratios P_{av}, Y_{avb} & $R(a,b)$ are "fixed points"

$$R(P,Y) = R$$
$$P(P,R) = P$$
$$Y(P,Y) = Y.$$



Lecture ~~4~~ ~~2~~ 5.2

Operators are "contractive" for probability-like $\underline{p}$ & $\underline{y}$:

Call ($\underline{p}$ & $\underline{Y}$) μ-probability-like on $\mathcal{D}_0$ $\leftarrow$ some set $\subseteq \mathbb{Z}^n_{\geq 0}$

(P1) $0 \leq p_{av}(\underline{d}) \leq \mu$ $\quad \forall \underline{d} \in \mathcal{D}_0$

(P2) $\sum_{v \neq a,b} Y_{avb}(\underline{d}) \leq \mu \cdot d_a$ $\quad \forall \underline{d} \in \mathcal{D}_0$, $a, b \in [n]$ distinct.

(P3) $0 \leq Y_{avb}(\underline{d}) \leq \mu \cdot p_{bv}(\underline{d})$ $\quad \forall \underline{d} \in \mathcal{D}_0$ & $a, v, b \in [n]$ distinct.

Exercise: Show that real (P & Y) is μ-probability like on $\mathcal{B}_{16 \log m}(\mathcal{D}_m)$, assuming $n \log^3 n \leq m = o(n^2)$ where $\mu = 5 \cdot \frac{m}{\binom{n}{2}}$.

Hint: Use simple switching lemma & Recursion for Y_{avb}.

Contraction lemma $\quad$ $(\underline{p}, \underline{y})$, $(\underline{p}', \underline{y}')$ are μ-probability like on $\mathcal{B}_2(\mathcal{D}_0)$

$\underline{r}, \underline{r}'$ fct$^{\text{ns}}$s., $\xi > 0$

(a) If $\underline{p} = \underline{p}'(1 \pm \xi)$ & $\underline{y} = \underline{y}'(1 \pm \xi)$ on $\mathcal{B}_1(\mathcal{D}_0)$

then $\mathcal{R}(\underline{p}, \underline{y}) = \mathcal{R}(\underline{p}', \underline{y}')(1 + O(\mu \cdot \xi))$ on $\mathcal{D}_0$

(b) If $\underline{p} = \underline{p}'(1 \pm \xi)$ on $\mathcal{B}_2(\mathcal{D}_0)$ & $\underline{r} = \underline{r}'(1 \pm \mu \xi)$ on $\mathcal{B}_1(\mathcal{D}_0)$

then $\mathcal{P}(\underline{p}, \underline{r}) = \mathcal{P}(\underline{p}', \underline{r}')(1 + O(\mu \xi))$ on $\mathcal{D}_0$

(c) If $\underline{p} = \underline{p}'(1 \pm \mu \xi)$ and $\underline{y} = \underline{y}'(1 \pm \xi)$ on $\mathcal{B}_2(\mathcal{D}_0)$

then $\mathcal{Y}(\underline{p}, \underline{y}) = \mathcal{Y}(\underline{p}', \underline{y}')(1 + O(\mu \xi))$ on $\mathcal{D}_0$

[w/o proof] e.g. $\mathcal{P}(p, r) = d_v \left(\sum r'(1 \pm \mu \xi) \cdot \frac{1 - p'(1 \pm \xi)}{1 - p'(1 \pm \xi)} \right)^{-1}$

$= d_v \left(\sum r' \frac{1 - p' + O(\mu \xi)}{1 - p' + O(\mu \xi)} (1 + O(\mu \xi)) \right)^{-1}$

$= \mathcal{P}(p', r')(1 + O(\mu \xi))$

Take $(\underline{p}, \underline{y}, \underline{r}) = (P, Y, R)$ and $(\underline{p}', \underline{y}', \underline{r}')$ are guessed formulae.

