An introduction to $\mathbb{A}^1$-homotopy theory using enumerative examples

Enumerative geometry: counts algebro-geometric objects over $\mathbb{C}$

Q: How many lines meet 4 lines in $\mathbb{P}^3$?

A: 2

Goal: to record arithmetic information about geometric objects over a field k

(whose # is fixed over $\bar{k}$, but not over k)

tool: $\mathbb{A}^1$-homotopy theory (Morel–Voevodsky)

cf. Krashen, Déglise lectures here PCMI

First some classical homotopy theory:

$$S^n = \{(x_0, \ldots, x_n) \mid \Sigma x_i^2 = 1\} = \mathbb{P}^n(\mathbb{R}) / \mathbb{P}^{n-1}(\mathbb{R})$$

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$$

pointed homotopy classes

Given $f: S^n \longrightarrow S^n$ and $p \in S^n$ s.t. $f^{-1}(y) = \{q_1, \dots, q_N\}$

$$\deg f = \sum_{i=1}^{N} \deg_{q_i} f$$

where $\deg_{q_i} f$ is the <u>local degree</u>:

Let $V \ni p$ be a small ball

Let $f^{-1}(V) \supset U \ni q_i$ be a small ball s.t. $f^{-1}(p) \cap U = q_i$

$$\begin{array}{ccc} U/\partial U \simeq S^n & & V/\partial V \simeq S^n \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ U/(U - q_i) & \xrightarrow{\bar{f}} & V/(V-p) \end{array}$$

$\deg_{q_i} f := \deg \bar{f}$

<u>Formula from differential topology</u>

Let $(x_1, \dots, x_n)$ be oriented coordinates near q_i

Let $(y_1, \dots, y_n)$ —— " —— p

Then $f = (f_1, \dots, f_n): \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$

$$\deg_{q_i} f = \begin{cases} +1 & J(q_i) > 0 \\ -1 & J(q_i) < 0 \end{cases}$$

<u>ex</u>: f is a polynomial map over $\mathbb{C}$, $J(q_i) \neq 0$

$\Rightarrow \deg_{q_i} f = 1$

$\forall q_i \Rightarrow \deg f = |f^{-1}(\text{point})|$

$= \text{the number of solutions}$
$\text{to } \{f_1 = f_2 = \ldots = f_n = 0\}$

ex: $f(x) \in \mathbb{C}[x]$ degree n

$f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$

$\deg f = n = |\{f=0\}|$

fundamental theorem of algebra

Rmk: We can also count solutions to $f=0$ where f is a section of a rank n vector bundle $V \underset{f}{\overset{p}{\rightleftarrows}} X$

$$e(V) = e(V, f) = \sum_{q \in \{f=0\}} \deg_q f$$

Ex: $Gr(1,3)$ = Grassmannian parameterizing dim 2 subspaces $W \subset \mathbb{C}^4$

or equivalently lines

$$\mathbb{P}W := \frac{W - \{0\}}{\substack{\lambda w \sim w \\ \forall w \in W \\ \lambda \in \mathbb{C}^*}} \subset \mathbb{P}\mathbb{C}^4$$

$$\mathbb{P}\mathbb{C}^4 = \mathbb{P}^3$$

$S \longrightarrow Gr(1,3)$ tautological bundle

rank 2

$$S_{[\mathbb{P}W]} = W$$

Let L_1, L_2, L_3, L_4 be four lines in $\mathbb{P}^3$

$$\{\text{lines meeting } L_1, L_2, L_3, L_4\} = \{f = 0\}$$

for f a section depending on the L_i

of $\bigoplus_{i=1}^{4} \Lambda^2 S^* \longrightarrow Gr(1,3)$

$\Rightarrow$ # lines meeting $L_1, L_2, L_3, L_4 = e\left(\bigoplus_{i=1}^{4} \Lambda^2 S^* \longrightarrow Gr(1,3)\right)$

In particular, this number is independent of the generically chosen lines!

$= 2$ (splitting principle, $H^*(Gr)$...)

