Lecture 2: Codes and homology

Goal: chain complex, homology, toric code in various spatial dimensions, RG/refined triangulation in 2d.

- A glimpse to the cellular homology.

Abstract definition of a chain complex.

A list of linear spaces with connecting maps such that the composition of two is zero.
Trivial example: two spaces, one map.

Over Z/2

Circle = Triangle: three edges, three vertices. One boundary map, two spaces.

2-sphere = tetrahedron: four faces, six edges, four vertices. Two boundary maps, three spaces.

Very nontrivial idea. Why would you consider formal linear combination of cells?

The homology is a topological invariant.
> Meaning:
> two spaces, homeomorphism between the two, then the homology groups are isomorphic.
> The subject is very mature, and any book would do a better job than my explanation.
> However, a useful mental picture is refinements.

- Syndrome, errors, stabilizers. --- an origin of three-term complex in quantum codes.

Let us restrict ourselves to CSS codes.
A Z-error will be detected by X-stabilizers. This detection is a linear map.

The group of stabilizers is a linear space, and its concrete realization as a Pauli operators is a linear map.

What is the composition?

Let us work on a cellulation of a surface.
> Suppose qubits are on the edges.
> Throw in a Z-error.
> What X-checks do we need to catch the end points of the Z-error?
> What Z-stabilizers can we introduce that doesn't violate any X-checks?

The toric code.
> Arbitrary cellulated surface.

Exercise:

Prove that from the group/space of all Pauli Z operators to the group of X-checks, the commutation relation defines a Z/2-linear map.

Repeat the same question over prime dimensional qudits with clock X and diagonal Z with Fourier phases.

- Toric code in higher dimensions.
  Prescription: Take any cell complex representing a manifold, declare which cells are qubits. X-checks are one dimension lower and Z-stabilizers are one dimension higher.

Let's work on the 3d.

Qubits are on edges,
X-checks are on vertices.
Z-stabilizers are on plaquettes.

Z-logical operators corresponds to homology cycles of the manifold. On 3-torus, the dimension is 3. This is the number of logical qubits.

Exercise:
what is the weight of a Z-stabilizer on the 4d toric code on hypercubic lattice where qubits are 2-cells? How many Z-stabilizers act on a given qubit? What are the left and right degrees of the Tanner graph with qubits on the left and stabilizers on the right?

Exercise:
In the 3d toric code, visualize two X-logical operators representing the same (co)homology class multiply to become stabilizers.

===== Refinement of the cellulation and entanglement RG --- an encoding map =====

Def. An (inverse) entanglement RG in the strictest sense is an isometry from a qubit system to another such that (1) it consists of geometrically local gates, and (2) the code space is mapped correspondingly.

Disentangling transformation. Apply a shallow unitary circuit and look for completely disentangled qubits and throw them away.

Let us construct such an isometry (or reversely disentangling transformation) for the 2d toric code.

There are calculations found in literature that works for a broader class of states, but here we take a shortcut that is closely related to "reversible measurements"

Let us recap the evolution of a stabilizer under Pauli measurements.

Let S be a Pauli stabilizer group. Let P be a Pauli operator, upon measuring P,

1. \pm P is an element of S.
   i. The measurement is deterministic and the stabilizer remains the same.

2. \pm P is not an element of S.
    i. P commutes with every element of S.
        1) The stabilizer group get enlarged upon measurement. (Exercise: prove it.)
    ii. P does not commute with some element of S. (Exercise: prove the following)
        1) A generating set for S can be chosen such that exactly one generator is not commuting with P.
        2) Measurement outcome is 50-50 random.
        3) Post-measurement stabilizer group is generated by those that commute with P and \pm P, where the sign depends on the measurement outcome.

The most interesting case for us is 2.ii. In that case, the underlying stabilizer group is changed. Although it is changed by a measurement, a nonunitary process, there exists a unitary that realizes the change. Let Q be the element of S anticommuting with P. Then, consider (P+Q)/\sqrt{2}. An easy calculation shows that the postmeasurement stabilizer group is achieved from the original by this unitary if the measurement outcome is +1.

Example: Disentangling a Bell pair. XX & ZZ. Measure IZ. Post measurement stabilizer group is generated by single-qubit operators. There are many choices of the anticommuting generator. Each choice gives a different unitary, but they all act the same on the stabilizer group.

Exercise:
Perform the analogous calculation with (P-Q)/\sqrt{2} . Is this a Clifford unitary?
What can you say about the postselection?

- Application to disentangling the toric code state on the honeycomb lattice.

X's for stars, and Z for plaquettes.

Draw a super-honeycomb lattice over the original. Measure each qubit that is not crossed by the superlattice edges in Z. This disentangles out those single qubits. Find unitary that realizes this transformation.

Iterating this transformation, we arrive at a sequence of toric codes where one is mapped to the next one, by a locality preserving isometry. But since the number of qubits is reduced exponentially fast, the whole network of circuit is best represented in an "expanding space." To compensate the expansion, one could insert some swap circuit, and this leads to a linear depth circuit overall. Therefore, we have generated the toric code state by a linear depth circuit.

Exercise:
Consider a quantum circuit consisting of mutually commuting gates where each acts on a ball of radius 172 on a D-dimensional Euclidean lattice. Prove that this is a quantum circuit of depth D+1 where each layer consists of nonoverlapping gates.