# SHARP ESTIMATES FOR SPANNING TREES

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ABSTRACT. We prove the following sharp estimate for the number of spanning trees of a graph in terms of its vertex-degrees: a simple graph $G$ on $n$ vertices has at most $(1/n^2) \prod_{v \in V(G)} (\deg(v) + 1)$ spanning trees. This result is tight (for complete graphs), and improves earlier estimates of Alon from 1990 and Kostochka from 1995 by a factor of about $1/n$ (for dense graphs). We additionally show that an analogous bound holds for the weighted spanning tree enumerator of a (nonnegatively) weighted graph as well.

## 1. INTRODUCTION

In this paper, we shall be concerned with enumerating spanning trees; this is a fairly old problem in enumerative combinatorics with its origins in Kirchoff's matrix-tree theorem [9]. This problem is closely connected to linear algebra and has applications in geometry, number theory and algebra; see [15] for a survey of many of the classical results, and [8, 13, 2, 3, 6] for more on applications.

Here, we shall aim to answer the following problem: writing $\tau(G)$ for the number of spanning trees of a simple graph $G$, what is the best-possible upper bound for $\tau(G)$ in terms of the vertex-degrees of $G$? Our main result, stated below, answers this question with a sharp bound.

**Theorem 1.1.** *For every simple graph $G$, we have*

$$\tau(G) \le \frac{1}{|V(G)|^2} \prod_{v \in V(G)} (\deg(v) + 1). \tag{1}$$

The most attractive feature of Theorem 1.1 is that the bound (1) cannot be improved in general; indeed, Cayley's formula (see [1]) states that the number of spanning trees of the (unweighted) complete graph $K_n$ is $n^{n-2}$, and this is precisely what the right-hand side of (1) yields in this case.

Our theorem is by no means the first result in this direction; Kostochka [12], generalising an argument of Alon [2], showed that every simple graph $G$ satisfies

$$\tau(G) \le \frac{1}{|V(G)| - 1} \prod_{v \in V(G)} \deg(v). \tag{2}$$

By way of comparison, our bound (1) improves on (2) by a factor of about $1/n$ for dense $n$-vertex graphs. On the other hand, for very sparse graphs, our estimate (1) is weaker than (2), and in this regime, very strong bounds (sharper than both of these estimates) are known for bounded-degree graphs; see [14, 2, 12], for example.

We shall give two proofs of Theorem 1.1, a combinatorial proof by induction on a more involved proposition, as well as a linear-algebraic proof based on Hadamard's inequality.

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Our linear-algebraic approach allows us to generalise Theorem 1.1 to weighted graphs. To state this generalisation, we need some definitions: given a nonnegative weight function $\alpha : E(G) \to \mathbb{R}_{\geq 0}$ on a simple graph $G$, we write $\tau_\alpha(G)$ for the *weighted spanning tree enumerator* of $G$ and $\deg_\alpha(v)$ for the *weighted degree* of a vertex $v \in V(G)$. In other words, $\tau_\alpha(G) = \sum_T \prod_{e \in E(T)} \alpha(e)$, where the sum is taken over all spanning trees $T$ of $G$, and $\deg_\alpha(v)$ is the sum of the weights of the edges incident to the vertex $v$. In this language, we have the following analogue of the bound (1) for nonnegatively weighted graphs.

**Theorem 1.2.** *For every simple graph $G$ equipped with a nonnegative weight function $\alpha : E(G) \to \mathbb{R}_{\geq 0}$, we have*

$$\tau_\alpha(G) \leq \frac{1}{|V(G)|^2} \prod_{v \in V(G)} (\deg_\alpha(v) + 1). \tag{3}$$

Of course, when the weight function is identically one, we recover the usual spanning tree count and vertex-degrees of a graph, so Theorem 1.2 readily implies Theorem 1.1.

