Lecture 3: Circuit complexity of code states

By counting, we know that a typical Boolean function and a typical unitary are hard to implement using elementary gates. But what can we say for specific instances? That's usually a hard question.

We will ask questions about the complexity of states. Classically, this doesn't make too much sense. A classical state is a bit string, each of which is always generated from the all 0 state by some bit flips. As a circuit, this has depth one. It is more appropriate to ask the question for a probability distribution over a bit string. Sometimes, some detailed knowledge of the mixture allows us to give nontrivial lower bound on the complexity.

Let's jump into the quantum setting. The setup is that we always start with a pure product state, and apply some quantum circuit consisting of layers of nonoverlapping local gates, until we arrive at a desired state.

- Light-cone of a quantum circuit.

Quantum circuit is there to conjugate observables. (Heisenberg picture)
In the brick wall diagram, it is obvious that an evolved observable does not depend on the gates that are far away. In particular, the observable only see a small patch of the state, or more precisely the reduced density matrix there.

Exercise: density matrix and linear functional.
Let rho be a density matrix of a bipartite system AB.
Consider the linear functional on the space of all hermitian operators acting on A defined by Tr(rho O_A).

1. Does there exist a matrix sigma such that Tr(rho (O_A \otimes Id_B)) = Tr(sigma O_A)?
2. Is such a matrix sigma unique?
3. How do you compute it?

Exercise: the nominal lightcone cannot in general be taken smaller. Find an example circuit and an observable such that evolved observable has nontrivial support on the boundary of the nominal lightcone.

- Lieb-Robinson bounds:

Let there be a d-dimensional lattice with some qudits per site.
A local Hamiltonian term is a hermitian operator acting on 2^d sites that form a unit cube.
A local Hamiltonian is a sum of local Hamiltonian terms.
We assume that a local Hamiltonian term has operator norm at most 1.

A local Hamiltonian in d dimensional lattice gives the time evolution operator e^{-itH} for states (Schroedinger picture),
Or equivalently, every observable (a hermitian operator) evolves by conjugation A \mapsto e^{itH}

Ae^{-itH} (Heisenberg picture).

These are axioms of quantum mechanics.

Let's work in the Heisenberg picture and determine spatial profile of a local operator.

Lemma A.5. *For any integer $k > 0$ and any $y \in \mathsf{Mat}$,*

$$\frac{\|[H, y]_k\|}{k!} \leq C\zeta^k |\mathrm{Supp}(y)| \|y\| \tag{44}$$

where $C, \zeta > 0$ depends only on d *and R.*

Exercise:
Assuming H and A are finite dimensional matrices,
prove that e^{itH} A e^{-itH} = \sum_{k >=0} (it)^k [H,A]_k / k! .
Here [H,A]_k = [H, [H,A]_{k-1}] with [H,A]_0 = A is the nested commutator.

1. Show that the RHS converges absolutely.
2. Hence, the LHS and RHS are both complex analytic in t. Match the derivatives.

- Correlation generation

Correlation between two observables refers to the value
<O_A O_B> - <O_A><O_B>.

For a product state and single-qubit operators, this quantity is zero.
A separable state may have nonzero correlation.

Proposition: Starting with a product state, evolve the state by depth t quantum circuit with 2-qubit local gates on a metric space. The correlation between two observables whose support is separated by distance greater than t is zero.

Exercise: Consider (\ket{00..0}+\ket{11…1})/\sqrt{2} on a line.

1. Find two observables at the ends of the line with nonzero correlation.
2. Prove that the cat state on a line is hard to generate.
3. Find a linear depth circuit using 1d local 2-qubit gates that creates the cat state.

- Another example:

A code state of the toric code.

Properties we will use: Local observable dressed by local projectors is proportional to the product of the local projectors

1. Decompose it in Pauli operator basis.
2. Any anticommuting piece dies away.
3. Any commuting piece survives, but look at Z and X separately, apply homological interpretation, make up the operator by the stabilizers. And, hence is a scalar.
4. Conclude

Zero correlation state: any pair of distant observables have zero correlation.

Corollary: the mutual information is zero. The reduced density operator vanishes.

- Large code distance implies deep encoding circuit.

We cannot rely on the correlation to lower bound the circuit complexity.
Instead, we look at the logical operator or code distance.

Trivial observation:
1. Every logical operator must act nontrivially on d or more qubits.
2. A single-site operator conjugated by depth t unitary circuit acts on a disk of radius t.
3. On toric code, the smallest enveloping ball for a logical operator has diameter L if the toric code has nonzero number of logical qubits.

Corollary: the toric code state with nonzero number of logical qubits is hard to generate.

Toric code state on a sphere --- Twist product

Mutual information for separated regions is zero. There is no encoded qubit, so no logical operator. Is this state really different from the product state? Does there exist a generating circuit of small depth?