$\mathbb{A}^1$-homotopy theory

Lannes / Morel : k field

degree for $f : \mathbb{P}^1 \to \mathbb{P}^1$

remembering more than the sign of the Jacobian

$\deg f$ valued in $GW(k)$ = Grothendieck–Witt group of k

$GW(k) :=$ group completion under $\perp$ of semi-ring $\perp, \otimes$ of nondegenerate, symmetric bilinear forms

Krashen: lectures

$W(k)$ = Witt group of k

$= GW(k) / \mathbb{Z}(\langle 1 \rangle + \langle -1 \rangle)$

$\mathrm{rank} : GW(k) \longrightarrow \mathbb{Z}$ $\quad \mathrm{rank}(V \times V \xrightarrow{\beta} k) = \dim_k V$

$$\begin{tikzcd} GW(k) \arrow[r] \arrow[d, "\mathrm{rank}"'] \arrow[dr, phantom, "\lrcorner"] & W(k) \arrow[d, "\mathrm{rank}"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

$GW(k)$ generators: $\langle a \rangle$ $\quad a \in k^* / (k^*)^2$

$\langle a \rangle$ element associated to bilinear form

$\langle a \rangle : k \times k \longrightarrow k$

$(x, y) \longmapsto axy$

relations: (1) $\langle a\rangle\langle b\rangle \simeq \langle ab\rangle$

(2) $\langle u\rangle+\langle v\rangle \simeq \langle uv(u+v)\rangle + \langle u+v\rangle \qquad u+v\neq 0$

(3) $\langle u\rangle + \langle -u\rangle = \langle 1\rangle + \langle -1\rangle = \mathbb{H}$

Exercise: (2) $\Rightarrow$ (3)

Exercise: $\langle 1\rangle+\langle -1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in $GW(k)$. watch characteristic 2.

Ex: $GW(\mathbb{C}) \overset{\text{rank}}{\cong} \mathbb{Z}$

Ex: $GW(\mathbb{R}) \cong \mathbb{Z}\times\mathbb{Z}$

Silvester's theorem

$B: V\times V \to \mathbb{C}$, there is a basis $\{v_1,\dots,v_r\}$ of V s.t.

Gram matrix $B(v_i,v_j)$ is $\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}$

Signature $B = \#1\text{'s} - \#(-1)\text{'s}$

$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r, "\text{rank}\times\text{sign}"] \arrow[dr, "\cong"'] & \mathbb{Z}\times\mathbb{Z} \\ & \{(r,s)\in\mathbb{Z}\times\mathbb{Z} \mid r+s\equiv 0 \bmod 2\} \cong \mathbb{Z}\times\mathbb{Z} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Ex: $GW(\mathbb{F}_q) \xrightarrow[\cong]{\text{rank}\times\text{disc}} \mathbb{Z}\times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$

Ex: (Springer's theorem) K complete discrete valued field
k residue field

e.g. $K = \mathbb{Q}_p$ $\quad K = \mathbb{F}_p((t))$, $k = \mathbb{F}_p$

Assume $\quad \operatorname{char} k \neq 2$

$$W(K) \cong W(k) \oplus W(k)$$

<u>Transfer</u>: $K \subseteq E$ finite separable $\quad Tr_{E/K} : GW(E) \to GW(K)$

$$Tr_{E/K}\left(V \times V \xrightarrow{B} E \right) = \left(V \times V \xrightarrow{B} E \xrightarrow[Tr_{E/K}]{} K \right)$$

Back to Lannes's formula:

$$f : \mathbb{P}^1_k \to \mathbb{P}^1_k \qquad p \in \mathbb{P}^1(k) \qquad f^{-1}(p) = \{q_1, \dots, q_N\}$$

s'pose $\quad J(q_i) = f'(q_i) \neq 0 \quad \forall i$

$$\deg f = \sum_{i=1}^{N} Tr_{k(q_i)/k} \langle J(q_i) \rangle$$

This doesn't depend on p!

(unstable)
$\mathbb{A}^1$-htpy
classes of maps

<u>Exercise</u>: 1) $\deg \left(\mathbb{P}^1 \xrightarrow[z \mapsto az]{[x:y] \mapsto [ax:y]} \mathbb{P}^1 \right) = \langle a \rangle$

2) $\deg \left(\mathbb{P}^1 \xrightarrow[z \mapsto z^2]{} \mathbb{P}^1 \right) = \langle 1 \rangle + \langle -1 \rangle$

<u>Morel</u>: $\deg^{\mathbb{A}^1} : \left[\mathbb{P}^n / \mathbb{P}^{n-1}, \mathbb{P}^n / \mathbb{P}^{n-1} \right]_{\mathbb{A}^1} \to GW(k)$

$$\begin{tikzcd}
{[S^n, S^n]} \arrow[d, "\deg"] & {[\mathbb{P}^n_{\mathbb{R}} / \mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{R}} / \mathbb{P}^{n-1}_{\mathbb{R}}]} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\
\mathbb{Z} & GW(\mathbb{R}) \arrow[l, "\text{signature}"'] \arrow[r, "\text{rank}"] & \mathbb{Z}
\end{tikzcd}$$

