

# Isadore M. Singer

## (1924–2021)

### *Edward Witten*

I first met Is Singer in 1979, while visiting the University of California at Berkeley, where he was a professor at the time. Singer was then running a very active seminar on quantum field theory and differential geometry. He had been one of the first mathematicians or physicists to appreciate the new opportunities for interaction of math and physics that were opened up by the emergence of the modern Standard Model of particle physics, which is based on nonabelian gauge theory.

Through his vision and his contributions, Is Singer played a major role in the emerging interaction of mathematics and quantum field theory. One of the early landmarks was the role in physics of the Atiyah–Singer index theorem. A puzzle in strong interaction physics was a missing symmetry—a symmetry of the classical action of the Standard Model that was not visible in experiment. It turned out that the solution to this puzzle revolves around “instantons” of four-dimensional gauge theory, and the Atiyah–Singer index theorem. This had become clear in 1976, through the work of Gerard ‘t Hooft as interpreted by Albert Schwarz. As a result, instantons and the index theorem were prominent at the interface of physics and differential geometry in the late 1970s, when I first met Singer. By this time, Singer, with Michael Atiyah and Nigel Hitchin, had developed the mathematical foundations of the study of instantons on a general four-manifold. All this was one of the topics in his seminar.

Within a few years, the many sides of the index theorem became familiar to physicists. For example, in 1982 it turned out that the mod 2 index theorem of Atiyah and Singer governs an obstruction to a certain generalization of the Standard Model. In 1984, Atiyah and Singer—in what proved to be their last joint paper—interpreted the “anomalies” of Steve Adler, John Bell, and Roman Jackiw in terms of the families index theorem (the index theorem for a family of elliptic differential operators). These anomalies are an important constraint on the consistency of the Standard Model and its possible extensions. With Henry McKean, Singer had initiated the heat kernel approach to the index theorem. This approach became familiar to physicists as it is natural in quantum field theory and is closely related to the supersymmetric proof of the index theorem. Over time, the influence of the index theorem in physics spread beyond elementary particle physics and relativistic quantum field theory, where it started. By now, the index theorem is also a familiar and important tool in condensed matter physics as well.

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Another aspect of Singer's work that has been important in physics involves the  $\eta$ -invariant that was introduced by Atiyah and Singer with Vijay Patodi. The Gauss–Bonnet formula for the Euler characteristic of a manifold with boundary has a boundary contribution that involves the integral of a local invariant involving the extrinsic curvature of the boundary. Atiyah, Patodi, and Singer showed that for a Dirac-like operator on a boundary, in general there is no local elliptic boundary and no local invariant analogous to the extrinsic curvature. Instead they introduced a “global” boundary condition and proved an index theorem in which a spectral invariant that they called  $\eta$  appears as a boundary contribution. All this is relevant in quantum field theory at multiple levels. The most basic is that the global APS boundary condition encodes the ground state of a quantum field theory; in fermion quantum field theory, the APS index theorem has a direct physical interpretation. The APS  $\eta$ -invariant has several other manifestations in quantum field theory that are loosely related to this. It controls “charge fractionalization,” an important phenomenon, first explored in the early 1970s by Jackiw and Claudio Rebbi (without knowing the relation to index theory and the  $\eta$ -invariant), in which the ground state of a quantum field carries fractional quantum numbers. The  $\eta$ -invariant is also important in a refined understanding of the Adler–Bell–Jackiw anomaly and its generalizations. All these facets of the  $\eta$ -invariant have manifestations in condensed matter physics as well as relativistic field theory.

Finally, I should mention the role in physics of Ray–Singer analytic torsion. In 1971, Daniel Ray and Singer discovered an invariant of a flat bundle on a compact manifold  $M$  that is expressed in terms of a regularized determinant of the Laplace operator acting on differential forms on  $M$ . Ray and Singer conjectured that their “analytic torsion” is equivalent to the “combinatorial torsion” of Reidemeister, and this was later proved by Jeff Cheeger and Werner Müller. Two years later, Ray and Singer formulated a version of analytic torsion for holomorphic vector bundles over a complex manifold. Here, there is no combinatorial counterpart of the analytic torsion. Analytic torsion for complex manifolds is important in string theory in multiple ways. The complex manifold is usually either the worldsheet of a string, or a factor in spacetime. Analytic torsion for real manifolds appears in physics primarily in topological field theory and in simple models of quantum gravity. When one expands the three-dimensional Chern–Simons topological field theory in perturbation theory, the leading approximation can be expressed in terms of the analytic torsion and the  $\eta$ -invariant. In higher orders, one runs into more subtle invariants, which were studied by

Singer and Scott Axelrod in two relatively well-known papers in the early 1990s. The most interesting application of analytic torsion to gravity involves the relation of a simple model of gravity in two dimensions that is known as Jackiw–Teitelboim or JT gravity to the volumes of moduli spaces of Riemann surfaces; the celebrated results of Maryam Mirzakhani on those volumes are also part of this story. The relation of JT gravity to the volumes is a special case of a more general statement about analytic torsion.

Apart from specific applications of analytic torsion in physics, there was a technical step in the work of Ray and Singer that has also been important. To define a regularized determinant of a nonnegative self-adjoint differential operator, they first defined the  $\zeta$  function  $\zeta(s) = \sum_i \lambda_i^{-s}$ , where  $\lambda_i$  are the eigenvalues of the operator, and then, assuming  $\zeta(s)$  has an analytic continuation that is holomorphic at  $s = 0$ , they define the determinant as  $\exp(-\zeta'(0))$ . The analytic continuation to  $s = 0$  can be made in fairly wide circumstances, using the ideas of McKean and Singer concerning the heat kernel. Determinants of differential operators were known to be important in physics, but the methods that physicists were using to define these determinants were less incisive than the  $\zeta$ -function regularization of Ray and Singer.  $\zeta$ -function regularization was taken into physics in the 1970s by Stuart Dowker and Raymond Critchley, followed by Stephen Hawking. I know from multiple discussions that Singer was particularly proud of this application of his work to physics.

Is developed many active collaborations with physicists—his collaborators included Orlando Alvarez, Laurent Baulieu, and Axelrod, among others. Is' interest in high energy physics was not limited to theory. He became very interested in what was going on in experiments where fundamental ideas of physics are tested. My wife and I were on sabbatical in the spring of 2009 at the European particle physics laboratory CERN when Is visited. We toured the ATLAS detector at CERN, one of the two detectors that discovered the Higgs particle three years later.

One thing that Is and I had in common, apart from an interest in math and physics, is that we both took up tennis relatively late in life (in our forties) and became passionate about the game. It ended up that we never got to play; the one time that this almost happened, Is ended up bowing out and leaving me on the court with Cumrun Vafa and his sons. But I am pretty sure that Is' tennis was on a different level from mine. He had been an athlete in his youth—a minor league baseball player for a time, in fact. I can remember many discussions about tennis. Perhaps 15 years ago, he told me that he had finally developed a strong serve. On another occasion, he expressed the ambition of eventually becoming the United States Tennis Association

champion in the oldest age group. Yet another time, he explained that he had assured the dean at MIT that he would retire once the dean could beat him in tennis.

It was a pleasure to know Is and to learn from him over the years.