Quantum Hamiltonian Complexity
Part II

Sandy Irani
Computer Science Department
UC Irvine

The class QMA (Quantum Merlin Arthur)

NP

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If $x \notin L$, then for every y, $C_n(x, y) = 0$.

$|y| \leq \text{poly}(x)$

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Boolean Satisfiability is NP-hard [Cook-Levin]

Start with a generic language L in NP

Is $x \in L$?

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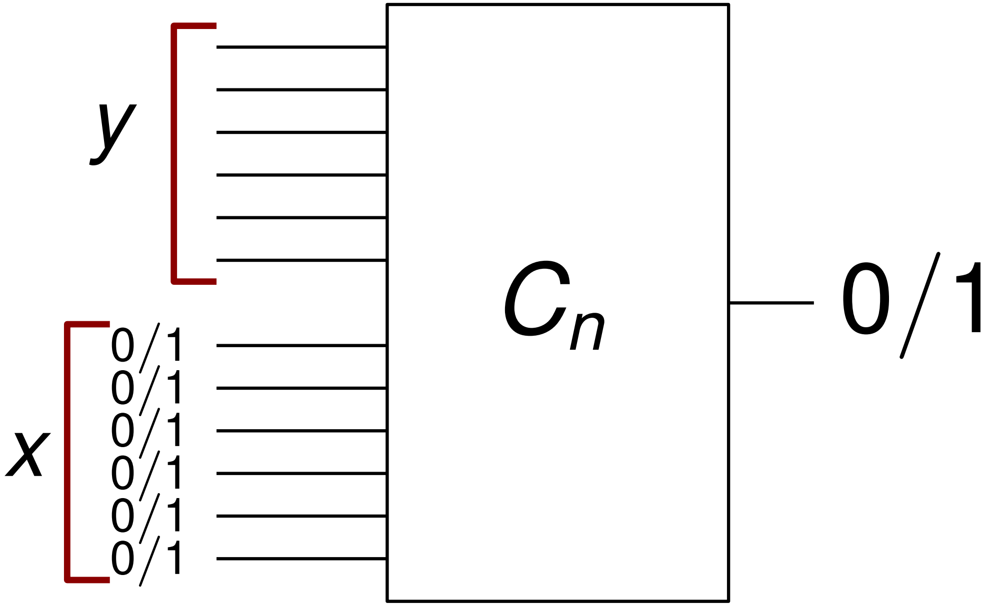

Is there a string y that causes this circuit to output 1?

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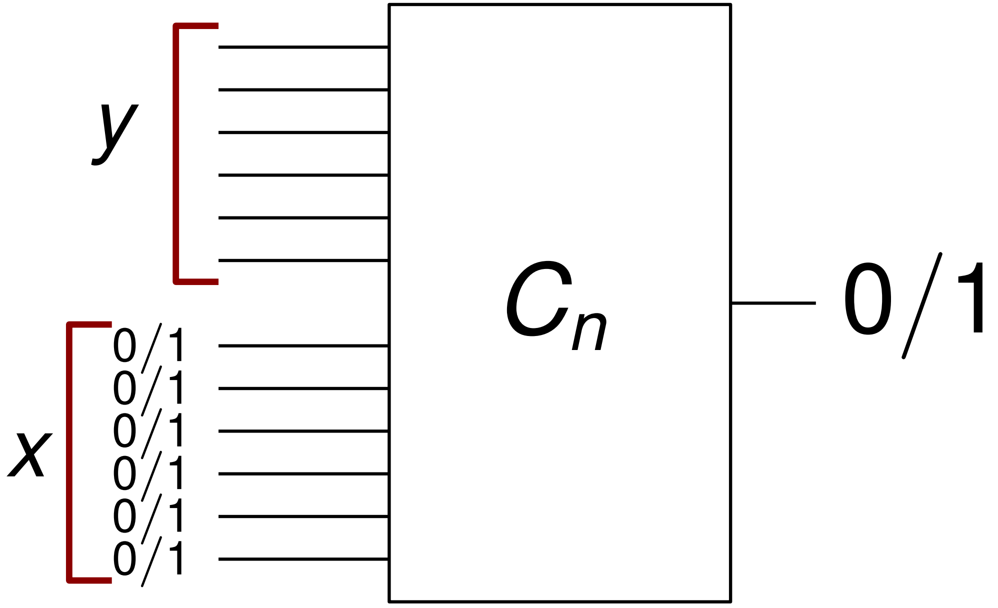

$\Rightarrow$

Boolean Formula: $\Phi_x(y)$

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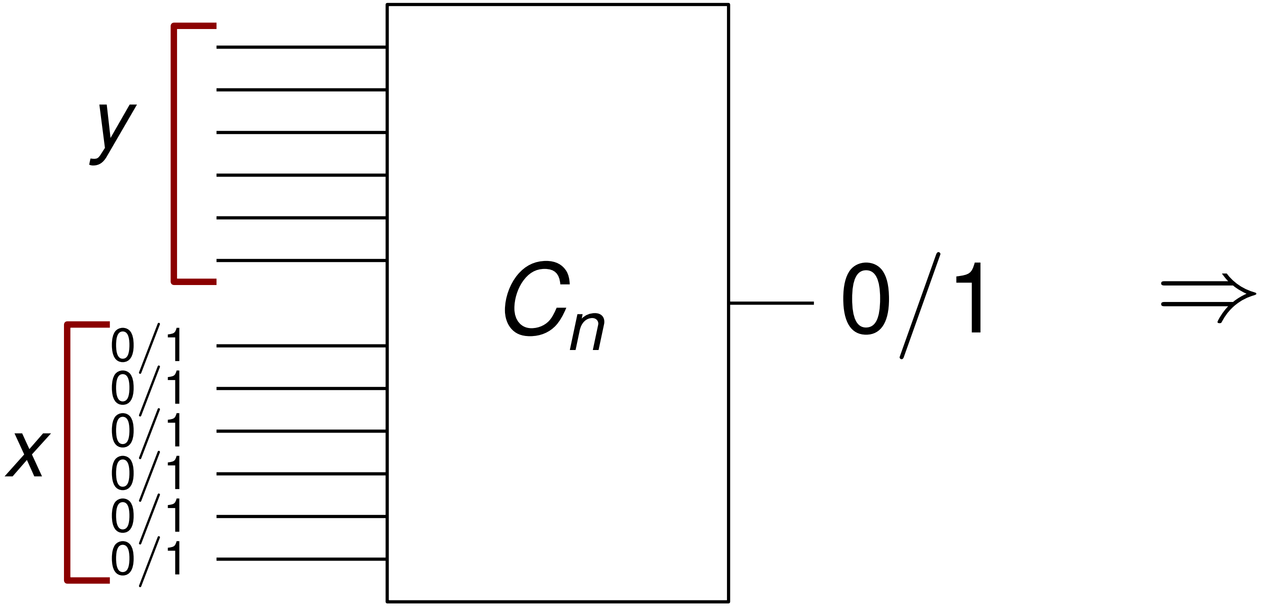

Boolean Formula:
$\Phi_x(y)$

Is there a string y that causes this circuit to output 1? $\Leftrightarrow$ Is $\Phi_x(y)$ satisfiable?

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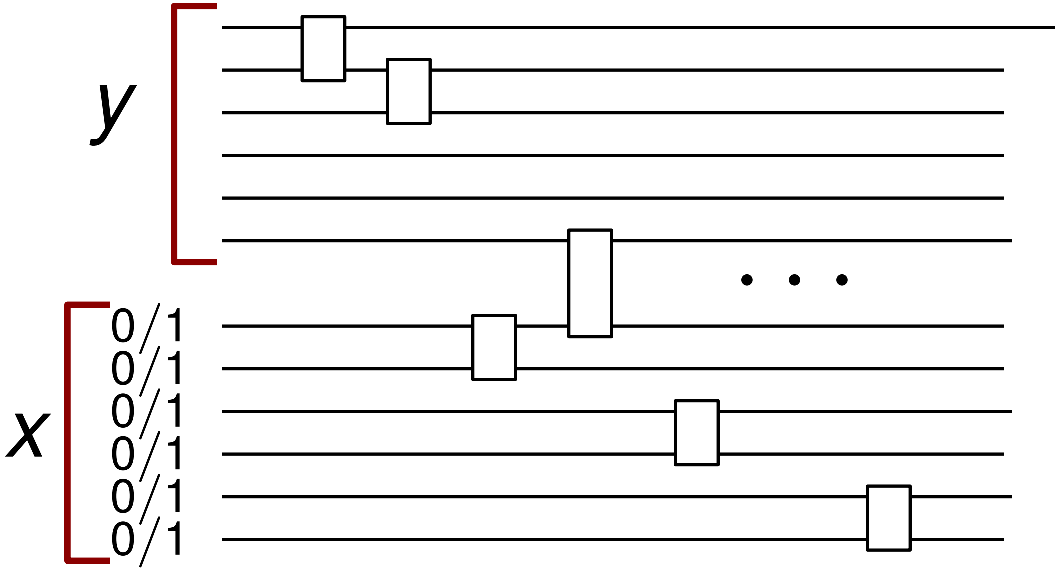
Can assume C_n is a reversible circuit.



