Continuation: An introduction to $\mathbb{A}^1$-homotopy theory using enumerative examples

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$$

Given $f: S^n \longrightarrow S^n$ and $p \in S^n$ s.t. $f^{-1}(y) = \{q_1, \dots, q_N\}$

$$\deg f = \sum_{i=1}^{N} \deg_{q_i} f$$

Local degree $\deg_{q_i} f$ can be computed:

Let $(x_1, \dots, x_n)$ be oriented coordinates near q_i

Let $(y_1, \dots, y_n)$ —— " —— p

Then $f = (f_1, \dots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$

$$\deg_x f = \begin{cases} 1 & \text{if } J(x) > 0 \\ -1 & \text{if } J(x) < 0 \\ ? & \text{if } J(x) = 0 \end{cases}$$

Q: What if the zeros of f are not multiplicity 1?

Eisenbud-Levine/Khimshiashvili Signature formula:

$$\deg_x f = \text{Signature } \omega^{EKL}$$

where ω^{EKL} is the isomorphism class of the following bilinear form:

$f=(f_1,\dots,f_n)$ $\quad Q := \dfrac{\mathbb{R}[x_1,\dots,x_n]_0}{\langle f_1,\dots,f_n\rangle}$ finite dimensional complete intersecn

$\Rightarrow Q$ Gorenstein, $\mathrm{Hom}_k(Q,k)\cong Q$ Better: distinguished iso, coming from distinguished socle element $\rightsquigarrow$ bilinear form

Schejea-Storch

Explicitly: $\quad \mathrm{Jac}\, f = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$

Choose $\eta: Q \to k$

s.t. $\eta(\mathrm{Jac}\, f) = \dim_k Q$

isomorphism class of ω^{EKL} does not depend on choice of η.

Ex: $f: \mathbb{A}^1 \to \mathbb{A}^1$ $\quad f(z)=z^2$

Eisenbud Q (Bulletin AMS '78): ω^{EKL} defined over a field k char $k \neq 2$ and acts as a "degree." Does ω^{EKL} have a topological interpretation?

Thm: (Kass-W) $\deg_x f = \omega^{EKL}$ in $GW(k)$ for X rational

(Brazelton-Burkland-McKean-Montoro-Opie) $k(x) \supseteq k$ separable

Exercises: 1) $f: \mathbb{A}^2_k \to \mathbb{A}^2_k$ $f(x,y) = (4x^3, 2y)$
char $k \neq 2$

Compute $\deg_0 f$

Detour on $\mathbb{A}^1$-Milnor numbers

$\mathbb{A}^1$-Milnor #'s char $k \neq 2$

The simplest singularity is a node, defined over $\overline{k}$ be a point $p \in X$ with $\hat{\mathcal{O}}_{X,p} \cong \frac{\overline{k}[[x_1, \ldots, x_n]]}{x_1^2 + \ldots + x_n^2 + \text{higher order}}$

hypersurface singularity $p \in \{f = 0\} = X$

If you vary X in a family $X_t = \{ f(x_1, \ldots, x_n) + a_1 x_1 + \ldots + a_n x_n = t \}$ the (potentially) more complicated singularity bifurcates into nodes

$R = \mathbb{C}$

For any $(a_1, \ldots, a_n)$ sufficiently close to 0

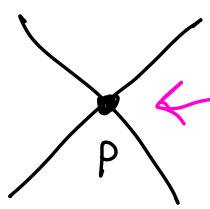
$\#$ nodes in family $= \deg_p^{top} \operatorname{grad} f$ (Milnor)

$\qquad \qquad \qquad \qquad \quad \| $

$\qquad \qquad \qquad \qquad \quad \mu_p$ "Milnor $\#$"