As mentioned earlier, our proof of Theorem 1.2 proceeds via Hadamard's inequality. Here, it is also worth mentioning that an alternate linear-algebraic proof of Theorem 1.1, that gives a different (and incomparable) estimate for multigraphs as compared to the one in Theorem 1.2, using Schur's inequality and some majorization theory was independently found by Chelpanov [5].

Our work here was motivated in part by a beautiful conjecture of Ehrenborg (see [6, 10]) asserting that for any simple bipartite graph $G$ with partition classes $A$ and $B$, we have

$$\tau(G) \leq \frac{1}{|A||B|} \prod_{v \in V(G)} \deg(v). \tag{4}$$

We are unfortunately unable to sharpen either Theorem 1.1 or 1.2 in any meaningful way for bipartite graphs. However, we remain optimistic — indeed, this is our rationale for supplying two proofs of Theorem 1.1 — that the techniques we introduce may be of some use in resolving Ehrenborg's conjecture.

This short paper is organised as follows. The proofs of Theorems 1.1 and 1.2 are given in Section 2. We then conclude with a discussion of some open problems in Section 3.

## 2. Proofs of the main result

In what follows, all graphs under consideration will be loopless. We shall deal with various types of graphs: a simple graph is one with at most one edge between any pair of vertices, a weighted graph is a simple graph along with a nonnegative real-valued weight function on its edges, and a multigraph is a weighted graph with nonnegative integer weights, which we alternately think of as an unweighted graph that allows multiple edges between pairs of vertices.

For a graph $G$, we write $V(G)$ and $E(G)$ for its vertex set and edge set respectively. For a subset $U \subset V(G)$, we denote the subgraph induced by $G$ on $U$ by $G[U]$. For a vertex $v \in V(G)$, we write $\deg(v)$ for the degree of $v$ in $G$ when $G$ is a simple graph or multigraph, and if $G$ is a weighted graph equipped with a weight function $\alpha$, we write $\deg_\alpha(v)$ for the weighted degree $v$ in $V(G)$, i.e., the sum of the weights of the edges incident to $v$. Finally, we write $\tau(G)$ for the number of spanning trees of $G$ when $G$ is a simple graph or multigraph, counting with multiplicity in the latter case, and if $G$ is a weighted graph equipped with a weight function $\alpha$, we write $\tau_\alpha(G)$ for the weighted spanning tree enumerator of $G$, i.e., the sum $\sum_T \prod_{e \in E(T)} \alpha(e)$ over all spanning trees $T$ of $G$.

For a weighted graph $G$ with a weight function $\alpha$ on its edges, we write $\mathcal{L}_\alpha(G)$ for the weighted Laplacian of $G$, which is the matrix whose rows and columns are indexed by the vertices of $G$ whose entries are given by

$$(\mathcal{L}_\alpha(G))_{u,v} = \begin{cases} \deg_\alpha(u) & \text{if } u = v, \text{ and} \\ -\alpha(\{u,v\}) & \text{if } u \neq v \end{cases}$$

for all $u, v \in V(G)$. We identify missing edges in a weighted graph with edges whose weight is 0.

Kirchoff's theorem [9] famously asserts that every cofactor of $\mathcal{L}_\alpha(G)$ equals $\tau_\alpha(G)$. We shall make use of the following result of Klee and Stamps [10, 11], generalising an older result of Temperley [16], for enumerating spanning trees via rank-one perturbations.

**Proposition 2.1.** *Let $G$ be a weighted graph on $n$ vertices equipped with a weight function $\alpha : E(G) \to \mathbb{R}_{\geq 0}$. For any vectors $\mathbf{u} = (u_i)_{i=1}^n$ and $\mathbf{v} = (v_i)_{i=1}^n$, we have*

$$\left(\sum_i u_i\right)\left(\sum_i v_i\right)\tau_\alpha(G) = \det(\mathcal{L}_\alpha(G) + \mathbf{u}\mathbf{v}^T).$$

*In particular, by taking $\mathbf{u} = \mathbf{v} = \mathbf{1}_n$ to be the all-ones vectors, we have*

$$\tau_\alpha(G) = \frac{\det(\mathcal{L}_\alpha(G) + \mathbf{1}_{n \times n})}{n^2},$$

*where $\mathbf{1}_{n \times n}$ denotes the $n \times n$ matrix of ones.* □

We shall also need Hadamard's inequality in the following formulation; see [7] for a proof.