- Cazanave, Brazelton-McKean-Pauli give formulas for $\deg^{\mathbb{A}^1}$(polynomials) in terms of Bézoutians

Points of view on $\mathbb{P}^n/\mathbb{P}^{n-1}$:

Notation: $X \wedge Y = X \times Y / ((X \times *) \cup (* \times Y))$

<u>ex</u>: In top, $S^n \wedge S^m \simeq S^{n+m}$, $(S^1)^{\wedge n} = S^n$

$$\Sigma_{S^1} X = S^1 \wedge X$$

<u>Ex</u>:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "z \mapsto z"] \arrow[d, "z \mapsto 1/z"'] & \mathbb{A}^1 \simeq * \arrow[d, hook] \\ \mathbb{A}^1 \simeq * \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

pushout $\Rightarrow$ $\Sigma \mathbb{G}_m \simeq \mathbb{P}^1$, $\Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

<u>Spheres</u>: S^1, $\mathbb{G}_m = \operatorname{spec} k[\frac{1}{z}, z] = \mathbb{A}^1 - \{0\}$

$$S^{p+q\alpha} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q} \simeq S^{p+q,q}$$

Ex: $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$

pf: Induction and

$$\begin{tikzcd}
(\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^{n-1} - \{0\}) \arrow[r] \arrow[d] & \mathbb{A}^1 \times (\mathbb{A}^{n-1} - \{0\}) \simeq \mathbb{A}^{n-1} - \{0\} \arrow[d] \\
(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n \arrow[r] & \mathbb{A}^n - \{0\}
\end{tikzcd}$$

pushout

$$(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n \wr\!\!\simeq \mathbb{A}^1 - \{0\}$$

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & \Sigma X \wedge Y = S^1 \wedge X \wedge Y
\end{tikzcd}$$

pushout

$\Rightarrow \mathbb{A}^n - \{0\} \simeq \Sigma (\mathbb{A}^n - \{0\}) \wedge (\mathbb{A}^1 - \{0\})$

Ex: $\mathbb{P}^n / \mathbb{P}^{n-1} \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$

pf: $\mathbb{P}^n / \mathbb{P}^{n-1} \simeq \mathbb{P}^n / \mathbb{P}^n - \{0\} \simeq \mathbb{A}^n / \mathbb{A}^n - \{0\}$

$$\simeq * / \mathbb{A}^n - \{0\} \simeq \operatorname{colim} \begin{tikzcd} \mathbb{A}^n - \{0\} \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

$$\simeq \Sigma (\mathbb{A}^n - \{0\})$$

- stable homotopy theory allows Σ^{-1}

useful for duality and representing cohomology theories

- $\mathbb{A}^1$-stable homotopy theory allows $\Sigma_{\mathbb{P}^1}^{-1}$

$SH(k)$ where $\Sigma_{\mathbb{P}^1}(-) = \mathbb{P}^1 \wedge (-)$

<u>Thm</u> (Morel, Hopkins-Morel) k field

Stably $[S^0, S^0] \cong [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \cong GW(k)$

$$\bigoplus_{n \in \mathbb{Z}} [S^0, \mathbb{G}_m^{\wedge n}] \cong K_*^{MW}(k)$$

We saw both $W(k)$ and

$K_*^M(k)$ = Milnor K-theory Krashen Lectures

and

$W(k)$

Combine to $K_*^{MW}(k)$ = Milnor-Witt K-theory

graded associative algebra

generators: $[u] \in K_1^{MW}(k)$ $u \in k^*$ $\eta \in K_{-1}^{MW}(k)$

relations: 1) Steinberg relation $[a][1-a] = 0$

2) $[ab] = [a] + [b] + \eta [a][b]$

3) $[a]\eta = \eta [a]$

4) $\eta h = 0$ for $h = \eta[-1] + 2$

<u>Rmk</u>: $GW(k) \cong K_0^{MW}(k)$

$$\langle a \rangle \longleftrightarrow 1 + \eta [a]$$

$$h = \langle 1 \rangle + \langle -1 \rangle \longleftrightarrow 1 + 1 + \eta [-1]$$

<u>About proof</u>: $[a] : S^0 \longrightarrow \mathbb{G}_m$

non base point $\longmapsto a$

$$\eta : \mathbb{A}^2 - 0 \longrightarrow \mathbb{P}^1$$
$$(x, y) \longmapsto [x, y]$$

<u>Rmk</u>: $\mathbb{C}$-pts: $\mathbb{C}^2 - 0 \simeq S^3 \to S^2$ Hopf map

$\mathbb{R}$-pts: $S^1 \xrightarrow{\cdot 2} S^1$

$\Rightarrow$ not nilpotent!