$$P^g_{av}(\underline{d}) := \frac{d_a d_v}{d(n-1)}\left(1 - \frac{(d_a-d)(d_v-d)}{d(n-1-d)}\right)$$

$$R^g_{ab}(\underline{d}) := \frac{d_a(n-d_b)}{d_b(n-d_a)}\left(1 + \frac{d_a-d_b}{d^2n^2}\sum_i (d_i-d)^2\right)$$

$$Y^g_{avb}(\underline{d}) := P^g_{av}(\underline{d})\, P^g_{bv}(\underline{d}-e_a-e_v)\left(1+\frac{1}{n}\right)$$

→ also μ-probability like on $B_{16\log n}(\mathcal{D}_m)$

<u>Lemma</u> (Guessed formula close to a fixed point)

(a) $\mathcal{R}(P^g, Y^g) = R^g(1+O(\eta))$ $\quad \eta = \frac{1}{dn} + \frac{\sqrt{\log n}\, d}{n^2}$

(b) $\mathcal{P}(P^g, R^g) = P^g(1+O(\eta))$

(c) $\mathcal{Y}(P^g, Y^g) = Y^g(1+O(\eta))$

on $B_{16\log n}(\mathcal{D}_m^*)$: $\mathcal{D}_m^* = \{\underline{d}\in\mathcal{D}_m : \sum_i (d-d_i)^2 \le 2dn\}$ & ass^{ing} $n(\log n)^c \le m = o\left(\frac{n}{(\log n)^{3/4}}\right)$.

Now, $P = P^g(1\pm\xi)$ & $Y = Y^g(1\pm\xi)$ for $\xi = 1$ on $\mathcal{D}_m$

Apply Contraction lemma:

$$R = \mathcal{R}(P,Y) = \mathcal{R}(P^g, Y^g)(1+O(\mu))$$
$$= R^g(1+O(\mu+\eta))$$

$$P = \mathcal{P}(P,R) = \mathcal{P}(P^g, \mathcal{R}(P^g,Y^g)) = \mathcal{P}(P^g,R^g)(1+O(\mu+\eta))$$
$$= P^g(1+O(\mu+\eta))$$

$$Y = \mathcal{Y}(P,Y) = \mathcal{Y}(P^g,Y^g)(1+O(\mu))$$
$$= Y^g(1+O(\mu+\eta))$$

1 round of application $P = P^g(1\pm\xi)$ & $Y = Y^g(1\pm\xi)$

to $P = P^g(1+O(\mu+\eta))$ & $Y = Y^g(1+O(\mu+\eta))$

do this $\log n$ times.

↳ $P = P^g\left(1+O(\mu+\eta)^{\log n} + O(\eta)\right)$ & $Y = Y^g\left(1+O(\mu+\eta)^{\log n} + O(\eta)\right)$

$\mu = \frac{d}{n}$

$\mu^{\log n} = o\left(\frac{1}{dn}\right)$

$$\to R = R^g\left(1 + o\left(\tfrac{1}{dn}\right) + \eta\right) \quad \text{on } B_1(\mathcal{D}_m^*)$$

Remark: wider degree spread $\mathcal{D}_m^+ = \{\underline{d} : \sum d_i = 2m \ \& \ |d_i - d| \le d^{1/2+\varepsilon}\}$ fixed $\varepsilon > 0$

& $d = \varepsilon n$ → need more precise guesses & Maple assisted calculation.

<u>Lecture 5.3</u>

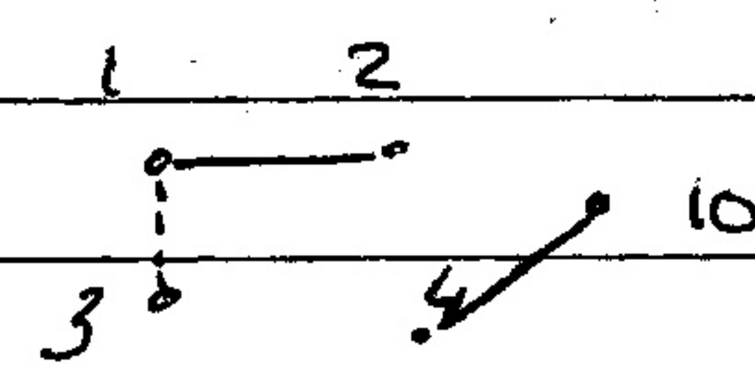
Methods for counting dense d-regular graphs — $d = \Omega\left(\frac{n}{\log n}\right)$

Consider $f(x_1,\dots,x_n) := \prod_{1\le i<j\le n} (1+x_ix_j)$.