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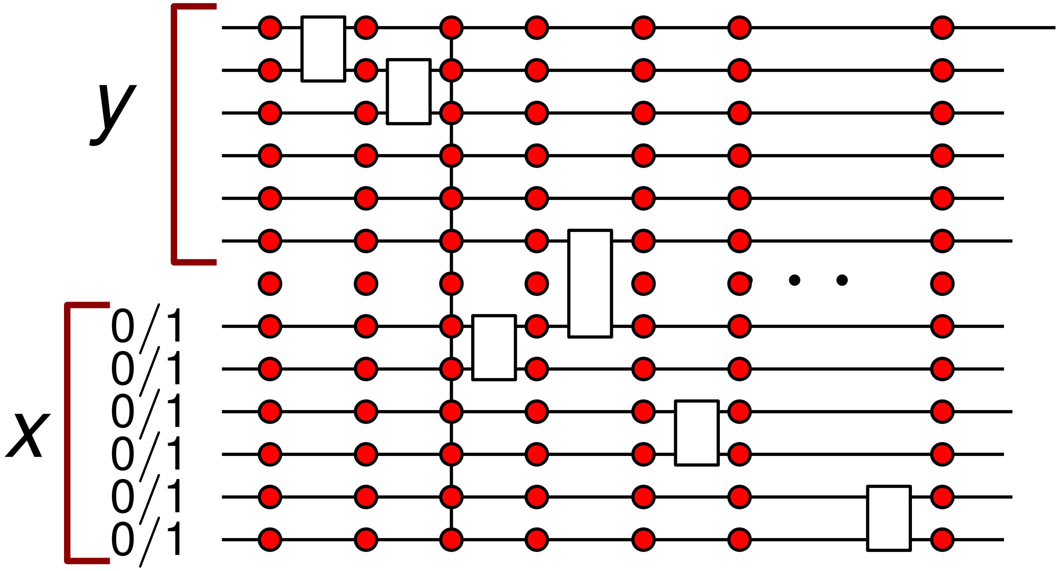

$m(T+1)$ Boolean variables

m = number of lines

T = number of gates

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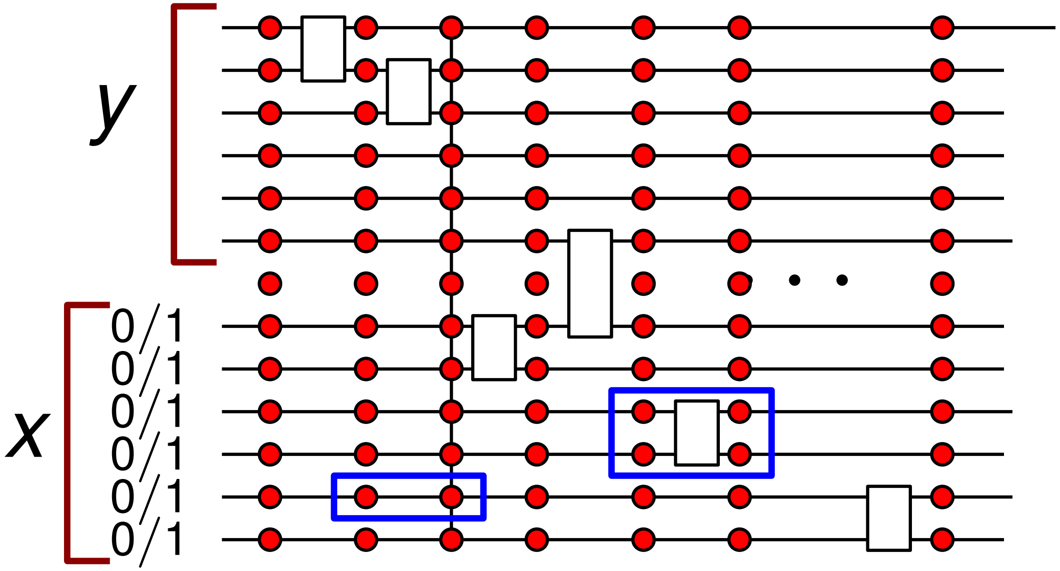

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Enforce correct computation by local constraints.

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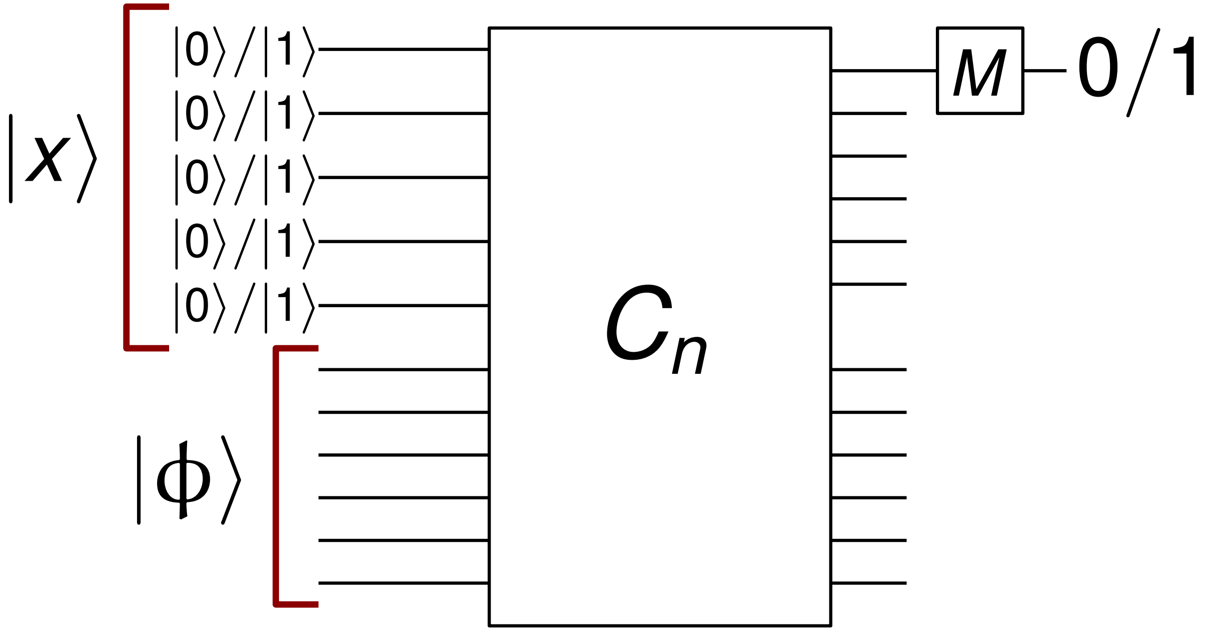

Is there a quantum state $\phi\rangle$
that causes this quantum circuit
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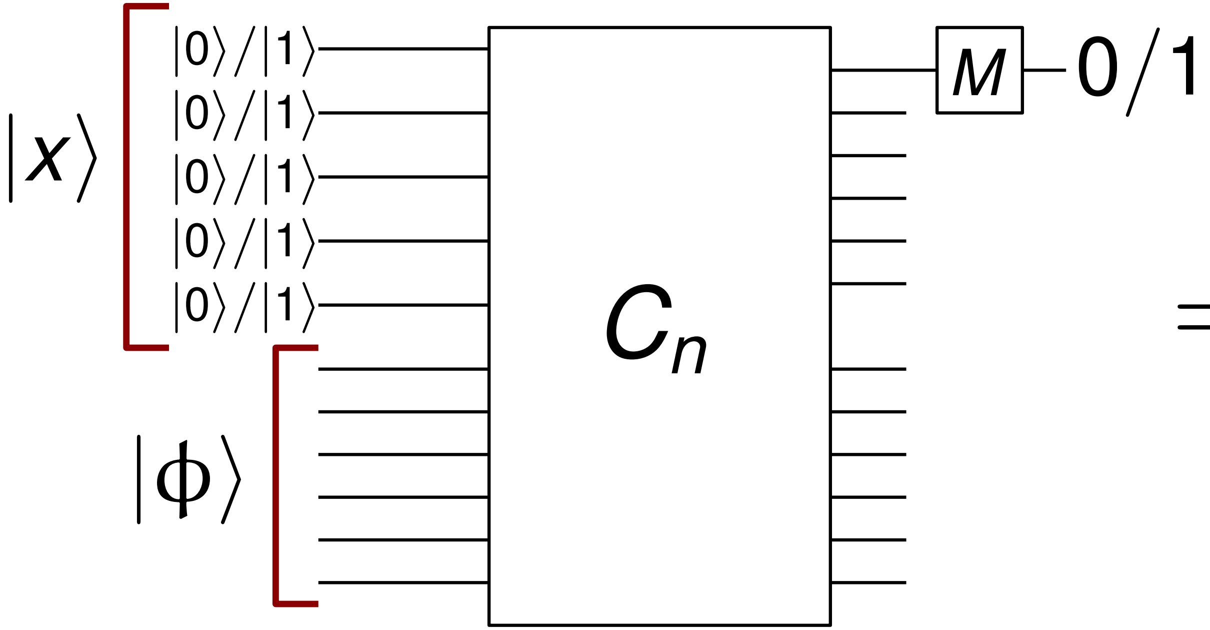

$\Rightarrow$ k-Local Hamiltonian: (H_x, E, Δ)

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Local Hamiltonian is QMA-hard [Kitaev 1995]