$S^1 \wedge (-)$

$$X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$$

$$\Sigma(X \vee Y) \longrightarrow \Sigma(X \times Y) \to \Sigma(X \wedge Y)$$

$$\Sigma \pi_X + \Sigma \pi_Y$$

$$\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$$

Lemma: In $SH(k)$,

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow[\text{mult}]{} & \mathbb{G}_m \\ \wr\| & & \| \\ \mathbb{G}_m \vee \mathbb{G}_m \vee (\mathbb{G}_m \wedge \mathbb{G}_m) & \xrightarrow[(1,1,\eta)]{} & \mathbb{G}_m \end{array}$$

Lemma $\Rightarrow$ $\left(\mathbb{P}^1 \xrightarrow[z \mapsto az]{} \mathbb{P}^1\right) = 1 + \eta[a]$ in $SH(k)$

because $\left(\mathbb{P}^1 \xrightarrow[z \mapsto az]{} \mathbb{P}^1\right) = \Sigma\left(\mathbb{G}_m \xrightarrow[z \mapsto az]{} \mathbb{G}_m\right)$

$= \Sigma\left(\mathbb{G}_m \times k \xrightarrow{1 \times a} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\text{mult}} \mathbb{G}_m\right)$

Lemma $\Rightarrow$ $[ab] = [a] + [b] + \eta[a][b]$

...

$$K^{MW}_*(k) \longrightarrow \bigoplus_{n \in \mathbb{Z}} [S^0, \mathbb{G}_m^{\wedge n}]$$

$\square$

Show isomorphism ...

In particular, $\deg: [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$

notation: $\bigoplus_{n\in\mathbb{Z}} [S^0, \mathbb{G}_m^{\wedge n}] \cong \bigoplus_n \pi_{n,n} \mathbb{S}$ 0-line

$\bigoplus_n \pi_{n+r,n} \mathbb{S}$ r-line

Röndigs – Spitzweck – Østraer compute 1-line
in terms of Hermitian and Milnor K-groups
k field char $\neq 2$
Annals of Math 2019

have information on 2-line

Big open problems:

- over more general rings? $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]$?

  Bachmann – Østraer '21 $\mathbb{Z}[\frac{1}{2}]$

  $\pi_{0,0}(\mathbb{S}) \otimes \mathbb{Z}_{(2)} \cong GW(\mathbb{Z}[\frac{1}{2}]) \otimes \mathbb{Z}_{(2)}$

- $\underline{\pi}_{*,*} \mathbb{S}$

- Freudenthal suspension theorem? which stable $\pi_{*,*}$ correspond to unstable $\pi_{*,*}$?

Here, what we'll do with such computations is count things, but there are other uses...

$\mathbb{A}^1$-Euler class

Barge-Morel, Fasel, Morel, Asok-Fasel, M. Levine, Kass-W., Déglise-Jin-Khan, Levine-Raksit, Bachmann-W.

X smooth $/k$ dim d

$V \to X$ rank r vector bundle

Def: V is oriented by (L, φ) where $L \to X$ is a line bundle and

$$\det V \xrightarrow{\cong} L^{\otimes 2}$$

Def: X is oriented if TX is

Def: $V \to X$ is relatively oriented if $\operatorname{Hom}(\det TX, \det V)$ is.