Expand : $2^{\binom{n}{2}}$ monomials, each corresponding to a graph on $[n]$

$(1+x_1x_2)(1+x_1x_3)\cdots\cdots(1+x_4x_{10})\cdots$

$x_1x_2\cdot 1\cdot x_4x_{10}$



Given a monomial $x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n} \to \alpha^i = d(v_i)$ in corresponding graph

$\to$ the coefficient of $x_1^d x_2^d \cdots x_n^d$ = # labelled d-regular graphs on $[n]$.

$= |\mathcal{G}(n,d)|$

<u>Cauchy's integral formula</u>

If $f : U \to \mathbb{C}$ holomorphic (our f is a polynomial, so infinitely diff'ble around every pt of $\mathbb{C}$.)

~~then f(z~~ for U open nbhd around 0, then

$$f(0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z}\,dz$$

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}\,dz \qquad ①$$

holomorphic $\Rightarrow$ analytic $\Rightarrow$

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 \dots \frac{f^{(n)}(0)}{n!}x^n \qquad ②$$

& multivariate version ①&② $\Rightarrow$ $[x^d]f(x) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{d+1}}\,dz$.

multivariate version

$$|\mathcal{G}(d,n)| = [x_1^d \cdots x_n^d] \prod_{1 \le i < j \le n} (1+x_i x_j) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{j<k} (1+z_j z_k)}{z_1^{d+1} z_2^{d+1} \cdots z_n^{d+1}} \, dz_1 \, dz_2 \cdots dz_n$$

→ they [McKay & Wormald] chose circles of radius $r = \sqrt{\mu / 1-\mu}$

$dz_i = i \cdot r \cdot e^{i\theta} d\theta$

$$\ldots = \frac{1}{(2\pi r^d)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \underbrace{\frac{\prod_{j<k} (1+r^2 e^{i(\theta_j+\theta_k)})}{e^{id(\theta_1+\cdots+\theta_n)}}} \, d\theta_1 \, d\theta_2 \cdots d\theta_n$$

[saddle point method]

try to find a pt. where integrand is largest
"saddle"