Over R? char $R \neq 2$

nodes come in different types



residue field L

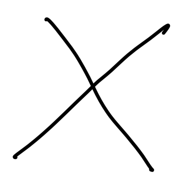
$R \subseteq L$ separable SGA 7 XV

tangent directions are defined over $L[\sqrt{-a}]$

e.g. $R = \mathbb{R}$

$x_1^2 + x_2^2 = 0$

non-split = non-rational tgt directions



$x_1^2 - x_2^2 = 0$

split = ratl tgt directions

Def: The <u>type</u> of a node p with $\hat{\mathcal{O}}_{X,p} \cong \dfrac{k(p)[[x_1,\dots,x_n]]}{\sum a_i x_i^2}$ is defined

$$\mathrm{type}(p) := \mathrm{Tr}_{k(p)/k} \langle 2^n \prod_{i=1}^{n} a_i \rangle \in GW(k)$$

Def: $\mathbb{A}^1$-Milnor #

$$\mu_p := \deg_p^{\mathbb{A}^1} \operatorname{grad} f \underset{\text{Kass-W.}}{=} \sum_{\substack{\text{nodes} \\ p \text{ in} \\ \text{family} \\ \text{for generic} \\ (a_1,\dots,a_n)}} \mathrm{type}(p)$$

Pauli-W. dynamic interpretation

Ex $f(x,y) = x^3 - y^2$ char $\neq 2,3$

$p = (0,0) \in \{f = 0\}$

$\operatorname{grad} f = (3x^2, -2y)$

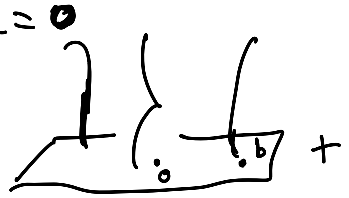
$$\deg \operatorname{grad} f = \deg(3x^2)\deg(-2y)$$

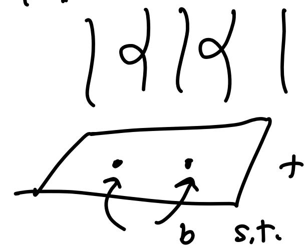
$$= \begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & \tfrac{1}{3} \\ x & \tfrac{1}{3} & 0 \end{array} \langle -2 \rangle = h \qquad \text{rank} = 2$$

$= \langle 1 \rangle + \langle -1 \rangle$ so $\mu = 2$

family: $y^2 = x^3 + ax + t$

$a = 0$ $\qquad\qquad\qquad\qquad$ $a \neq 0$





s.t. Disc $x^3 + ax + t = 0$

$\Leftrightarrow \quad -4a^3 - 27b^2 = 0$

bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split node

over $\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \not\Rightarrow$ can't bifurcate into 2 split or 2 non-split nodes

Exercise : compute $\mu^{\mathbb{A}^1}$ for "ADE singularities"

singularity	equation
A_n	$x_1^2 + x_2^{n+1}$
D_n	$x_2(x_1^2 + x_2^{n-2})$
E_6	$x_1^3 + x_2^4$

E_7	$x_1(x_1^2 + x_2^3)$
E_8	$x_1^3 + x_2^5$

which ones are <u>not</u> multiples of $\mathbb{H}$?

- The classical Milnor number can also be computed using the Euler characteristic χ of the Milnor fiber, and is related to conductor formulas

  M. Levine – Lehalleur – Srinivas

  R. Azouri

  have subtle results on $GW(k)$-enrichments

fun reference Milnor numbers: P. Orlik "The multiplicity of a holomorphic map at an isolated critical point"

$\mathbb{A}^1$-Euler characteristic $\chi^{\mathbb{A}^1}$

X sm, projective $/k$

Last time: A vector bundle $V \to X$ is relatively oriented by (L, φ) where $L \to X$ is a line bundle, and φ is an isomorphism $L^{\otimes 2} \underset{\varphi}{\cong} Hom(\det TX, \det V)$