Ex: $X = \mathbb{P}^n$ or $Gr(m,n)$ = Grassmannian parametrizing $(\mathbb{P}^m)$'s in $\mathbb{P}^n$

$$\omega_X = \det T^*X = \mathcal{O}(-n-1)$$

$\Rightarrow X$ is orientable $\Leftrightarrow$ n is odd

Ex: $\mathcal{O}(n)$ on $\mathbb{P}^1$ is relatively orientable $\Leftrightarrow$ n even

Euler "number" in $GW(k)$

Suppose X/k is a smooth, proper scheme dim d

and $V \longrightarrow X$ is a rank d vector bundle, with section $f: X \to V$
relatively oriented by (L, φ)
equipped with a section f such that

- $\{f=0\}$ consists of multiplicity 1 zeros

or equiv • $\forall x$ s.t. $f(x)=0$,
the map $Tf: T_xX \to T_{(x,0)}V \cong T_xX \oplus V_x \longrightarrow V_x$
is non-vanishing

<u>Def</u>: The Euler number of (V, φ) with respect to f is

$$n(V,\varphi,f) := \sum_{\substack{x \in X \\ \text{s.t. } f(x)=0}} \deg_x f$$

where $\deg_x f$ can be computed by

definition omitted

- choose local Nisnevich coordinates on X

and

a local trivialization of V which are compatible with the orientation φ

definition omitted

Then $f : \mathbb{A}^d \longrightarrow \mathbb{A}^d$

with $Jf = \det \frac{\partial f_i}{\partial x_j}$

$$Jf(x) \neq 0 \in k(x)$$

$$\deg_x f = \mathrm{Tr}_{k(x)/k} \langle Jf(x) \rangle$$

or equiv • $T_x f \in \mathrm{Hom}(T_x X, V_x)$

$Jf(x) = \det T_x f \in \mathrm{Hom}(\det T_x X, \det V_x)$ $\quad \det = \Lambda^r$

The orientation gives an isomorphism

$$\mathrm{Hom}(\det T_x X, \det V_x) \cong L_x^{\otimes 2}$$

$$J\ f(x) \in L_x^{\otimes 2}$$

induces a well-defined element of $R(x)/(R(x)^*)^2$ by choosing a trivialization of L_x

$\Rightarrow$ Well-defined $\langle J\ f(x)\rangle \in GW(k(x))$

Agrees with computing $\langle \det \frac{\partial f_i}{\partial x_j} \rangle$ in local coordinates as above

Q: What if the zeros of f are not multiplicity 1?

Even in differential topology? $\mathbb{R}$ manifolds

$$\deg_x f = \begin{cases} 1 & \text{if } J(x) > 0 \\ -1 & \text{if } J(x) < 0 \\ ? & \text{if } J(x) = 0 \end{cases}$$

<u>Eisenbud-Levine/Khimshiashvili Signature formula</u>:

$$\deg_x f = \text{Signature } \omega^{EKL}$$

where ω^{EKL} is the isomorphism class of the following bilinear form:

$f = (f_1, \ldots, f_n)$ $\quad Q := \dfrac{\mathbb{R}[x_1, \ldots, x_n]_0}{\langle f_1, \ldots, f_n \rangle}$ finite dimensional complete intersecti

$\Rightarrow Q$ Gorenstein, $\quad Hom_k(Q,k) \cong Q$

Better: distinguished iso, coming from distinguished socle element $\rightsquigarrow$ bilinear form

Explicitly:
$$\mathrm{Jac}\, f = \det\left(\frac{\partial f_i}{\partial x_j}\right)$$

Choose $\eta: Q \longrightarrow k$

s.t. $\eta(\mathrm{Jac}\, f) = \dim_k Q$

$\omega^{EKL}: Q \times Q \longrightarrow k \qquad \omega^{EKL}(a,b) = \eta(ab)$

isomorphism class of ω^{EKL} does not depend on choice of η.

Ex: $f: \mathbb{A}^1 \to \mathbb{A}^1 \qquad f(z) = z^2$

Eisenbud Q: ω^{EKL} defined over a field k, char $k \neq 2$ and acts as "degree." Does ω^{EKL} have top interpretation?

Thm: (Kass-W) $\quad \deg_x f = \omega^{EKL}$ in $GW(k)$ for x rational

$k(x) \supseteq k$ separable

(Brazelton - Burklund - McKean - Montoro - Opie)

Exercises: 1) $f: \mathbb{A}^2_k \to \mathbb{A}^2_k \quad f(x,y) = (4x^3, 2y)$

char $k \neq 2$

Compute $\deg_0 f$

Detour on $\mathbb{A}^1$-Milnor numbers

$\mathbb{A}^1$-Milnor #'s

char $k \neq 2$

hypersurface singularity $p \in \{f = 0\}$ bifurcates into nodes

$\cong x_1^2 + \dots + x_n^2 = 0$ over k^s

in family $f(x_1, \dots, x_n) + a_1 x_1 + \dots + a_n x_n = t$
parameterized by t, fibers are smooth
or have nodes