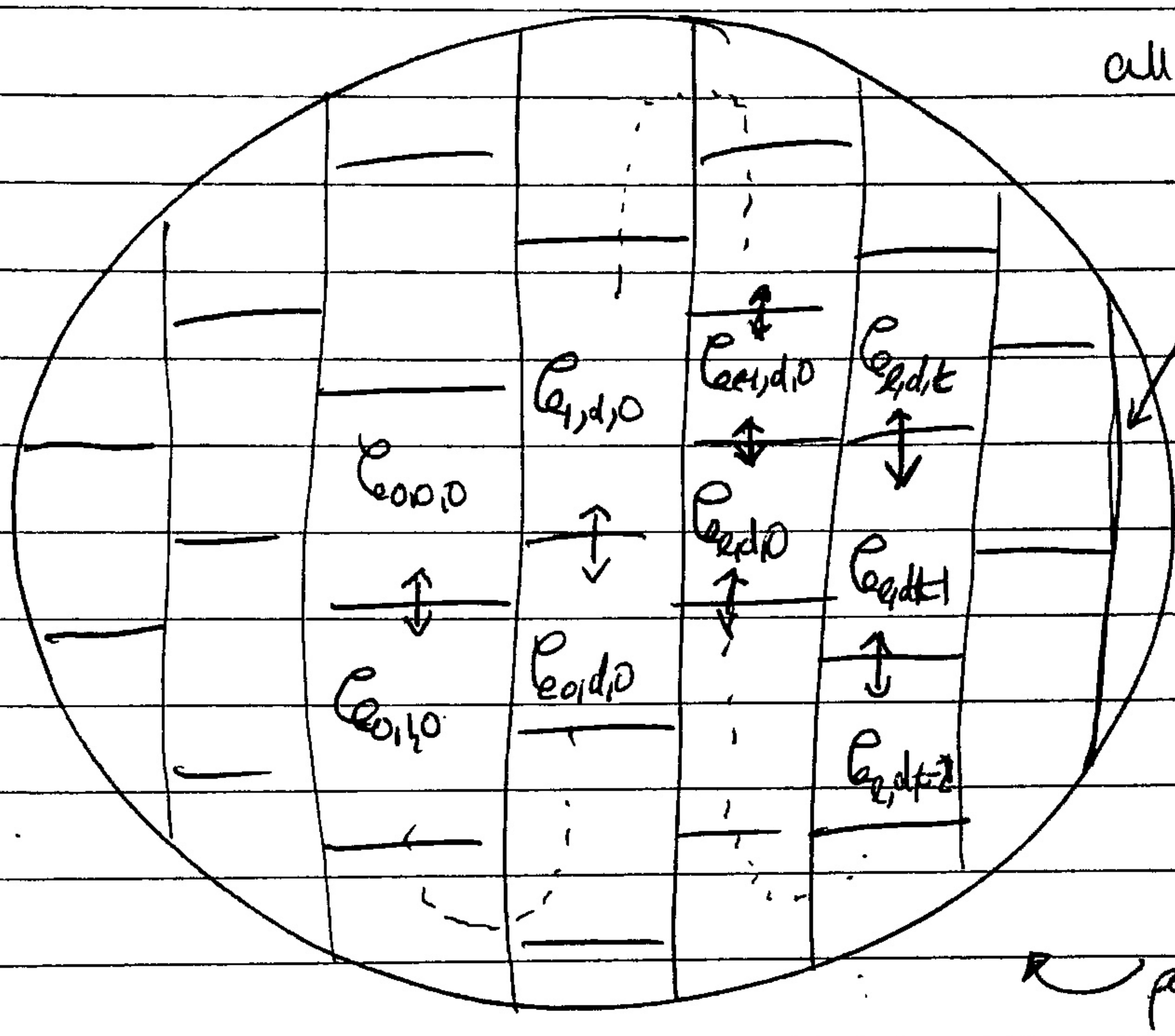
zoom in & show that, for large n, the contr^n from around that saddle point dominates the integral.

instead of integrating $\frac{\Pi(\ldots)}{e^{\ldots}}$ use quadratic app^{x}n around saddle point.

Lecture 5.4 (in case there is time)

For $\Delta^3 = o(M_1)$ or $d^2 = o(n)$. Recall configuration m

all pairings.



Step 1: very few have or or $> \log n$ or $> \log n$ or $> \log n$

partition the rest into $\leq \log^3 n$ classes

$\mathcal{C}_{\ell,d,t}$ — exactly ℓ loops, d doubles & t triples

$$\Pr(\text{Simple}) = \frac{|\mathcal{C}_{0,0,0}|}{\sum |\mathcal{C}_{\ell,d,t}|}$$

$$\frac{|\mathcal{C}_{\ell,d,t}|}{|\mathcal{C}_{0,0,0}|} = \frac{|\mathcal{C}_{\ell,d,t}|}{|\mathcal{C}_{\ell,d,t-1}|} \cdots \frac{|\mathcal{C}_{\ell,d,1}|}{|\mathcal{C}_{\ell,d,0}|} \cdot \frac{|\mathcal{C}_{\ell,d,0}|}{|\mathcal{C}_{\ell-1,d,0}|} \cdots \frac{|\mathcal{C}_{1,d,0}|}{|\mathcal{C}_{0,d,0}|} \cdots \frac{|\mathcal{C}_{0,1,0}|}{|\mathcal{C}_{0,0,0}|}$$

~~switch away~~ exactly 1 triple edge

switch away exactly 1 double edge