- TX has a canonical relative orientation

Since $Hom(\det TX, \det TX) \cong \mathcal{O} \cong \mathcal{O}^{\otimes 2}$

- It follows that we may define

Euler number

$$\chi^{\mathbb{A}^1}(X) := n(TX) \in GW(k)$$

<u>Thm:</u> (M. Levine) $\chi^{\mathbb{A}^1}(X)$ equals

the categorical Euler characteristic

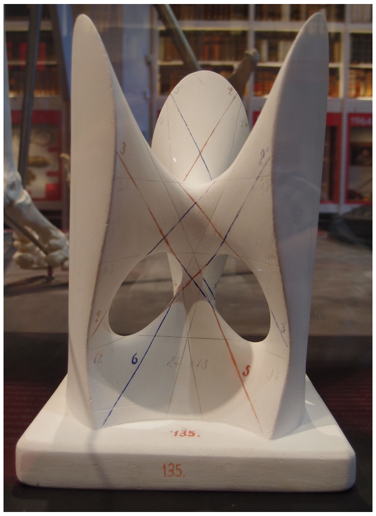
definition omitted

reference: M. Levine "Aspects of Enumerative Geometry with Quadratic Forms"

<u>Ex</u> (M. Levine - Lehalleur - Srinivas)

$X \subset \mathbb{P}^{n+1}$ $\quad$ n even

$X = \{f = 0\}$ for $f \in k[x_0, \dots, x_n]$



Clebsch cubic surface

photo Zaazig
model in Collection of Universität Göttingen

homogeneous degree e

Define B_{Jac} to be the restriction of

$$Q \times Q \xrightarrow{w_{EKL}} k$$

$$Q = \frac{k[x_0, x_1, \ldots, x_n]}{\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \rangle}$$

to $\bigoplus_{q=0}^{n} Q_{(q+1)e-n-2}$ ← The degree $(q+1)e-n-2$ part

$$\chi(X) = \langle e \rangle + \langle -e \rangle B_{Jac} + \frac{n}{2} h$$

$$h = \langle 1 \rangle + \langle -1 \rangle$$

Ex: $C = \{ [x_0, x_1, x_2, x_3] : \Sigma x_i^3 = (\Sigma x_i)^3 \} \subset \mathbb{P}^3_{\mathbb{Q}}$

Clebsch cubic surface $\quad \chi(C) = 9$

$$\chi^{A^1}(C) = 2h + \langle -10 \rangle + \langle -6 \rangle + \langle -21 \rangle + \langle -14 \rangle + \langle -2 \rangle$$

Computed with Macaulay 2

<u>Cohomology</u>: $SH(k)$ stable $\mathbb{A}^1$-homotopy theory

$\rightsquigarrow$ cohomology theories H on smooth k-schemes X

<u>ex</u>: $H =$

$H\mathbb{Z}$ motivic coh

$H\tilde{\mathbb{Z}}$ generalized motivic coh

K K theory

KO Hermitian K-theory

$$H^n(X) = \pi_{-n} \operatorname{Hom}(X, H) = [X, \Sigma^n H]$$

$V \to X$ vector bundle

$$H^V(X) = [X, Th(V) \wedge H]$$

$Th(V) = V^* / V^* - 0$ "twisted suspension"

<u>Ex</u>: $V = \mathcal{O}_X^{\oplus n}$ = trivial bundle of rank n on X

then write H^n for H^v

$$H\mathbb{Z}^n(X) \cong H^{2n}_{mot}(X, \mathbb{Z}(n)) \cong H^{2n,n}_{mot}(X)$$

$$\cong CH^n(X)$$ Chow group of algebraic cycles codim n modulo rational equivalence

"geometric degrees"

$$H\tilde{\mathbb{Z}}^n(X) \cong \tilde{CH}^n(X)$$ "Chow-Witt groups" or "oriented Chow groups"

formal sums of codim n subvarieties Z with coefficients in $GW(k(Z))$ subject to conditions mod equivalence

ref: Barge-Morel "Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels"

Bachman-Calmès-Déglise-Fasel-Østraer "Milnor-Witt Motives"