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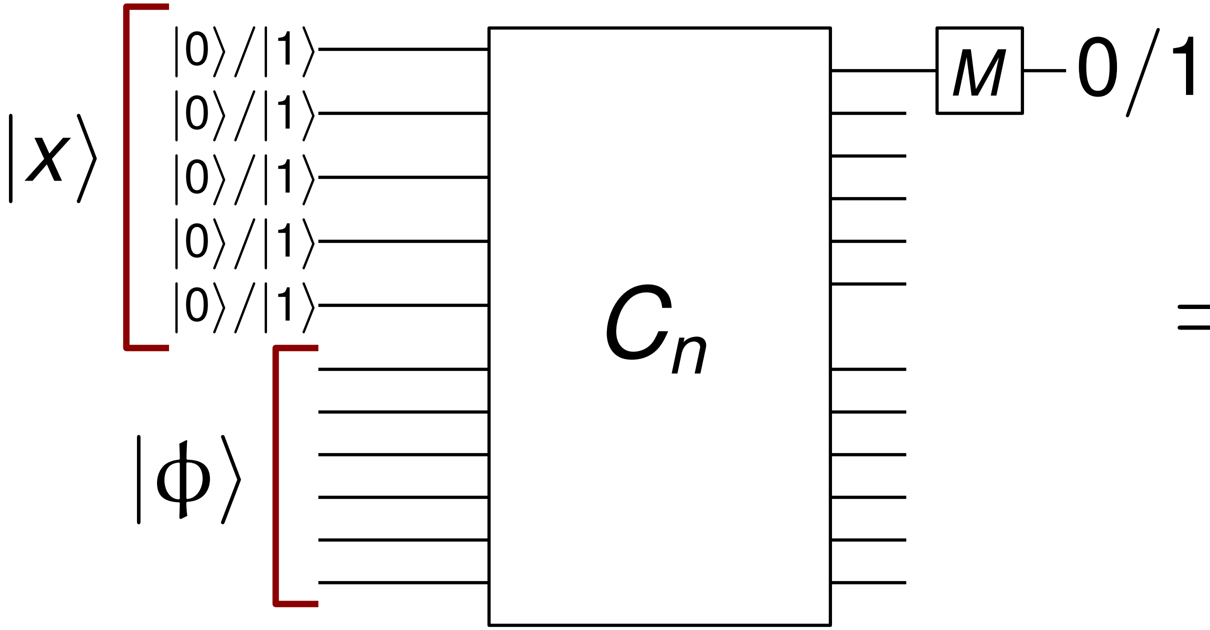


$\Rightarrow$ *k*-Local Hamiltonian: (H_x, E, Δ)

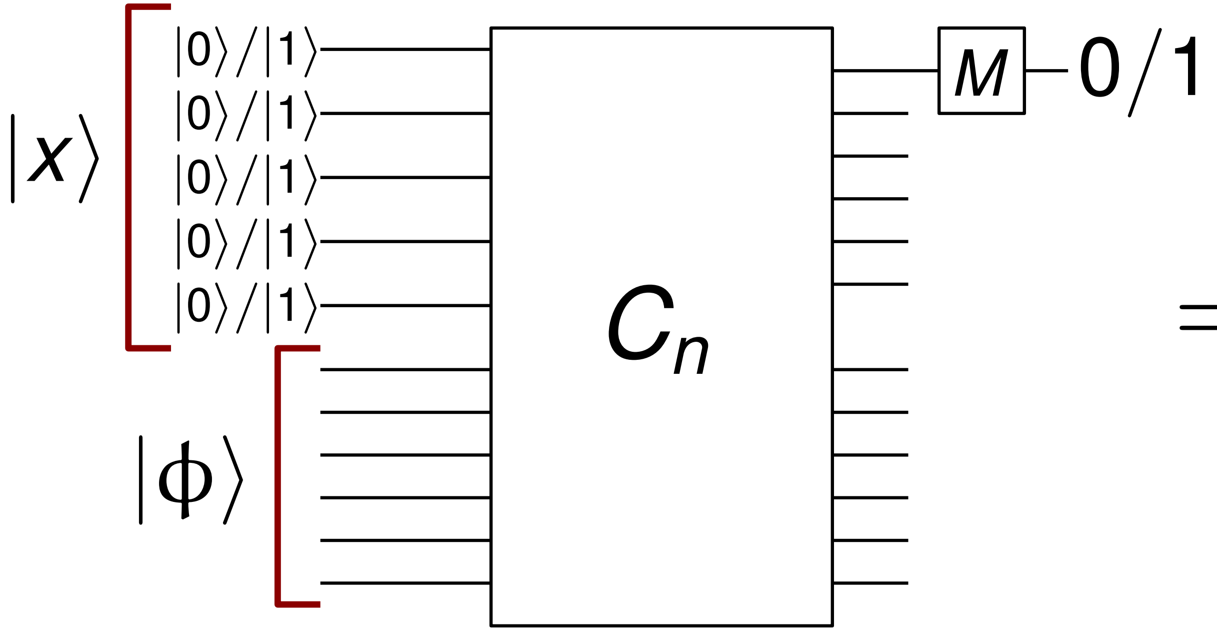
Is there a quantum state $\phi\rangle$ that causes this quantum circuit to output 1 with high probability? $\Leftrightarrow$ Is the ground energy of H_x $\leq E$ or $\geq E + \Delta$?

Local Hamiltonian is QMA-hard [Kitaev 1995]

Start with a generic language L in QMA

Is $x \in L$?

Start with $k = O(\log n)$
Then improve to $k = 5$



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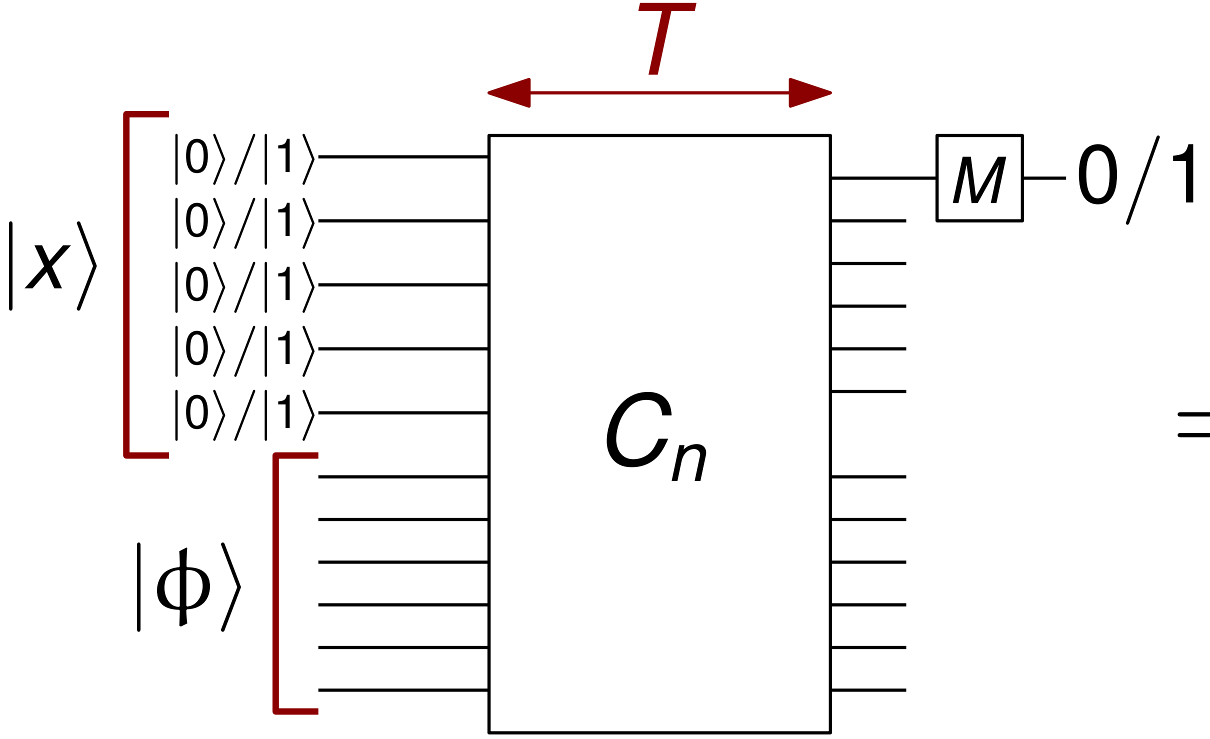

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Computational History States - Classical and Quantum

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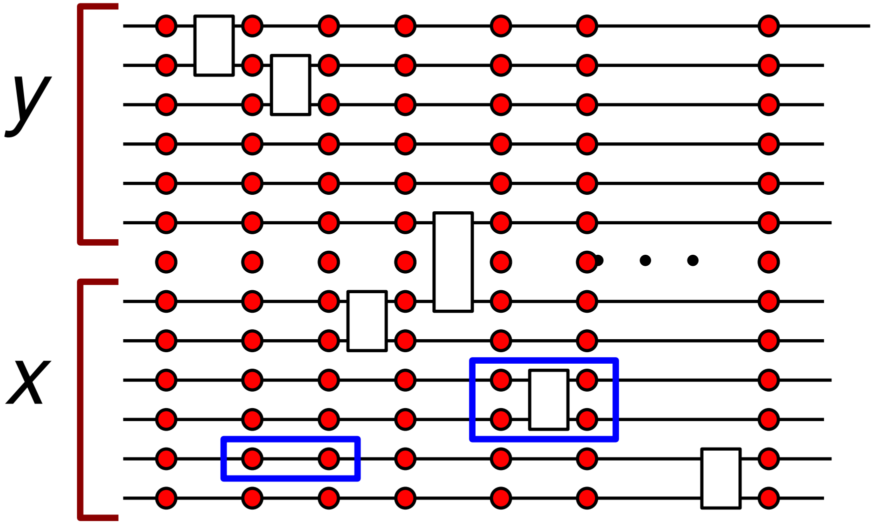
Reversible Circuit



