Lecture 1: Quantum Singleton bound and consequences

- Classical Singleton bound

  Linear code on n bits of minimal distance d
  Puncture $d-1$ coordinates. There shouldn't be any collision; if there were, a pair would differ only in $d-1$ coordinates, and minimal distance becomes $\le d-1$.
  So we have an injection from the code into a space of dimension $n-(d-1)$.
  $k \le n - (d-1)$.

  Saturated by repetition code.

- Cleaning lemma:

If a region does not support any nontrivial logical operator, the complementary region supports a complete set of logical operators.

An information theoretic argument exists [1610.06169] which gives a much general result, but we treat Pauli stabilizer codes only.

In analogy with the classical code, consider a truncation map acting on the linear space of logical operators in regards to a region that is correctable. The dimension of the logical operators modulo stabilizers in a region A is $(\Pi_A S)^\perp / S_A$ where Π is the restriction map and S_A is the group of all stabilizers supported on A. We want this number to be $2k$ whenever the complement of A supports no nontrivial logical operator.

When A and B are disjoint, it is obvious that S_A is disjoint from S_B. Since we are dealing with vector space, there is a direct summand S' such that $S = S_A \oplus S_B \oplus S'$. Consider Π_A on S'. If this map had some kernel, then that element is supported on B, in which case it would not be in S'. So, Π_A on S' is injective, and $\dim(\Pi_A S) = \dim(S_A) + \dim(S')$. Solving a linear equation gives dimension counting, do $\dim(\Pi_A S)^\perp / S_A = 2$(number of qubits in A) $- 2 \dim S_A - \dim S'$. On the other hand, the absence of nontrivial logical operator on B means that 2(number of qubits in B) $- 2 \dim S_B - \dim S' = 0$. Adding the two, we have the quantity we want is $2 n_A + 2n_B - 2 \dim S = 2k$, as promised.

Exercise: (CSS cleaning lemma) Prove that, for any CSS code on finitely many qubits, if a set of qubits does not support any nontrivial X-logical operator, then the complementary region supports a complete set of representatives of Z-logical operators.

- Quantum Singleton bound [Cerf--Cleve :quant-ph/9702032] :
  Statement
  Let k be the $\log_2$ of the dimension of a nontrivial code space which is assumed > 0. Suppose the subspace has distance d, meaning that any operator supported on $d-1$ or fewer qubits, if sandwiched by the projector, it is a scalar multiple of the projector. Then,

$k \le n - 2(d-1)$.

Proof

First, we need to show that $n \ge 2(d-1)$. Use Cleaning lemma.

If not, we can partition the system ABC such that $|AB| < d$ and $|BC| < d$, but then by the cleaning lemma we have two sets of complete representatives of logical operators on C and A. This violates the no-cloning theorem.

Second, we prove the bound.

Start with a reference system that is maximally entangled with the code space.

Take two correctable regions. The mutual information must be zero with correctable regions.

Take two regions A and B, each of which consists of d-1 qubits. Apply the subadditivity.

Exercise: If there is no correlation between two parties, then the mutual information is zero.

Exercise: Show that the subadditivity is nothing but the nonnegativity of the mutual information, which follows from the nonnegativity of the relative entropy.

- Union lemma :for local codes, if two regions are r-apart and each correctable, then the union is correctable.
  Proof: any logical operator on the union gives two logical operators on each region by the locality. Both are by assumption trivial.

- Application to codes with no "local" logical operator.

  Suppose that any region whose r-neighborhood is a topologically a ball does not support any logical operator.

  Sphere: north and south hemisphere, k=0.

  2-Torus: k = bounded by local qubit density, not on system size.
  High genus torus: k is bounded by a linear function in genus.
  Higher dimensional torus : k = volume of codimension 2.

Exercise: What is the maximum number of logical qubits on nonorientable surfaces?

- Bravyi-Terhal bound:

It is an argumente in 1d. Divide the line into O(r) segments. Let A be the union of every other segment, and B be the complement. If every segment is correctable, then the union is correctable, and by quantum Singleton bound argument, the whole system is correctable, encoding no logical qubit.

Higher dimensional implication: d = O(L^{D-1})

- One dimensional codes are repetition code.

Assumptions: 1d system. Each site holds some finite number of qubits. Stabilizers act on neighboring two sites.

Claim: there is a Clifford unitary for each site by which the stabilizer group is mapped to that of the repetition codes and some completely disentangled qubits and Bell pairs.

Proof:

1. Remove any single site stabilizers.

2. Analyze two-site stabilizer group S:
    a. Let L be the group of left tensor factors. And R be the group of right tensor factors.
    b. If for some elelement A in L, there were two elements B,B in R such that AB and AB' were both in S, then (AB)(AB') = BB' is a single-site stabilizer, which must be trivial. So, we have a map from L to R. By definition this map is surjective. By the same argument it has to be injective.
    c. If L is nonabelian, then there is a Bell pair.
    d. Remove all Bell pairs. Then, L and R are both abelian.

3. Now that all left and right groups (L,R) are abelian, some onsite Clifford will diagonalize them all. The code is reduced to a classical code.

4. If a X-logical operator was identity on a site, then the left factor and the right factor are both logical operators. (Test the commutation with stabilizers)

5. Lemma: For any interval L, if no proper subset of L supports any X-logical operator, then the space \calX of all X-logical operators is a direct sum \calX = \calX(L) \oplus \calY such that \Pi_k(\calY) \cap \Pi_k \calX(L) = 0 for all k \in L. Here, \calX(L) is the set of all X-logical operators supported on L and \Pi_k is the restriction map on site k.
    a. The assumption means that \calX(L) \cap \sum_{k \in L} \calX(except k) = 0.
        i. By 4, the sum of space is a sum of logical operator spaces supported on semi-infinite half lines. So, if the intersection in the claim had an element x, then this element must be a sum of elements x_left and x_right that are supported on a half line to the left and another to the right. The boundaries of these two lines do not have to coincide. But x being supported on L, each of x_left and x_right must be supported on L. By assumption, each must vanish.
    b. Find a subspace \calY of X-logical operators that complements \calX(L) and includes \sum_{k \in L} \calX(except k).
    c. Then, \calY is a desired subspace.
        i. Take z in \Pi_k(\calY) \cap \Pi_k \calX(L). So, z = \Pi_k(y) = \Pi_k(x) for some x,y. But then x-y is in \calX(except k), and hence in \calY, so x is in \calY. By construction, \calX and \calY have zero intersection.

6. Recursively use the lemma to extract repetition codes, starting from smallest intervals.