$k = \mathbb{C}$

For any $(a_1, \dots, a_n)$ sufficiently close to 0

$$\#\text{ nodes in family} = \deg_p^{top} \operatorname{grad} f \quad \text{(Milnor)}$$

$$\| \quad \mu_p \quad \text{"Milnor \#"}$$

Over k? char $k \neq 2$

nodes at k-rat'l points:

$$\operatorname{type}(x_1^2 + a x_2^2 = 0) := \langle a \rangle$$

$$\operatorname{type}\left(\sum a_i x_i^2 = 0\right) := \langle 2^n \prod a_i \rangle$$

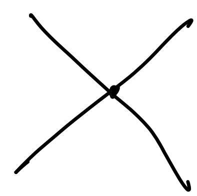
node at p
with $\kappa(p) = L$ $\quad$ $\text{type}\left(\sum a_i x_i^2 = 0 \right) := \mathrm{Tr}_{L/k} \langle 2^n \prod a_i \rangle$

Fact: $k \subseteq L$ is separable

Pictures $k = \mathbb{R}$ $\quad$ $\mathbb{R}$-rat'l nodes

$x_1^2 + x_2^2 = 0$

non-split =
non-rat'l
tgt directions



$x_1^2 - x_2^2 = 0$

split =
rat'l tgt directions

<u>Def</u>: $\mathbb{A}^1$-Milnor #

$$\mu_p := \deg_p^{\mathbb{A}^1} \operatorname{grad} f \underset{\text{Kass W.}}{=} \sum_{\substack{\text{nodes} \\ p \text{ in} \\ \text{family} \\ \text{for generic} \\ (a_1, \ldots, a_n)}} \text{type}(p)$$

Ex $\quad f(x,y) = x^3 - y^2 \qquad$ char $\neq 2, 3$

$p = (0,0) \in \{ f = 0 \}$

$\operatorname{grad} f = (3x^2, -2y)$

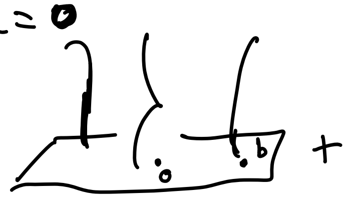
$$\deg \operatorname{grad} f = \deg(3x^2)\,\deg(-2y)$$

$$= \begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & \tfrac{1}{3} \\ x & \tfrac{1}{3} & 0 \end{array} \; \langle -2 \rangle = h \qquad \text{rank} = 2$$

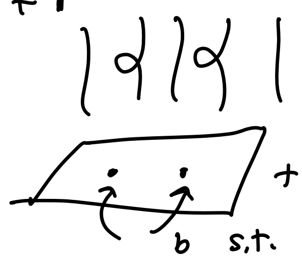
$= \langle 1 \rangle + \langle -1 \rangle$ so $\mu = 2$

family: $\quad y^2 = x^3 + ax + t$

$a = 0$



$a \neq 0$



s.t. Disc $x^3 + ax + t = 0$

$\Leftrightarrow \quad -4a^3 - 27b^2 = 0$

bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split node

over $\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \Rightarrow$ can't bifurcate into 2 split or 2 non-split nodes

Exercise: compute $\mu^{\mathbb{A}^1}$ for "ADE singularities"

singularity	equation
A_n	$x_1^2 + x_2^{n+1}$
D_n	$x_2(x_1^2 + x_2^{n-2})$
E_6	$x_1^3 + x_2^4$
E_7	$x_1(x_1^2 + x_2^3)$
E_8	$x_1^3 + x_2^5$

which ones are not multiples of $\mathbb{H}$?

Rmk: M. Levine has a beautiful count of the nodes in a pencil.

nearby cycles

Rmk: R. Azouri computes $\chi_c^{\mathbb{A}^1}(\Psi)$

$\mu_p^{\mathbb{A}^1}$ relation to motivic Milnor fiber Denef–Loeser

We'll stop to detour here;

Before we were counting, say, lines meeting 3 lines in space.

Saw count $= n(V, f)$

<u>Q</u>: Why is Euler # $n(V, f)$ independent of the section?

<u>A1</u>: Sections with isolated zeros can often be connected by families of such sections parameterized over $\mathbb{A}^1$

$$GW(k[x]) \cong GW(k)$$

pain in neck

<u>A2</u>: Euler number is pushforward of an Euler class

$$n(V, f) = \pi_* \, e(V, f)$$

<u>Euler class</u> next up

then: enumerative examples