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Rigid cocycles and singular moduli for real quadratic fields

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These are some exercises to accompany the course on rigid cocycles and singular moduli of real quadratic fields, held at the PCMI Summer School 2022. The teaching assistant is James Rickards (`james.rickards@colorado.edu`).

(1) (Only for those who enjoy it) Celebrate this year's PCMI summer school theme "number theory informed by computation" by recreating some observations described in Zagier's 1983 letter to Gross:

- Compute the factorisation of the integer
$$j\left(\frac{1+\sqrt{-43}}{2}\right)-j\left(\frac{1+\sqrt{-163}}{2}\right).$$
- Compute the factorisations of the positive integers of the form
$$\frac{43\cdot 163-x^2}{4} \qquad \text{for } x\in\mathbf{Z}.$$
- Compute the Legendre symbols
$$\left(\frac{-43}{q}\right)=\left(\frac{-163}{q}\right)$$
of the primes q that occur, and formulate your own conjecture. Finally, compare your conclusions to those of Gross–Zagier [10].

(2) Prove that the set of reduced forms of a fixed discriminant $D\neq 0$ is finite. Let $F=\langle a,b,c\rangle$ be of non-square discriminant D. Show that

- If F is *definite*, then F is reduced if and only if
$$r(F)\in\mathcal{D},$$
where $\mathcal{D}$ is the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ in the Poincaré upper half plane $\mathfrak{H}_\infty := \{z\in\mathbf{C} : \mathrm{Im}(z)>0\}$.

- If F is *indefinite*, then F is reduced if and only if $\left|\sqrt{D}-2|a|\right|<b<\sqrt{D}$. Show that this is furthermore equivalent to the following condition on the roots:
$$r'(F)r(F)<0 \quad\text{and}\quad |r(F)|<1<|r'(F)|.$$

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(3) Compute the set Σ_F of nearly reduced forms in the $SL_2(\mathbf{Z})$-orbit of the quadratic form F, which we defined by

$$\Sigma_F := \left\{ \langle a, b, c \rangle \sim F \ : \ ac < 0 \right\},$$

for the forms $F = \langle -1, 4, 4 \rangle, \langle 5, 11, -5 \rangle$, and $\langle -4825, -15989, -13246 \rangle$. Characterise all forms F of non-square discriminant $D > 0$ for which the sets Σ_F are symmetric under the involutions

$$\begin{array}{rcccc} s_1 & : & \langle a, b, c \rangle & \longmapsto & \langle -a, -b, -c \rangle \\ s_2 & : & \langle a, b, c \rangle & \longmapsto & \langle a, -b, c \rangle \end{array}$$

in terms of the associated class in the (narrow) Picard group of $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$.

(4) Consider a multiplicative cocycle

$$\Theta \in Z^1\left(SL_2(\mathbf{Z}), \mathbf{C}(z)^\times\right).$$

Its *value* at a form $G \in \mathcal{F}_D$ with $D > 0$ non-square is defined by

$$\Theta[G] := \Theta(\gamma_G)(r(G))$$

where γ_G is the automorph of G. Show that for a fixed Θ, the value $\Theta[G]$ only depends on the $SL_2(\mathbf{Z})$-orbit of G.

(5) (Warm-up cocycle I) For any $c = (r, s) \in \mathbf{P}^1(\mathbf{Q})$ define the function

$$L(c) := \frac{s}{sz - r}$$

and consider the map

$$\begin{array}{rccc} p_c: & GL_2(\mathbf{Q}) & \longrightarrow & \mathbf{C}(z), \\ & \gamma & \longmapsto & L(c) - L(\gamma c). \end{array}$$

Show that
- this is a 1-cocycle,
- its cohomology class

  $$[p_c] \in H^1(GL_2(\mathbf{Q}), \mathbf{C}(z))$$

  is independent of the choice of cusp c,
- if $F \in \mathcal{F}_D$ for $D > 0$ non-square, the multiplicative lift $p^\times$ of its restriction to the subgroup $SL_2(\mathbf{Z})$ has value at F equal to

  $$p^\times[F] = \varepsilon_D^{12}$$

  where $\varepsilon_D > 1$ is the fundamental unit of norm 1 in the quadratic order $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of discriminant D.

(6) (Warm-up cocycle II) For any $c = (r, s) \in \mathbf{P}^1(\mathbf{Q})$ define the function

$$N(c) := \frac{1}{(sz - r)^2},$$

where choose $\gcd(r,s)=1$, and consider the map

$$\begin{array}{rccc} q_c: & \mathrm{SL}_2(\mathbf{Q}) & \longrightarrow & \mathbf{C}(z), \\ & \gamma & \longmapsto & N(c)-N(\gamma c). \end{array}$$

Show that
- this is a 1-cocycle,
- its cohomology class
$$[q_c] \in H^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z))$$
is independent of the choice of cusp c.

Remark. Note that this cocycle is not in the image of dlog.

(7) (Knopp cocycle) Choose a cusp $c \in \mathbf{P}^1(\mathbf{Q})$. Define the map
$$\mathrm{kn}_{c,F} : \mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathbf{C}(z)$$
by setting
$$\mathrm{kn}_{c,F}(\gamma) = \sum_{Q\sim F} \frac{\mathrm{sgn}_{c,Q}(\gamma)}{z-r(Q)}.$$
where the numerator is defined by
$$\mathrm{sgn}_{c,Q}(\gamma) := \begin{cases} 1 & \text{if } Q(c) > 0 > Q(\gamma c), \\ -1 & \text{if } Q(c) < 0 < Q(\gamma c), \\ 0 & \text{else.} \end{cases}$$

- Show that $\mathrm{kn}_{c,F}$ is a 1-cocycle.
- Show that its cohomology class
$$[\mathrm{kn}_{c,F}] \in H^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z))$$
is independent of the choice of cusp c.
- Show that the multiplicative lift
$$\mathrm{kn}_F^{\times} \in Z^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times})$$
satisfies
$$\mathrm{kn}_F^{\times}(T) = \varepsilon_D^{12},$$
where $\varepsilon_D > 1$ is the fundamental unit of norm 1 in the quadratic order $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of discriminant D.

(8) Define the map $t : \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}_{\geqslant 0}$ by
$$t(x) := \begin{cases} x-1 & \text{if } x \geqslant 1 \\ x/(x-1) & \text{if } 0 < x < 1 \end{cases}$$
Show that the periodic orbits for iteration of t are the sets $\{0\}$ and
$$\mathcal{S}_F := \{r(Q) \ : \ Q \in \Sigma_F\},$$

for $F \in \mathcal{F}_D$, with $D > 0$ non-square.

(9) Use the previous exercise to show that the cocycles p, q, kn_F introduced above generate the group of parabolic (additive) rational cocycles

$$Z^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\right).$$

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