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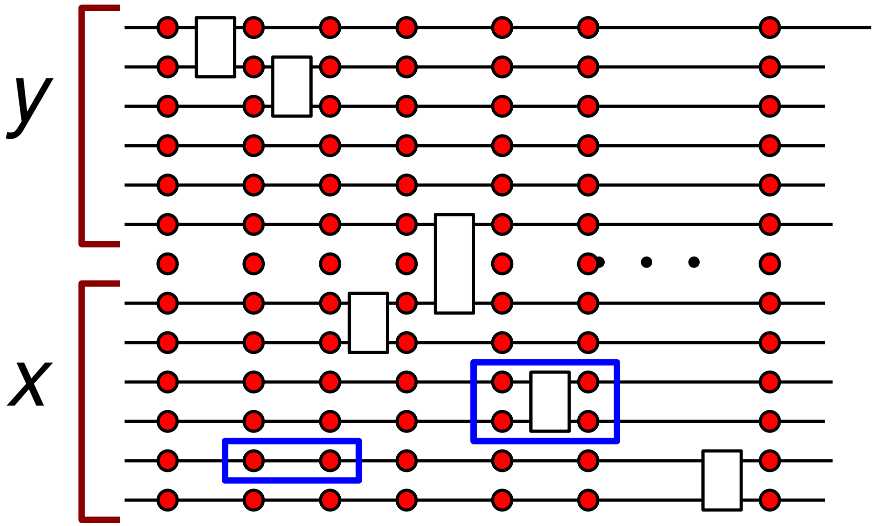

Quantum
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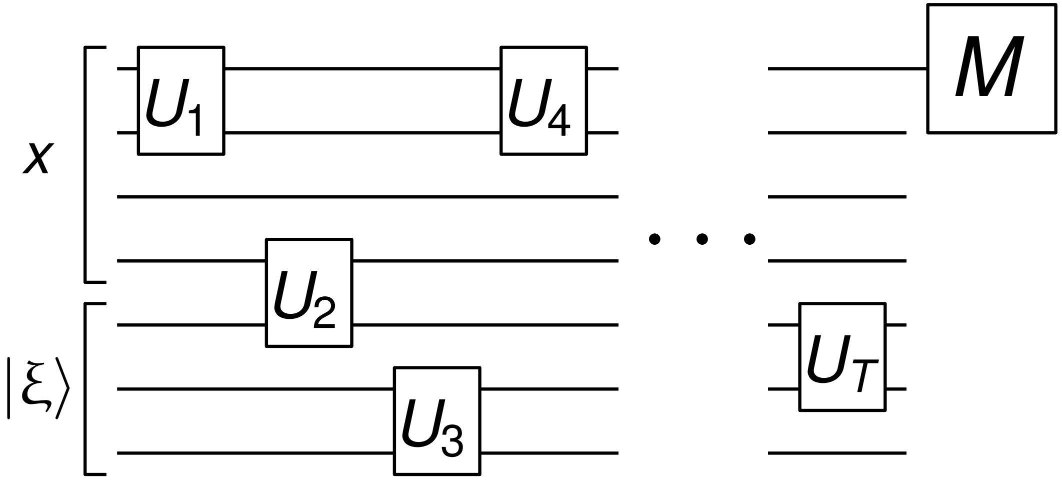

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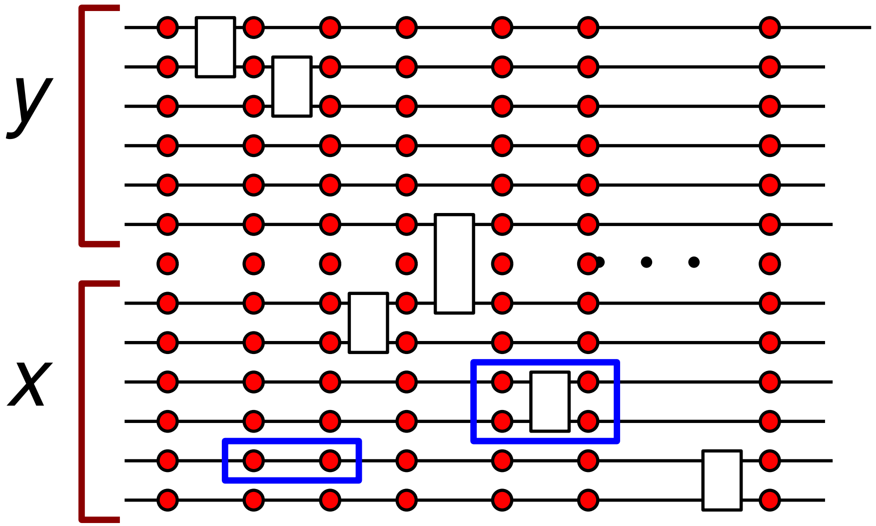

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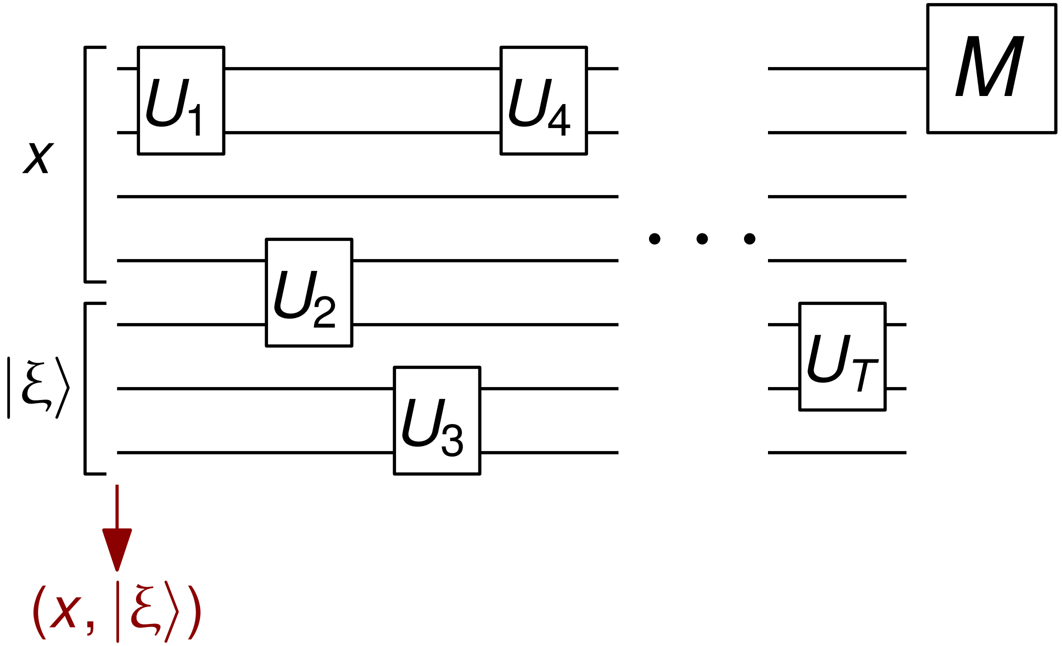

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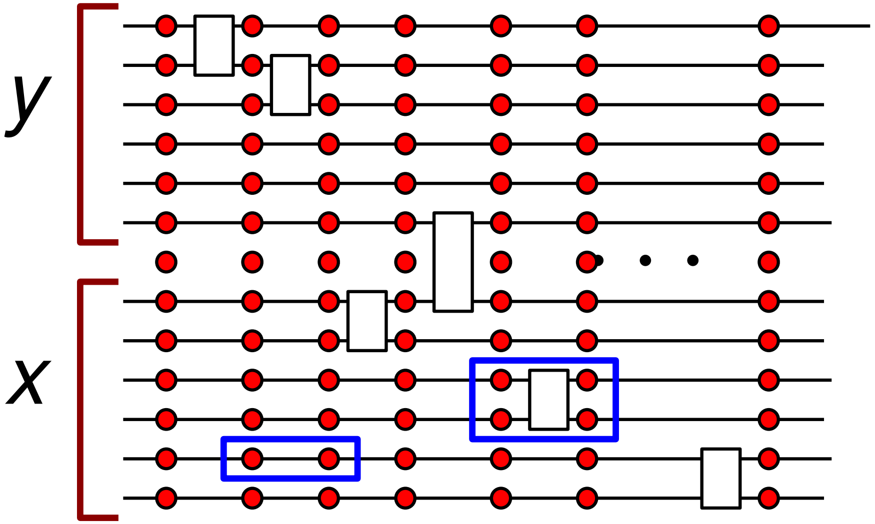
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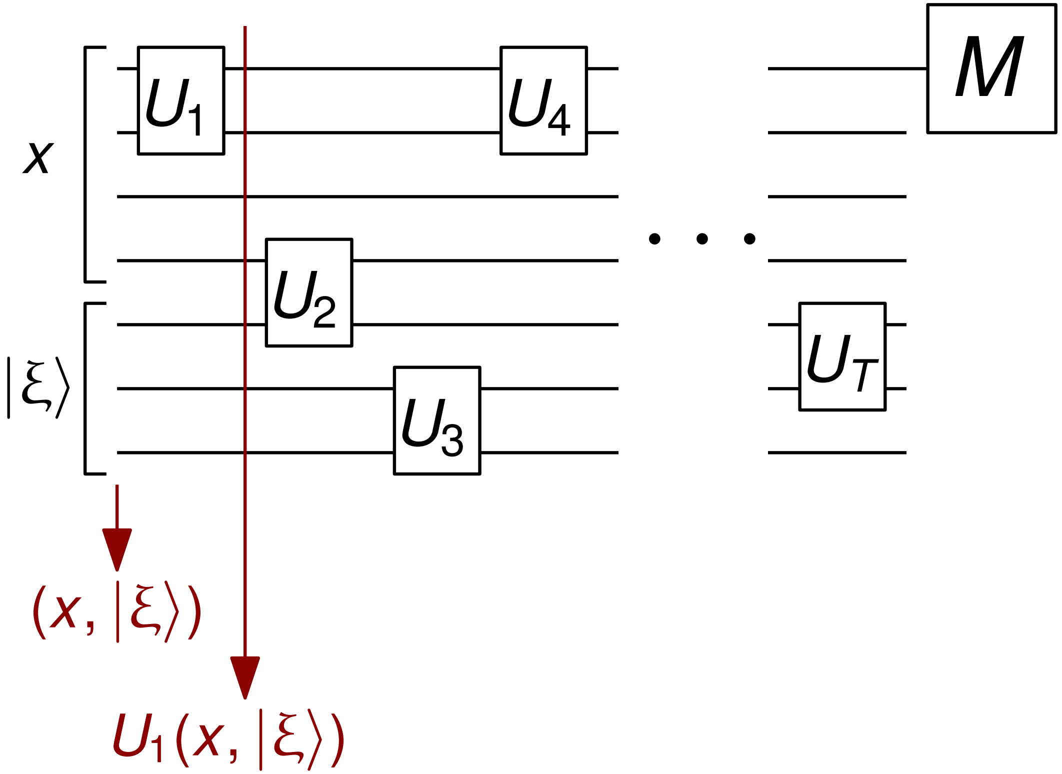
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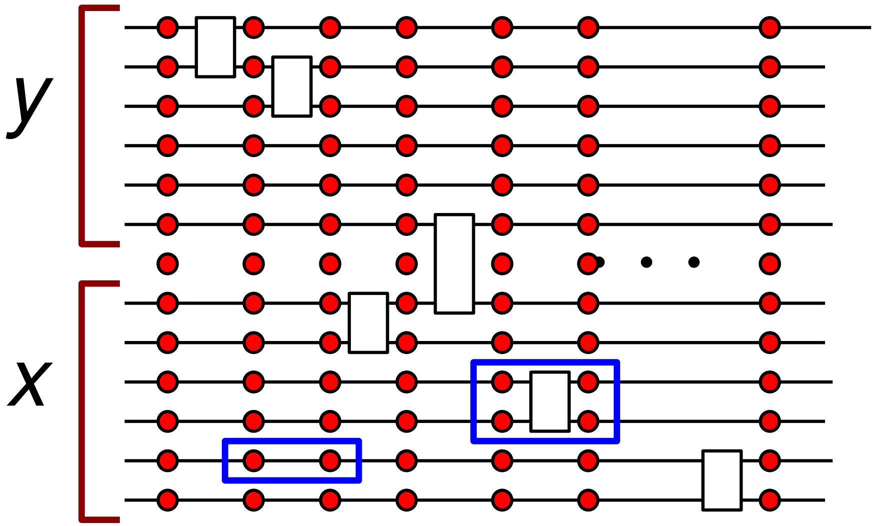

