PCMI 2023 | *Instructor: Sandy Irani*
TA: Kunal Marwaha

Quantum Hamiltonian Complexity: Day 2

August 1, 2023

topics: the Circuit to Hamiltonian construction in detail

1. Show that the cat state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ can not be the unique ground state of any local Hamiltonian.

2. Let A be the following 2-qubit operator:

$$A = \frac{1}{2}(I \otimes |0\rangle\langle 0| + I \otimes |1\rangle\langle 1| - H \otimes |0\rangle\langle 1| - H \otimes |1\rangle\langle 0|)$$

   (a) Calculate the expected value of A for each of the following states:
      i. $|\phi_1\rangle = |0\rangle|0\rangle$
      ii. $|\phi_2\rangle = |+\rangle|1\rangle$
      iii. $|\phi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |+\rangle|1\rangle)$

      (Note: A is the H_{prop} term for a one qubit circuit consisting of a Hadamard gate.

   (b) Show that A is Hermitian and positive semi-definite. One of the previous states was an eigenvector with eigenvalue 0. Can you find an orthogonal eigenstate also with eigenvalue 0?

3. Why do we need a forward propagation term *and* a backward propagation term? Why can't we have just one?

4. The clock construction shown in lecture has been improved to show that the 2-local Hamiltonian problem is QMA-hard. How hard is the 1-local Hamiltonian problem?

5. In class we proved the following Lemma:

   Lemma: *Let H_1 and H_2 be two Hermitian positive semi-defininte matrices whose second eigenvalue is at least λ. Let N_1 and N_2 be the null spaces of the two Hamiltonians and suppose that the angle between N_1 and N_2 is at least θ. Then the smallest eigenvalue of $H_1 + H_2$ is at least $\lambda \sin^2(\theta/2)$.*

   (a) What lower bound does the lemma give when N_1 and N_2 are orthogonal subspaces?

   (b) Can you prove a better lower bound for this case?

6. (Challenge) We proved that for Hermitian and positive semi-definite matrices H_1 and H_2 with null spaces N_1 and N_2, if the second eigenvalue of each is at least λ and the angle between N_1 and N_2 is at least θ, then the smallest eigenvalue of $H_1 + H_2$ is at least $\lambda \sin^2(\theta/2)$. What happens if there are three matrices? k matrices? (Suppose the dimension is d; i.e. $H \in \mathbb{C}^{d \times d}$.) (Note: I'm not sure of the answer)