**Proposition 2.2.** *The determinant of a positive semidefinite matrix is no larger than the product of its diagonal entries.* □

We are now ready to give the first proof of Theorem 1.1 via the more general Theorem 1.2.

*Proof of Theorem* 1.2. Let $n = |V(G)|$ be the number of vertices of the graph $G$. We start by verifying that the matrix $\mathcal{L}_\alpha(G) + \mathbf{1}_{n \times n}$ is positive semidefinite; indeed, this follows from observing that

$$x^T(\mathcal{L}_\alpha(G) + \mathbf{1}_{n \times n})x = x^T \mathcal{L}_\alpha(G)x + x^T \mathbf{1}_{n \times n}x = \sum_{u,v \in V(G)} \left(\frac{\alpha(\{u,v\})}{2}\right)(x_u - x_v)^2 + \left(\sum_{v \in V(G)} x_v\right)^2 \geq 0$$

for all $x \in \mathbb{R}^{V(G)}$.

We are now done since

$$\tau_\alpha(G) = \frac{\det(\mathcal{L}_\alpha(G) + \mathbf{1}_{n \times n})}{|V(G)|^2} \leq \frac{1}{|V(G)|^2} \prod_{v \in V(G)} (\deg_\alpha(v) + 1),$$

the first equality following from Proposition 2.1, and the second inequality from Proposition 2.2; this establishes the required bound. □

Our second proof of Theorem 1.1 is combinatorial and proceeds by induction. We may deduce Theorem 1.1 from the following stronger — albeit somewhat technical — result about a special class of multigraphs.

**Theorem 2.3.** *Let $G$ be a multigraph on $n$ vertices and suppose that there is a vertex $u \in V(G)$ such that $G[V(G) \setminus \{u\}]$ is simple; furthermore, let $V(G) \setminus \{u\} = \{v_1, v_2, \ldots, v_{n-1}\}$, and for $1 \leq i \leq n-1$, let $d_i$ denote the degree of $v_i$ in $G[V(G) \setminus \{u\}]$ and let $m_i$ denote the number of edges between $u$ and $v_i$. Then, we have*

$$\tau(G) \leq \left(1 - \sum_{i=1}^{n-1} \frac{1}{n-1+m_i}\right) \prod_{i=1}^{n-1} (d_i + m_i + 1). \tag{5}$$

Theorem 2.3 yields the following result — itself slightly stronger than Theorem 1.1 — when specialised to simple graphs.

**Corollary 2.4.** *For every simple graph $G$ on $n$ vertices and any vertex $u \in V(G)$, we have*

$$\tau(G) \leq \frac{\deg(u)}{n(n-1)} \prod_{v \in V(G) \setminus \{u\}} (\deg(v) + 1) \leq \frac{1}{n^2} \prod_{v \in V(G)} (\deg(v) + 1). \tag{6}$$

*Proof.* The second inequality in the result follows from the fact that $\deg(u)/(n-1) \leq (\deg(u)+1)/n$ since $\deg(u) \leq n-1$ (as $G$ is simple); it therefore suffices to prove the first inequality.

