Machines take me by surprise with great frequency.– Alan Turing

Problem Set 6

PCMI USS, Summer 2023

1. **(Unambiguous State Discrimination)** Last week, your were gifted a qubit $|\psi\rangle$ and told that there is a 50% chance that $|\psi\rangle = |0\rangle$ and a 50% chance that $|\psi\rangle = |+\rangle$. At that time, you investigated the probability with which these states can be distinguished. In the intervening days, however, you've had an epiphany, and you've decided never to be wrong again. As a result of your new way of living, you wish to come up with a protocol that satisfies these properties

   - If the state $|\psi\rangle = |0\rangle$, then with some positive probability, you answer "Zero state." Otherwise, you answer "Don't know."
   - If the state $|\psi\rangle = |+\rangle$, then with some positive probability, you answer "Plus state." Otherwise, you answer "Don't know."

   (a) Show that if your strategy is to measure $|\psi\rangle$ and an orthogonal basis of $\mathbf{C}^2$, you cannot achieve both objectives.

   (b) Consider instead this strategy: You flip a coin. If you see heads, you measure in the computational basis. If you see tails, you measure in the Hadamard basis. Show that you may now achieve both of your objectives. With what probability does your strategy correctly identify the state $|\psi\rangle$?

   (c) Now investigate whether there is a better strategy that deploys a more general POVM. If you want to think about this on your own first, do so now. Otherwise, read on for some suggestions.

   (d) Let $E_0 = |1\rangle\langle 1|$ and $E_1 = |-\rangle\langle -|$ be the projections onto the 1 state and the minus state. Let $B = E_0 + E_1$. Write the matrix form (yellow) for B. Compute the eigenvalues of B.

   (e) Let c denoter a real number. Consider the operator $E_2 = I - cE_0 - cE_1$. Since you found the eigenvalues of $E_0 + E_1$, it should be easy to find the eigenvalues of E_2. What is the largest value of c for which E_2 is a positive operator (or equivalently, E_2 has nonnegative eigenvalues.)

   (f) Let c be the value you found in (e). Explain why $\mathcal{E} = \{cE_0, cE_1, E_2\}$ fits the definition of a POVM.

   (g) Suppose this POVM is used to measure $|0\rangle$. What are the possible results of this measurement and their probabilities?

   (h) Repeat for $|+\rangle$.

   (i) Use the measurment $\mathcal{E}$ to come up with a strategy identifying our unknown state $|\psi\rangle$ that satisfies both objectives. With what probability does your strategy succeed? How does this compare with your answer to (b)?

(j) Generalize in any way you want or can.

2. (**Quantum Swap**)

   (a) Your friend Jeff is very excited about a cool 2-qubit circuit U that has the following effect on the computational basis:

   $$|00\rangle \longmapsto |00\rangle, \quad |01\rangle \longmapsto |10\rangle, \quad |10\rangle \longmapsto |01\rangle, \quad |11\rangle \longmapsto |11\rangle.$$

   Write the matrix form (yellow) of U. Is U unitary?

   (b) Jeff claims that for any 2-qubit product state, U swaps the two qubits. That is, for any qubits $|\psi\rangle$ and $|\phi\rangle$, we have

   $$U|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle.$$

   Is Jeff correct?

   (c) Create a quantum circuit to perform this operation using only familiar single qubit operations and CNOT (the controlled-NOT). (You may be able to get away with only CNOT's).

3. In class, we considered the question of distinguishing the three quantum states $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$, and $|\psi_3\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$. We gave a reasonable strategy involving a measurement in the computational basis that succeeded with probability 7/12. Then we gave a better strategy involving a non-orthogonal POVM, achieving a success probability of 2/3. We then proved that this strategy is optimal. We left open the question of whether a POVM was necessary or whether you can get away with a measurement in an orthogonal basis. For this question, I would like you to investigate how well you can do starting by measuring in an orthogonal basis other than the computational basis.

   (a) Is there a strategy which starts with an orthogonal measurement and has success probability more than 7/12? (You might try the $|+\rangle, |-\rangle$ basis, for example, or a different basis, for starters.). Can you achieve 2/3 in this way?

   (b) Now suppose that the states live in three dimensions; that is suppose that the $|\psi_i\rangle$ are vectors in $\mathbf{C}^3$ with basis $|0\rangle, |1\rangle, |2\rangle$. Using an orthogonal measurement on $\mathbf{C}^3$, with what probability can the states be distinguished?

   (c) How about if the $|\psi_i\rangle$ live in $\mathbf{C}^4$, i.e, are two qubit states with the second qubit equal to $|0\rangle$ in each case? (That is, $|\psi_1\rangle = |00\rangle$, $|\psi_2\rangle = -\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|10\rangle$, and $\psi_3\rangle = -\frac{1}{2}|00\rangle - \frac{\sqrt{3}}{2}|10\rangle$.)

4. (a) Two random unit vectors in $\mathbb{R}^n$ are chosen. How big do you expect their inner product to be? What does this mean geometrically? [Implicit here is the question of what "random unit vectors" means and how you might generate them.]

   (b) How does your answer to compare to the inner products we found for the important vectors involved in Grover's Algorithm?