$K^0(X) \cong$ group completion vector bundles on X

$KO^0(X) \cong$ group completion vector bundles on X with non-degenerate, symmetric form on X

"representable"

$H\mathbb{Z}, H\tilde{\mathbb{Z}}, K, KO \in SH(k)$

cohomology with support: $Z \hookrightarrow X$ closed subscheme

<u>Def</u>: $H_Z^V(X) := [X/X-Z, Th(V) \wedge H]$

Euler class

H cohomology theory, $\mathbb{S} \to H$ ring

$V \rightleftarrows X$ vector bundle with a section f

$$\begin{tikzcd} V \arrow[r] & X \arrow[l, bend left, "f"] \end{tikzcd}$$

<u>Def</u>: (The Euler class of V with respect to f)

$e^H(V, f) \in H^{V^*}_{\{f=0\}}(X)$ is the htpy class of

$$X/X-\{f=0\} \xrightarrow{f} \frac{V}{V-0} \wedge H$$

Euler number

Def: $f: X \to S$ is lci or a "local complete intersection morphism" if it locally factors as

$$U \xhookrightarrow{i} P \xrightarrow[\text{smooth}]{p} S$$

closed immersion determined by a koszul-regular sequence

Properties:

- well behaved cotangent complex L_f
- $L_i \simeq N_U^* P[1]$ conormal bundle, i.e. dual of normal bundle
- $L_p \simeq \Omega_{P/S} \simeq T_P^*$
- L_{pi} determined by $i^* L_p \to L_{pi} \to L_i$
- coherent / serre duality related to L_f

pushforward : $p: X \to S$ proper, lci

Becker-Göttlieb transfer $\Sigma_+^\infty S \to Th(L_p)$

$\rightsquigarrow p_* : H^{L_p}(X) \to H^0(S)$

ex: X sm/R $p_* : H^{T^* X}(X) \to H^0(R)$

oriented theories: • H is GL-oriented if

$$H^V_Z(X) \cong H^n_Z(X) \qquad n = \operatorname{rank} V$$

"the twisted suspension is just a suspension to the eyes of H"

ex: $H\mathbb{Z}$, K non-ex: $H\tilde{\mathbb{Z}}$, KO

• H is SL-oriented if
SL$_c$-oriented

$$H^V_Z(X) \cong H^{V'}_Z(X)$$

whenever V and V' have the same rank and $\det V \cong \det V' \otimes L^{\otimes 2}$

$L \to X$ line bundle

Ananyevskiy

ex: $H\tilde{\mathbb{Z}}$, KO

• For V relatively oriented on $X \xrightarrow{p} k$ $\quad p$ sm proper and H an SL_c oriented theory

$$H^{V^*}(X) \cong H^{T^*X}(X)$$

Let f be any section of V, for example $f=0$

$$\begin{array}{ccccc} H^{V^*}_{\{f=0\}}(X) & \xrightarrow{\ 2 = \text{forget support}\ } & H^{V^*}(X) \cong H^{T^*(X)}(X) & \xrightarrow{\ p_*\ } & H^0(k) \\ \Psi & & \Psi & & \Psi \\ e^H(V,f) & \longmapsto & e^H(V) & \longmapsto & n^H(V) \end{array}$$

$$2(e^H(V,f)) = X \to \frac{X}{\{f=0\}} \to Th(V) \wedge H$$

since any two sections f_1, f_2 of V are connected by (A^1)'s, $2(e^H(V,f_1)) = 2(e^H(V,f_2))$

Let $e^H(V) = 2(e^H(V,f))$

<u>Def</u>: The <u>Euler number</u> $n^H(V)$ of V in $H^0(k)$ is $n^H(V) := p_* e^H(V)$

- This agrees with our previous definition for $H = H\tilde{\mathbb{Z}}$ or KO

- So we have shown $n(V) = \sum_{\substack{x \text{ s.t.} \\ f(x)=0}} \deg_x f$

is independent of the choice of section f!

reference: Déglise-Jin-Khan, Bachmann-W.