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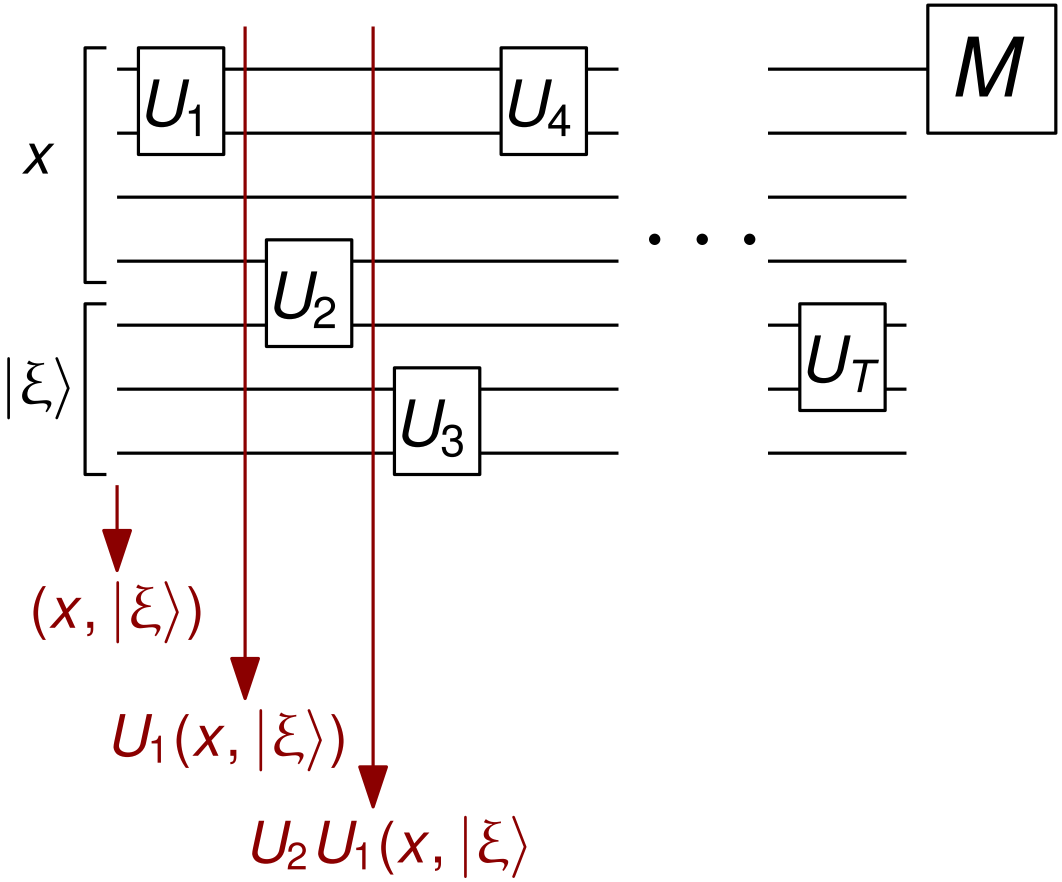

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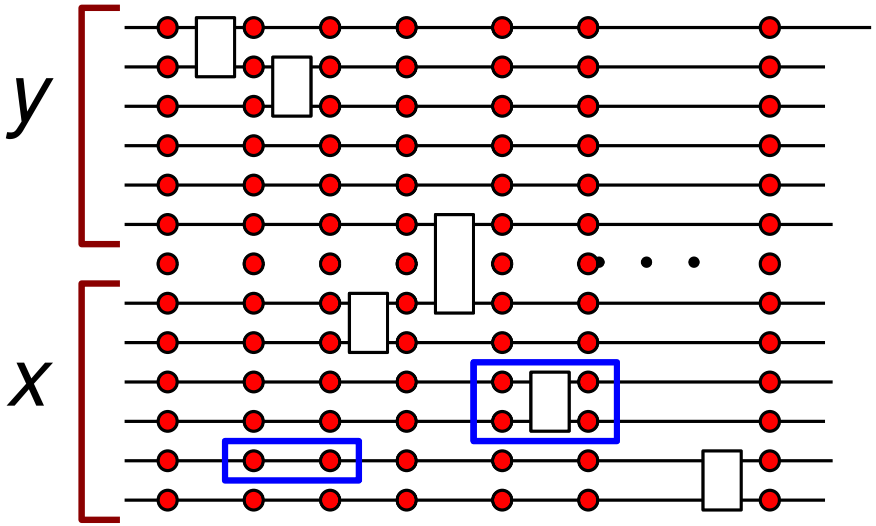

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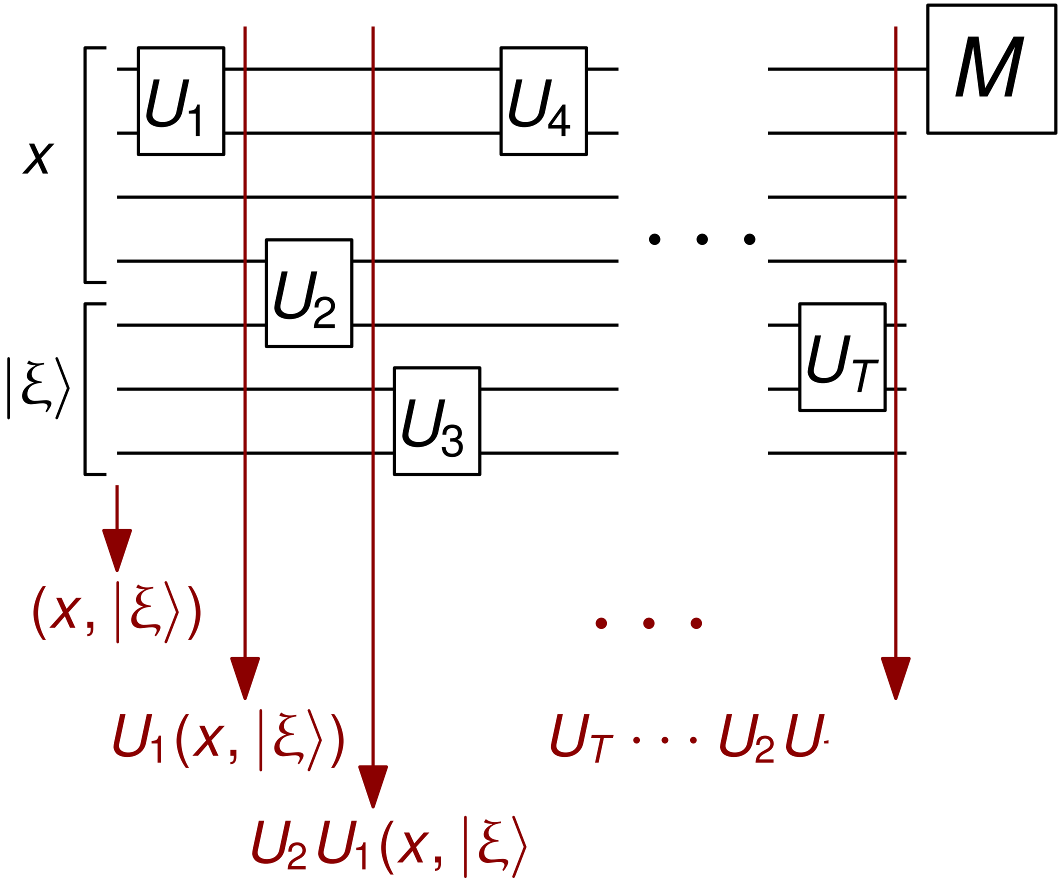