As in the statement of Theorem 2.3, let $V(G) \setminus \{u\} = \{v_1, v_2, \ldots, v_{n-1}\}$, let $d_i$ denote the degree of $v_i$ in $G[V(G) \setminus \{u\}]$, and let $m_i \in \{0, 1\}$ denote the number of edges between $u$ and $v_i$. Then $\deg(u) = m_1 + m_2 + \cdots + m_{n-1}$, and for each $v_i$, we have $\deg(v_i) = d_i + m_i$, so in this language, the first inequality in (6) may be rewritten as

$$\tau(G) \leq \frac{m_1 + m_2 + \cdots + m_{n-1}}{n(n-1)} \prod_{i=1}^{n-1} (d_i + m_i + 1). \tag{7}$$

Since $m_i \in \{0, 1\}$ (as $G$ is simple), it is easy to see that

$$\frac{1}{n-1+m_i} = \frac{1}{n-1} - \frac{m_i}{n(n-1)},$$

whence we have

$$1 - \sum_{i=1}^{n-1} \frac{1}{n-1+m_i} = 1 - \sum_{i=1}^{n-1} \left(\frac{1}{n-1} - \frac{m_i}{n(n-1)}\right) = \frac{m_1 + m_2 + \cdots + m_{n-1}}{n(n-1)}.$$

With this identity in hand, it is now clear that Theorem 2.3 implies the bound (7), and the corollary follows. $\square$

We now proceed to give the proof of Theorem 2.3.

*Proof of Theorem* 2.3. We proceed by induction on the number of edges of $G$. Let us treat our base case, namely where $m_1 = m_2 = \cdots = m_{n-1} = 0$. If $n = 1$, then both sides of the inequality are equal to one. If $n \geq 2$, having $m_1 = m_2 = \cdots = m_{n-1} = 0$ means that $u$ is an isolated vertex, so $\tau(G) = 0$; the right-hand side of the inequality is also seen to be zero in this case when $n \geq 2$.

We may now assume that $m_i \geq 1$ for some $i \in \{1, \ldots, n-1\}$. Relabelling the vertices if necessary, we suppose that $m_{n-1} \geq 1$ is the maximum of $m_1, m_2, \ldots, m_{n-1}$, and we then colour one of the $m_{n-1}$ edges between $u$ and $v_{n-1}$ red.

Now, let $G'$ be the multigraph obtained from $G$ by deleting the red edge between $u$ and $v_{n-1}$, and let $G^*$ be the multigraph obtained from $G$ by contracting $u$ and $v_{n-1}$ into one vertex (which we again call $u$ in $G^*$). For each $1 \leq j \leq n-2$, the number of edges in $G^*$ between $u$ and $v_j$ is the sum of the number of edges between

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$u$ and $v_j$ in $G$ and the number of edges between $v_{n-1}$ and $v_j$ in $G$; consequently, for each $1 \leq j \leq n-2$, the degree of $v_j$ in $G^*$ is the same as its degree in $G$ (namely, $d_j + m_j$).

Clearly, $\tau(G) = \tau(G') + \tau(G^*)$; indeed, $\tau(G')$ is the number of spanning trees of $G$ that do not use the red edge, whereas $\tau(G^*)$ is the number of spanning trees of $G$ that use the red edge. Note that both $G'$ and $G^*$ have fewer edges than $G$.

By applying the induction hypothesis to $G'$, we get

$$\tau(G') \leq \left(1 - \sum_{j=1}^{n-2} \frac{1}{n-1+m_j} - \frac{1}{n-2+m_{n-1}}\right)\left((d_{n-1}+m_{n-1}) \cdot \prod_{j=1}^{n-2}(d_j+m_j+1)\right). \tag{8}$$

For $1 \leq j \leq n-2$, let $\delta_j \in \{0,1\}$ denote the number of edges in $G$ between $v_j$ and $v_{n-1}$, so that the number of edges in $G^*$ between $u$ and $v_j$ equals $m_j + \delta_j$. By applying the induction hypothesis to $G^*$, we get

$$\tau(G^*) \leq \left(1 - \sum_{j=1}^{n-2} \frac{1}{n-2+m_j+\delta_j}\right)\prod_{j=1}^{n-2}(d_j+m_j+1). \tag{9}$$