An Arithmetic Count of the lines on a Smooth Cubic Surface

joint w Jesse Kass

A <u>cubic surface</u> is $X \subseteq \mathbb{P}^3$

$$X = \{[w,x,y,z] \mid f(w,x,y,z) = 0\}$$

where f is homogeneous of degree 3

<u>Thm</u> (Salmon + Cayley 1849): Any smooth cubic surface over $\mathbb{C}$ contains exactly 27 lines

<u>Ex</u>: Fermat cubic surface

$$f(w,x,y,z) = w^3 + x^3 + y^3 + z^3$$

lines: $\{[S,-S,T,-T] \ : \ [S,T] \in \mathbb{P}^1(\mathbb{C})\} \subseteq X$

or $\omega^3, \lambda^3 = -1$

$$\{[S, \lambda S, T, \omega T] : [S,T] \in \mathbb{P}^1(\mathbb{C})\} \subseteq X$$

permute variables: $\binom{4}{2}/2 = 3$ ways

Total: $3 \times 3 \times 3 = 27$ lines

Proof for any smooth cubic surface over $\mathbb{C}$:

$Gr(1,3)$ = Grassmannian parameterizing lines in $\mathbb{P}^3$

equivalently $W \subseteq k^4$ $\dim W = 2$

$$S \longrightarrow Gr(1,3) \quad \text{tautological bundle}$$

$$S_{[\mathbb{P}W]} = W$$

$$\mathrm{Sym}^3 S^* \longrightarrow Gr(1,3) \quad \text{rank 4 vector bundle}$$

$$V = \mathrm{Sym}^3 S^*_{[\mathbb{P}W]} = \mathrm{Sym}^3 W^*$$

$=$ cubic polynomials on W

f determines a section σ_f of V

$$\sigma_f([\mathbb{P}W]) = f|_W$$

$\sigma_f([\mathbb{P}W]) = 0 \iff$ line L corresponding to W is in X

So we have reduced to counting:
The zeros of a section of V

$$n(V) = \sum_{\substack{\text{lines } L \\ \text{s.t. } \sigma_f(L)=0}} \deg_L \sigma_f$$

Euler number
same as Euler
Class using ordinary
Singular homology
of $\mathbb{C}$-points

Fact: cubic surface smooth $\Rightarrow$ all zeros have multiplicity 1

Since σ_f is over $\mathbb{C}$, σ_f preserves orientation $\Rightarrow$ $\deg_L \sigma_f = 1$

$\Rightarrow \quad n(V) = \#$ lines

and in particular is <u>independent</u> of f!

compute $n(V) = 27$ from splitting principle, $H^*(Gr(1,3))$

or from the above example

Cubic surfaces over $\mathbb{R}$: (Schläfli 1863)

There can be 3, 7, 15, or 27 lines

Segre (1942) distinguished between

hyperbolic and elliptic lines

$L \subseteq X$

L real line

<u>Aside</u>: $\mathrm{Aut}(L) \cong \mathbb{P}GL_2\mathbb{R}$

$z \mapsto \frac{az+b}{cz+d} \leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$I \in \mathbb{P}GL_2\mathbb{R}$

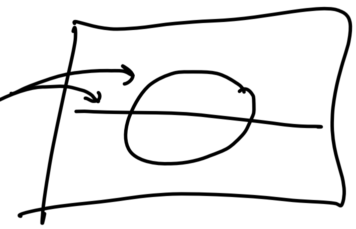
$\mathrm{Fix}(I) = \{z \mid I(z) = z\}$

$I \in \mathbb{P}GL_2\mathbb{R}$ is hyperbolic if $\mathrm{Fix}\{I\}$ is 2 real pts

—— " —— elliptic if $\mathrm{Fix}\{I\}$ is a conjugate pair of complex pts

- L gives rise to an involution $I : L \to L$

$I(p)$ defined :



