PCMI 2023 | *Instructor: Sandy Irani*
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Quantum Hamiltonian Complexity: Day 3

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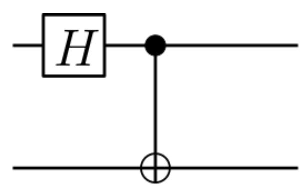
topics: connection between QMA-completeness and adiabatic computation, more QMA-hardness results and techniques, the thermodynamic limit

1. Consider the following circuit:



   (a) Express H_{prop} for this circuit using a 2-qubit unary clock.

   (b) What would be the H_{init} term be for enforcing that the input to the circuit is $x = 10$?

2. The term to "advance" a unary clock from time $t-1$ to t is

$$H_t = \frac{1}{2}(|100\rangle\langle 100| + |110\rangle\langle 110| - |100\rangle\langle 110| - |110\rangle\langle 100|)$$

   applied to qubits $t-1$, t, and $t+1$. What are the terms H_1 and H_t?

3. Suppose we have a 3-qubit unitary clock.

   (a) Give the configuration graph for this clock.

   (b) Give an expression for H_{valid} which gives an energy penalty to any clock state that is invalid.

   (c) Give the matrix for $H_{prop} + H_{valid}$ over the standard basis of clock states.

   (d) Prove that H_{prop} is invariant on the span of any set of clock states that form a connected component in the configuration graph for the clock.

4. In this question, we will analyze the Hamiltonian corresponding to the circuit-to-Hamiltonian construction for a circuit where the input is $|0\rangle^{\otimes n}$. This corresponds to the construction discussed in lecture showing that the model of adiabatic computation is polynomially equivalent to the circuit model. There is a register of n qubits for the computation and a register with T qubits for the clock. $\mathcal{H}_{comp}$ is the Hilbert space for the computation register. The clock construction will be the standard unary clock discussed in lecture.

(a) Let V be the set of T-bit strings corresponding to valid clock states. Define:

$$\mathcal{H}_{valid} = \text{Span}\{|x\rangle : x \in V\} \otimes \mathcal{H}_{comp}$$

$$\mathcal{H}_{invalid} = \text{Span}\{|x\rangle : x \in \{0,1\}^n - V\} \otimes \mathcal{H}_{comp}$$

Prove that H_{prop} is invariant on $\mathcal{H}_{valid}$ and $\mathcal{H}_{invalid}$.

(b) Let H_{valid} be the set of terms which give an energy penalty to any clock state that is invalid. Prove that the smallest eigenvalue of $H_{prop} + H_{valid}$ on the subspace $\mathcal{H}_{invalid}$ is at least 1.

(c) We will denote the set of valid clock states by $|0\rangle, |1\rangle, \ldots, |T\rangle$. For $x \in \{0,1\}^n$, let $|\Phi(x,t)\rangle$ denote the state resulting from starting with input $|x\rangle$ and executing t steps of the circuit. Prove that the following set forms an orthonormal basis for $\mathcal{H}_{valid}$.

$$\mathcal{B} = \{|\Phi(x,t)\rangle \otimes |t\rangle : 0 \leq t \leq T, x \in \{0,1\}^n\}$$

(d) When $H_{prop} + H_{valid}$ is expressed as a matrix in the basis above, it is block diagonal. What are the blocks?

(e) Express the term H_{init} that applies an energy penalty if the input state to the circuit is not $|0\rangle^{\otimes n}$. Argue that H_{init} is diagonal in the basis $\mathcal{B}$.

(f) Prove that the smallest eigenvalue of $H_{prop}+H_{valid}+H_{init}$ for each of the blocks except for one is $\Omega(1/T^2)$.

(g) What are the smallest and second smallest eigenvalue for the other block?

(h) Argue that the spectral gap of $H_{prop} + H_{valid} + H_{init}$ over the entire space is $\Omega(1/T^2)$.

(i) Suppose a state is prepared that is the ground state of $H_{prop}+H_{valid}+H_{init}$. What measurement should be performed to guarantee that the state in the computation register corresponds to the correct output of the circuit? What is the probability of success?

(j) Suppose instead we add T identity gates to the end of the circuit and apply the same construction to the new circuit, resulting in a new Hamiltonian. Now describe a measurement that can be applied to the ground state of the new Hamiltonian to guarantee that the resulting state in the computation register corresponds to the correct output of the circuit? What is the probability of success?

5. Give a polynomial time algorithm to compute the ground energy of a classical Hamiltonian (diagonal in the standard basis) on a 1D chain of particles.