Given that $\tau(G) = \tau(G') + \tau(G^*)$, it suffices to prove that the sum of the right-hand sides of (8) and (9) is at most the right-hand side of (5). After dividing by $(d_1+m_1+1)(d_2+m_2+1)\cdots(d_{n-2}+m_{n-2}+1)$, this is equivalent to showing that

$$\left(1 - \sum_{j=1}^{n-2} \frac{1}{n-1+m_j} - \frac{1}{n-2+m_{n-1}}\right)(d_{n-1}+m_{n-1}) + \left(1 - \sum_{j=1}^{n-2} \frac{1}{n-2+m_j+\delta_j}\right)$$

$$\leq \left(1 - \sum_{j=1}^{n-2} \frac{1}{n-1+m_j} - \frac{1}{n-1+m_{n-1}}\right)(d_{n-1}+m_{n-1}+1),$$

which may be rewritten after further rearrangement as

$$\sum_{j=1}^{n-2}\left(\frac{1}{n-1+m_j} - \frac{1}{n-2+m_j+\delta_j}\right) \leq \frac{d_{n-1}+m_{n-1}}{(n-2+m_{n-1})(n-1+m_{n-1})} - \frac{1}{n-1+m_{n-1}}. \tag{10}$$

Recall that $m_{n-1}$ is the maximum of $m_1, m_2, \ldots, m_{n-1}$, so we have

$$\frac{1}{(n-2+m_j)(n-1+m_j)} \geq \frac{1}{(n-2+m_{n-1})(n-1+m_{n-1})} \tag{11}$$

for each $1 \leq j \leq n-2$. Note that there are exactly $d_{n-1}$ different $j \in \{1,2,\ldots,n-2\}$ with $\delta_j = 1$, and $n-2-d_{n-1}$ different $j \in \{1,2,\ldots,n-2\}$ with $\delta_j = 0$. Summing (11) over all $j \in \{1,2,\ldots,n-2\}$ for which $\delta_j = 0$, we obtain

$$\sum_{\{j\,:\,\delta_j=0\}} \frac{1}{(n-2+m_j)(n-1+m_j)} \geq \frac{n-2-d_{n-1}}{(n-2+m_{n-1})(n-1+m_{n-1})}. \tag{12}$$

Recalling that $\delta_j \in \{0,1\}$ for each $j \in \{1,2,\ldots,n-2\}$, we now conclude that

$$\sum_{j=1}^{n-2} \frac{1}{n-2+m_j+\delta_j} - \frac{1}{n-1+m_j} = \sum_{\{j\,:\,\delta_j=0\}} \frac{1}{n-2+m_j+\delta_j} - \frac{1}{n-1+m_j}$$

$$= \sum_{\{j\,:\,\delta_j=0\}} \frac{1}{(n-2+m_j)(n-1+m_j)}$$

$$\geq \frac{n-2-d_{n-1}}{(n-2+m_{n-1})(n-1+m_{n-1})}$$

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$$= \frac{1}{n-1+m_{n-1}} - \frac{d_{n-1}+m_{n-1}}{(n-2+m_{n-1})(n-1+m_{n-1})}, \qquad (13)$$

which is precisely the inequality (10) we needed to establish; this completes the proof. $\qquad \square$

## 3. Concluding remarks

We remain cautiously optimistic that some of the ideas introduced here may be relevant in resolving Ehrenborg's conjecture (see (4)) which, to us, appears to be the main outstanding problem in this area.

For example, it is plausible that Ehrenborg's conjecture admits a proof by induction via some analogue of Theorem 2.3 for bipartite graphs. The main difficulty here lies in the identification of such a bipartite analogue, and all our attempts to find such a statement have so far been unsuccessful. More generally, any proof of Ehrenborg's conjecture must grapple with the fact that the most natural generalisations of this conjecture to weighted bipartite graphs are false.

We close by mentioning that it was shown by Bozkurt [4] via a linear-algebraic argument that every simple bipartite graph $G$ satisfies

$$\tau(G) \leq \frac{1}{|E(G)|} \prod_{v \in V(G)} \deg(v),$$

and this remains the state of the art.

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