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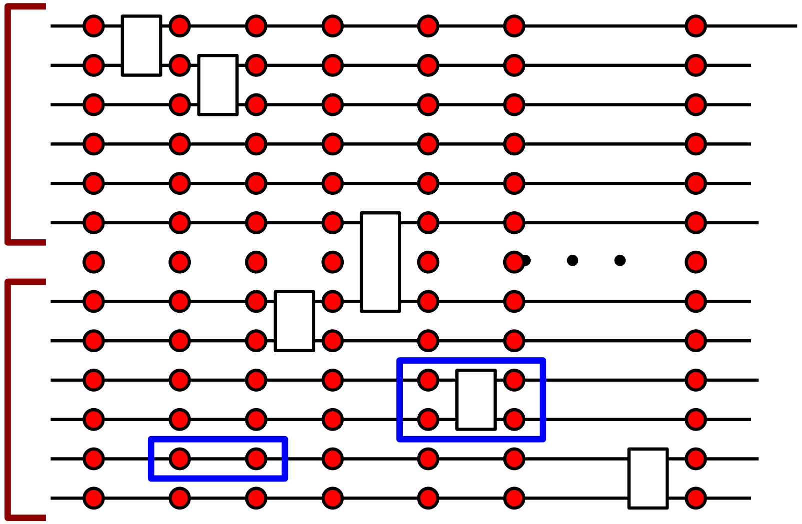

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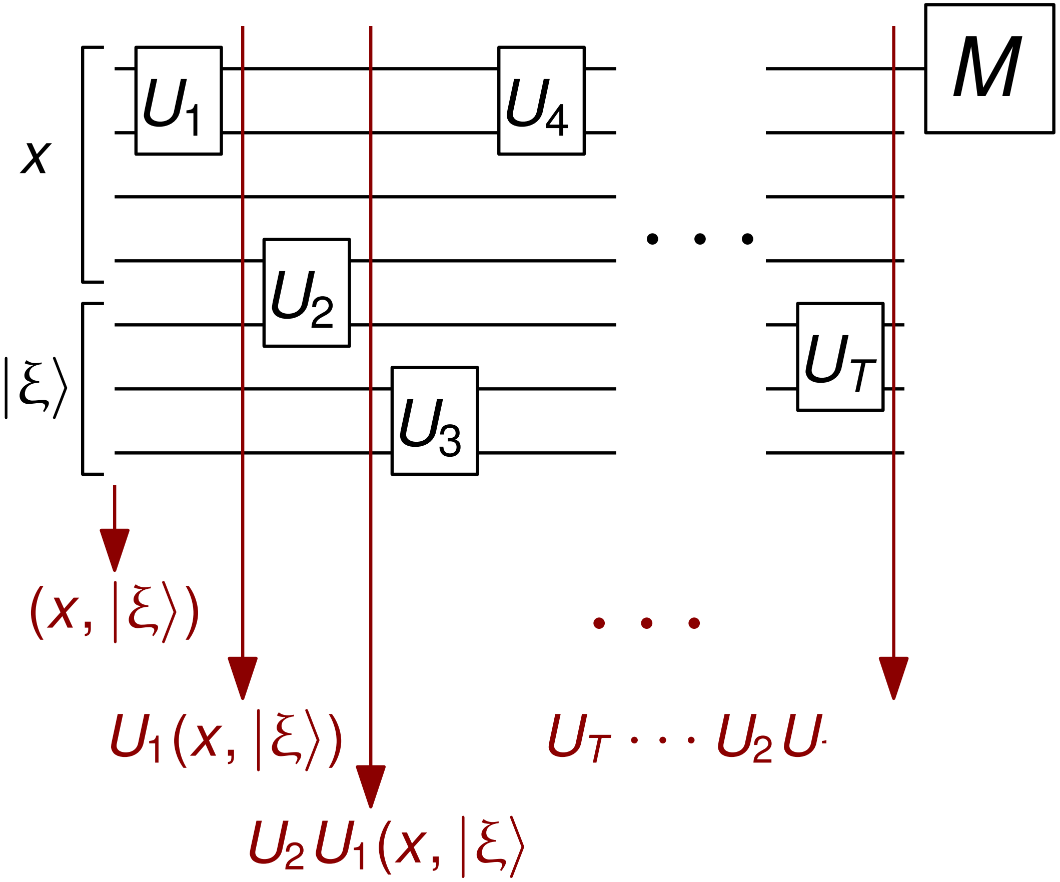

Idea: Store the history of the computation in quantum superposition.

Need for Superposition

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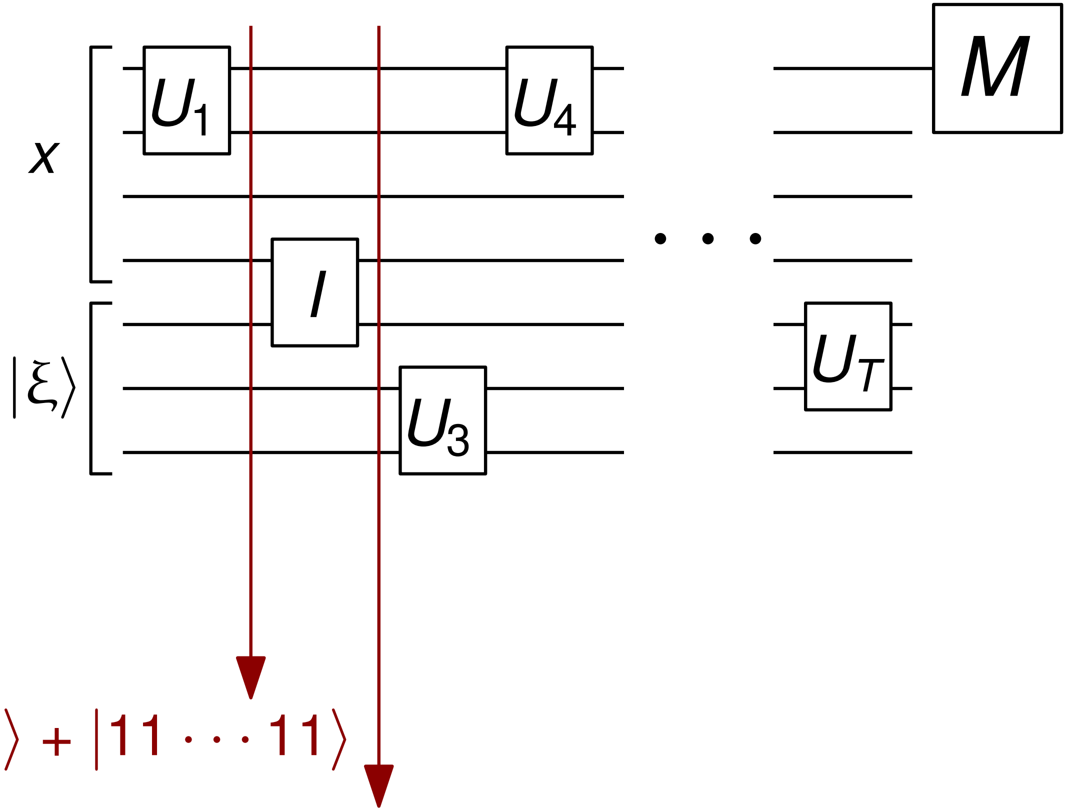

$|00\cdots00\rangle + |11\cdots11\rangle$

$|00\cdots00\rangle - |11\cdots11\rangle$

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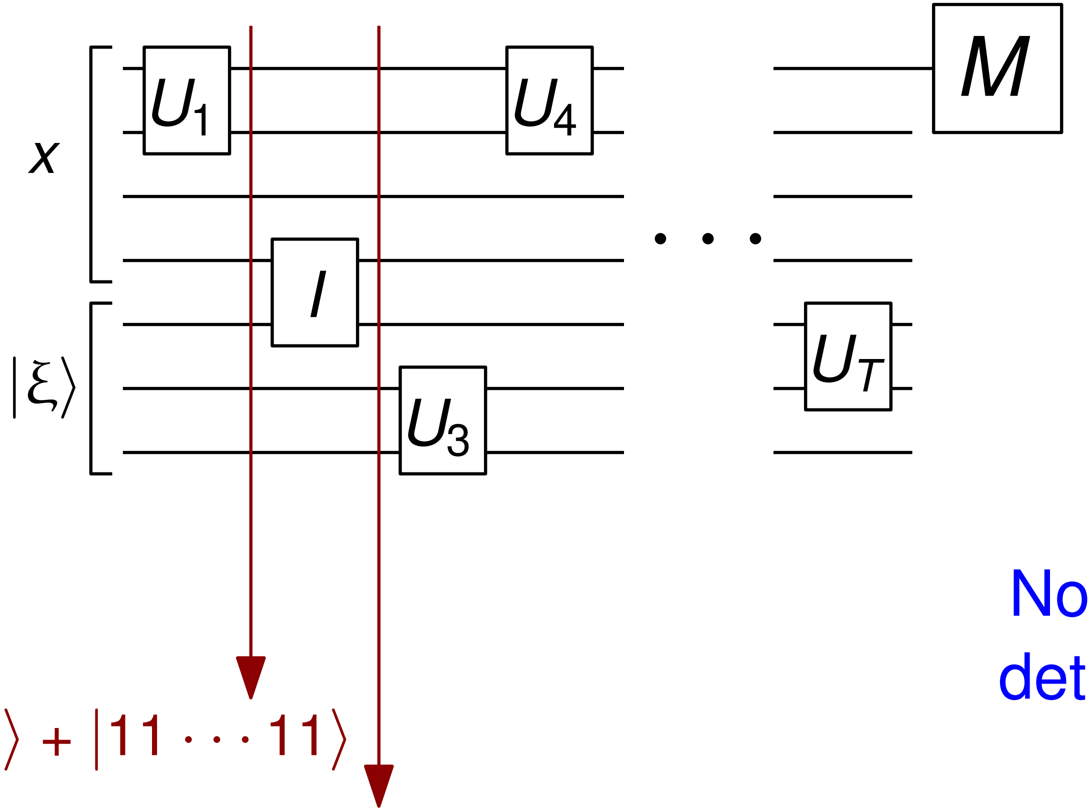

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$|00\cdots00\rangle - |11\cdots11\rangle$

No local check can
detect this problem.