T_pX

$T_pX \cap X$
$= L \cup C$

degree $3-1=2$
Bézout

$L \cap C = \{p, q\}$

$I(p) := q$

L is hyperbolic/elliptic when I is

Alternatively use (S)pin structures:

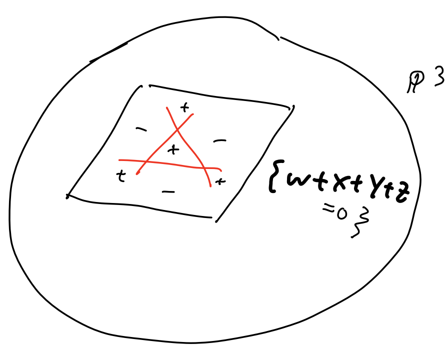
$$L \longrightarrow \text{planes in } \mathbb{P}^3 \text{ containing } L$$

$$p \longmapsto T_p X$$

elliptic : spins around
hyperbolic: doesn't

e.g.

Fermat surface is above + or below −



$w^3+x^3+y^3+z^3=0$

all $\mathbb{R}$-lines hyperbolic

Thm: (Segre + Okonek–Teleman '14, Finashin–Kharlamov '12, Benedetti–Silhol '94, Horev–Solomon '12)

$$\#\text{ hyperbolic lines} - \#\text{ elliptic lines} = 3$$

Q: What about $k = \mathbb{F}_p, \mathbb{Q}_p, \mathbb{Q}, \ldots$?

Hirschfield notes hyperbolic / elliptic makes sense

A: The above proof works in $\mathbb{A}^1$-homotopy theory. We obtain:

$$\text{line } L \subseteq \underset{\text{cubic surface}}{X} \subseteq \mathbb{P}^3$$

Def: the *type* of L is $\langle D \rangle \in GW(k(L))$ where $D \in k(L)^* / (k(L)^*)^2$ is s.t.

$\operatorname{Fix} I$ is a conjugate pair of points in $k(L)[\sqrt{D}]$

$$\Leftrightarrow \langle D \rangle = \langle ab - cd \rangle \qquad I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Leftrightarrow$ $\langle D \rangle = \langle -1 \rangle \deg^{\mathbb{A}^1} I$

Fact: $k \subseteq k(L)$ is separable

$$\underset{k(L)/k}{\mathrm{Tr}} : GW(k(L)) \longrightarrow GW(k)$$

$$[V \times V \to k(L)] \mapsto [V \times V \to k(L) \xrightarrow{\mathrm{Tr}_{k(L)/k}} k]$$

Thm (Kass-W.) Let k be a field, char $k \neq 2$. Let X be a smooth cubic surface in $\mathbb{P}^3$. Then

$$\sum_{\substack{\text{lines} \\ L \subseteq X}} \mathrm{Tr}_{k(L)/k} \, \mathrm{Type}(L) = 15\langle 1 \rangle + 12 \langle -1 \rangle$$

Rmk:

- $k = \mathbb{C}$: Apply rk $\rightsquigarrow$ 27 lines over $\mathbb{C}$
- $k = \mathbb{R}$: Apply signature $\rightsquigarrow$ # hyperbolic − # elliptic = 3

- $k = \mathbb{F}_q$. $\mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 = \{1, u\}$

So natural to say type $= \langle 1 \rangle \Leftrightarrow$ hyperbolic
type $= \langle u \rangle \Leftrightarrow$ elliptic

and Hirschfield did

<u>Cor</u> $\#$ elliptic lines defined over $\mathbb{F}_{q^{2a+1}}$ $+$ $\#$ hyperbolic lines defined over $\mathbb{F}_{q^{2a}}$ $\equiv 0 \pmod 2$

e.g. when all lines defined over k

$\#$ elliptic lines $\equiv 0 \pmod 2$

- Pauli analogue for Quintic 3-folds

- For Euler numbers, see M. Levine "Witt valued characteristic classes"

Bachmann-W. "$\mathbb{A}^1$-Euler classes : 6 functors..."