Consider a bipartite system, and we introduce a bilinear map that looks like a multiplication, but exactly a multiplication. Any bipartite operator is a bipartite vector, so it is always written as

$O = \sum_{k} A_k \otimes B_k.$

Bring another operator

$O' = \sum_{k'} A'_{k'} \otimes B'_{k'}$

Definition. The twist product $O \infty O'$ is

$\sum_{k,k'} A_k A'_{k'} \otimes B'_{k'} B_k$

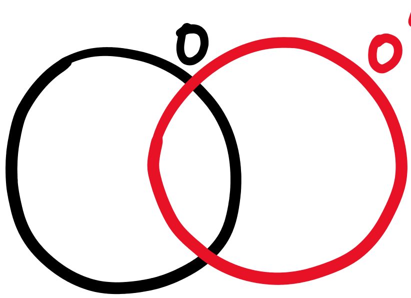
Where the multiplication order in the second tensor component is reversed.

Exercise: Prove that
1. The twist product is bilinear.
2. The twist product is well defined, independent of how one writes O,O' as a sum of tensor products.

Lemma: Suppose there is a "middle" tensor component M such that at least one of O and O' acts on C by identity. So, O and O' are operators on tripartite system AMB. Then, the twist product of O and O' by the bipartition A/MB is the same as that by AM/B.

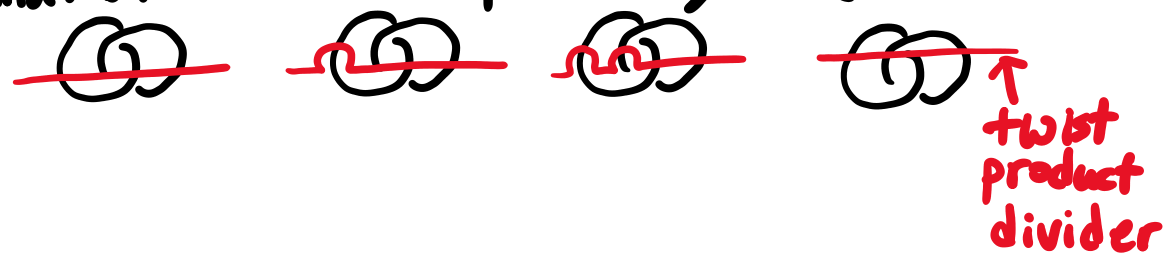
Proof) Obvious by the fact that the twist product is independent of how one writes O, O' as a sum of tensor products.

Now suppose that the supports of O and O' intersect along a disjoint union of two far separated regions, as in the following figure.

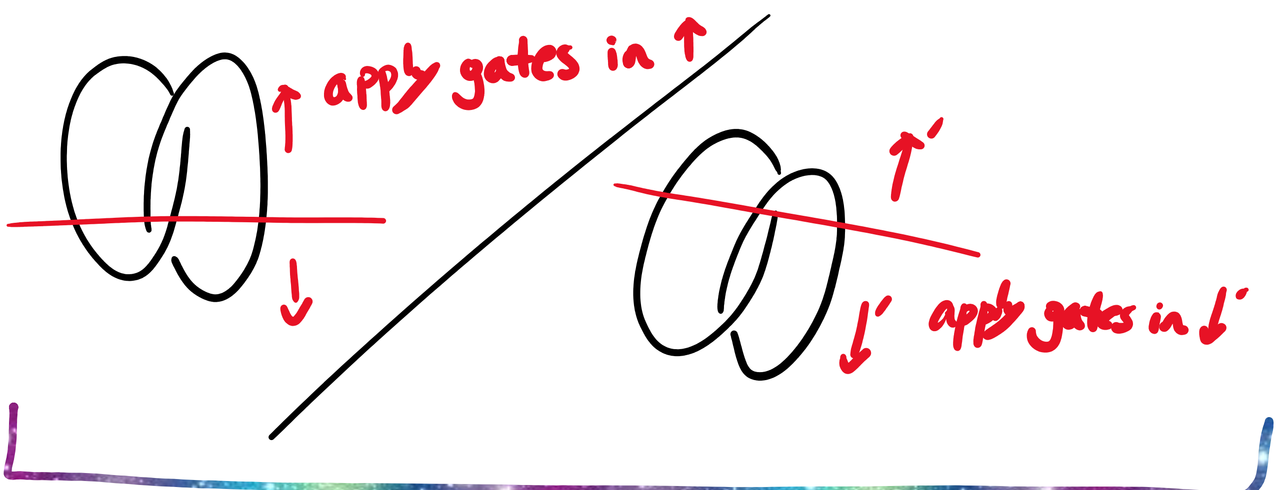


Then, the shallow circuit deformation of the twist product is the twist product of circuit deformations.

Lemma: Divider can be placed anywhere in between



It follows that the value of a twist product can be used to diagnose if the underlying state requires a deep circuit.



Applicable if $\mathrm{supp}(O) \cap \mathrm{supp}(O') = \begin{pmatrix} \circ \\ \circ \end{pmatrix} \sqcup \begin{pmatrix} \circ \\ O \end{pmatrix}$