The Computation State

The constraints of the *k*-local Hamiltonian ensure that the ground state is a superposition of:

$$\begin{array}{rl} |x\rangle|\xi\rangle & |00\cdots00\rangle \\ U_1|x\rangle|\xi\rangle & |00\cdots01\rangle \\ U_2U_1|x\rangle|\xi\rangle & |00\cdots10\rangle \\ & \vdots \\ \underbrace{U_T\cdots U_2U_1|x\rangle|\xi\rangle}_{\text{Computation Qubits}} & \underset{\text{Clock Qubits}}{|11\cdots11\rangle} \end{array}$$

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\end{array}$$

$$\frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}\underbrace{U_tU_{t-1}\cdots U_2U_1|x\rangle|\xi\rangle}_{\text{Computation Register}}\otimes|t\rangle$$

Clock
Register
s qubits
$T = 2^s - 1$

A Simpler Problem

Find a Hamiltonian over s qubits, whose ground state is:

$$\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \qquad \text{where } T = 2^s - 1$$

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Find a Hamiltonian over *s* qubits, whose ground state is:

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$$H = \sum_{j=0}^{T-1} \frac{1}{2} \left(|t\rangle\langle t| + |t+1\rangle\langle t+1| - |t\rangle\langle t+1| - |t+1\rangle\langle t| \right)$$

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Each term:

$$\begin{array}{c|cc} & t & t+1 \\ \hline t & 1/2 & -1/2 \\ t+1 & 1/2 & 1/2 \end{array}$$

$(1, 1)$, eigenvalue 0

$(1, -1)$, eigenvalue 1

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Sum:

$$\frac{1}{2} \cdot \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ 0 & -1 & 2 & -1 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & -1 & 2 & -1 & 0 \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 1 \end{bmatrix}$$

A Simpler Problem

Find a Hamiltonian over *s* qubits, whose ground state is:

For t in binary
s-bit operation:
$s \approx \log T = O(\log n)$

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The Propagation Matrix

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$T+1$
rows

$T+1$ columns

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Eigenvector: $\frac{1}{\sqrt{T+1}}(1, 1, \ldots, , 1)$ Eigenvalue: 0

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$T + 1$ rows

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$$A = I - B$$

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

The Propagation Matrix

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$T+1$ rows

$T+1$ columns

Eigenvector: $\frac{1}{\sqrt{T+1}}(1, 1, \ldots, , 1)$ Eigenvalue: 0

$$A = I - B$$

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

λ: eigenvalue of B

$\Rightarrow 1 - \lambda$ eigenvalue of H_{prop}

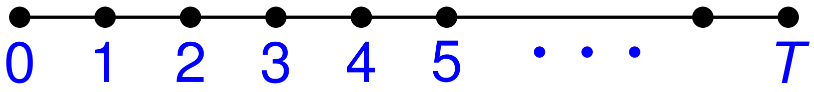
Random Walk Matrix

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

Random Walk Matrix

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

Matrix *B* describes a random walk on a line:



Random Walk Matrix

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

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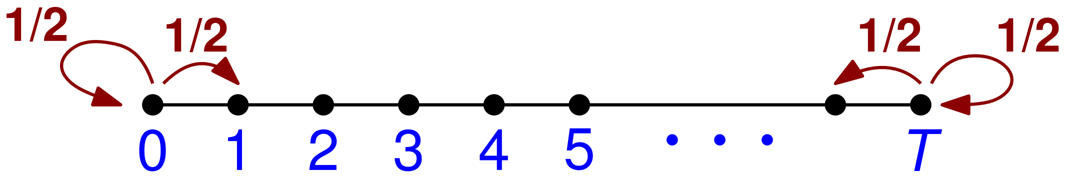

Random Walk Matrix

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & \\ 1/2 & 0 & 1/2 & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

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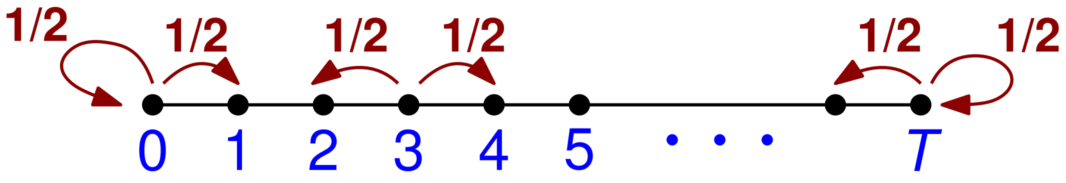

Random Walk Matrix

$$B = \begin{bmatrix} 1/2 & 1/2 & & & & & & & \\ 1/2 & 0 & 1/2 & & & & & & \\ 0 & 1/2 & 0 & 1/2 & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & 1/2 & 0 & 1/2 & 0 \\ & & & & & 1/2 & 0 & 1/2 \\ & & & & & & 1/2 & 1/2 \end{bmatrix}$$

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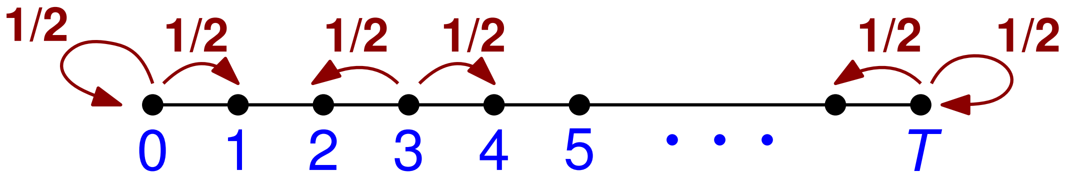

Maximum eigenvalue $\lambda_T = 1$

Second largest eigenvalue related to convergence time: $1 - \frac{1}{2(T+1)^2} \leq \lambda_{T-1}$

For $A = I - B$: $\lambda_0 = 0$ and $\lambda_1 \geq \frac{1}{2(T+1)^2}$

Propogation Matrix H_{prop}

Target Ground State: $$\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t U_{t-1} \cdots U_2 U_1 |x\rangle|\xi\rangle \otimes |t\rangle$$

Propogation Matrix H_{prop}

Target
Ground State: $\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t U_{t-1} \cdots U_2 U_1 |x\rangle|\xi\rangle \otimes |t\rangle$

For any $|\Phi\rangle$:

$|\Phi\rangle|t-1\rangle$ and $U_t|\Phi\rangle|t\rangle$ have the same amplitude.

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$$H_t =$$

$$\frac{1}{2}\left[\underbrace{I \otimes |t\rangle\langle t| + I \otimes |t-1\rangle\langle t-1|}_{\text{Diagonal terms}} - \underbrace{U_t \otimes |t\rangle\langle t-1|}_{\text{Forward Propogation}} - \underbrace{U_t^\dagger \otimes |t-1\rangle\langle t|}_{\text{Backward Propogation}}\right]$$

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$$\begin{array}{c|cc} & |\Phi\rangle|t-1\rangle & U_t|\Phi\rangle|t\rangle \\ \hline |\Phi\rangle|t-1\rangle & 1/2 & -1/2 \\ U_t|\Phi\rangle|t\rangle & -1/2 & 1/2 \end{array}$$

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Propogation Matrix H_{prop}, cont.

Express H_{prop} in special basis of $\mathcal{H} = \mathcal{H}_{input} \bigotimes \mathcal{H}_{witness} \bigotimes \mathcal{H}_{clock}$

Orthonormal basis for input register: $\{|a_j\rangle\}$

Orthonormal basis for witness register: $\{|\xi_k\rangle\}$

Propogation Matrix H_{prop}, cont.

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Orthonormal basis for witness register: $\{|\xi_k\rangle\}$

One possible basis for $\mathcal{H}$:

$$\{|a_j\rangle\} \otimes \{\xi_k\rangle\} \otimes \{\ |t\rangle\}$$

Propogation Matrix H_{prop}, cont.

Express H_{prop} in special basis of $\mathcal{H} = \mathcal{H}_{input} \otimes \mathcal{H}_{witness} \otimes \mathcal{H}_{clock}$

Orthonormal basis for input register: $\{|a_j\rangle\}$
Orthonormal basis for witness register: $\{|\xi_k\rangle\}$

For each j and k, $\mathcal{H}_{j,k}$ is the space spanned by:

$$|a_j\rangle|\xi_k\rangle \otimes |0\rangle$$

$$U_1|a_j\rangle|\xi_k\rangle \otimes |1\rangle$$

$$U_2U_1|a_j\rangle|\xi_k\rangle \otimes |2\rangle$$

$$\vdots$$

$$U_T \cdots U_2U_1|a_j\rangle|\xi_k\rangle \otimes |T\rangle$$

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The $\mathcal{H}_{j,k}$'s are mutually orthogonal:

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$$\vdots$$

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The $\mathcal{H}_{j,k}$'s are mutually orthogonal:

If $(j,k) \neq (j',k')$:

$$|a_j\rangle|\xi_k\rangle \otimes |t\rangle \quad \perp \quad |a_{j'}\rangle|\xi_{k'}\rangle \otimes |t\rangle$$

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If $(j, k) \neq (j', k')$:

$$|a_j\rangle|\xi_k\rangle \otimes |t\rangle \quad \perp \quad |a_{j'}\rangle|\xi_{k'}\rangle \otimes |t\rangle$$

So:

$$U_t \cdots U_1|a_j\rangle|\xi_k\rangle \otimes |t\rangle$$
$$\perp$$
$$U_t \cdots U_1|a_{j'}\rangle|\xi_{k'}\rangle \otimes |t\rangle$$

Propogation Matrix H_{prop}, cont.

$$H_t = \frac{1}{2}\left[I \otimes |t\rangle\langle t| + I \otimes |t-1\rangle\langle t-1| + U_t \otimes |t\rangle\langle t-1| - U_t^{\dagger} \otimes |t-1\rangle\langle t|\right]$$

$$H_{prop} = \sum_{t=1}^{T} H_t$$

Propogation Matrix H_{prop}, cont.

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$$\vdots$$
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H_{prop} is closed on each $\mathcal{H}_{j,k}$

$$H_{prop}\big|_{\mathcal{H}_{j,k}} = A =$$

$$\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & & & & & & \\
-\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
 & & & \ddots & & & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
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 & & & & & & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

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$$H_{prop} = \sum_{t=1}^{T} H_t$$

$\mathcal{H}_{j,k}$ is the span of:

$$\left.\begin{array}{r} |a_j\rangle|\xi_k\rangle \otimes |0\rangle \\ U_1|a_j\rangle|\xi_k\rangle \otimes |1\rangle \end{array}\right] H_1$$

$$U_2 U_1|a_j\rangle|\xi_k\rangle \otimes |2\rangle$$

$$\vdots$$

$$U_T \cdots U_2 U_1|a_j\rangle|\xi_k\rangle \otimes |T\rangle$$

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0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
 & & & \ddots & & & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
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$$\vdots$$
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H_2

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$$H_{prop}\big|_{\mathcal{H}_{j,k}} = A =$$

$$\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & & & & & \\
-\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\
& & & \ddots & & & \\
& & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
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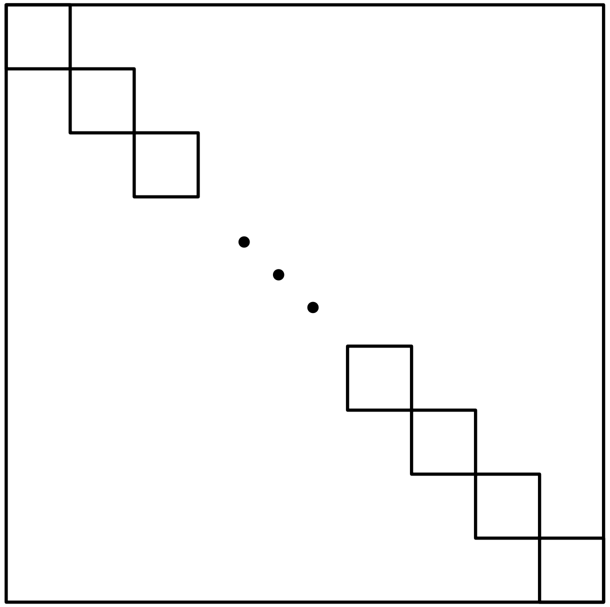
& & & & & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

Propogation Matrix H_{prop}, cont.

$H_{prop} =$



Propogation Matrix H_{prop}, cont.

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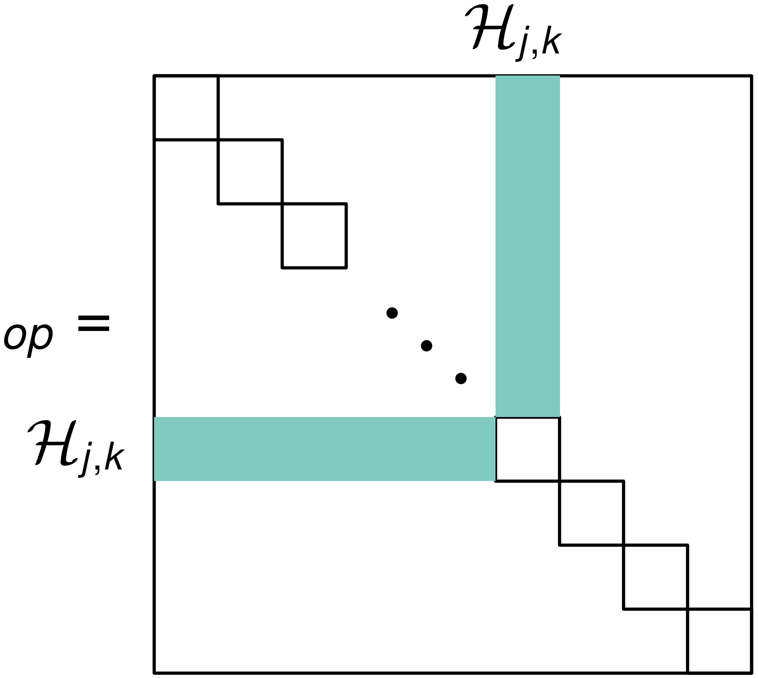
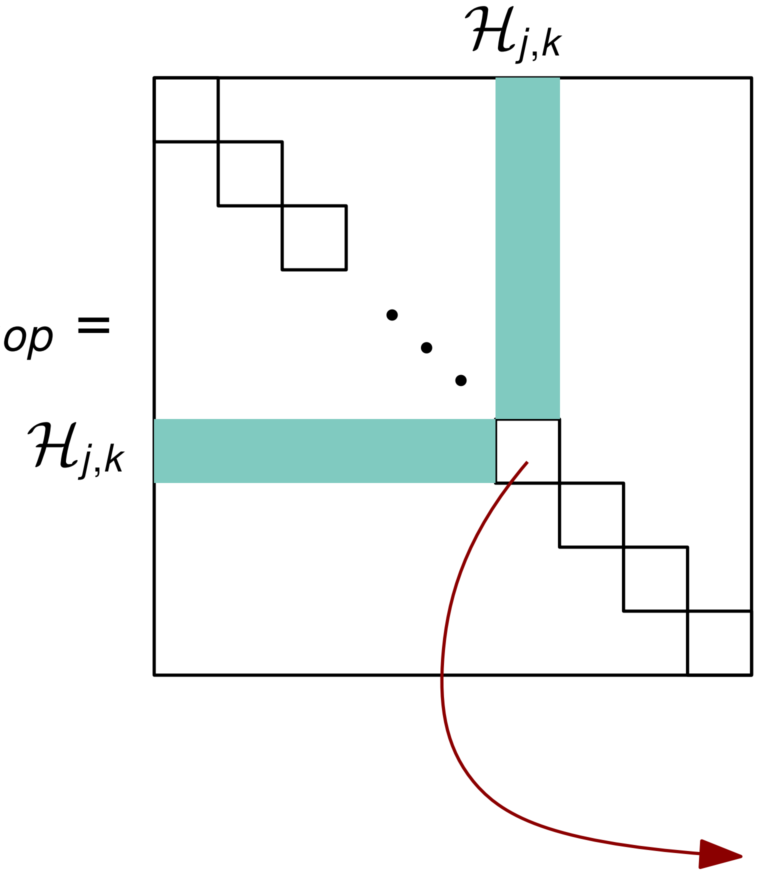


Propogation Matrix H_{prop}, cont.

$H_{prop} =$



$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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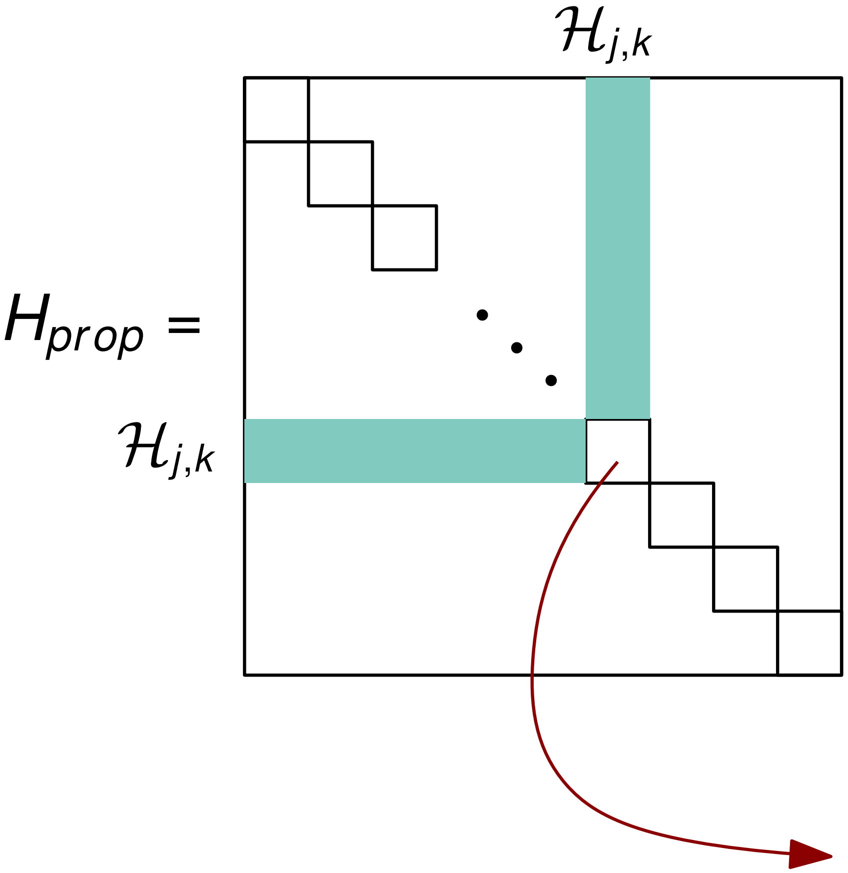
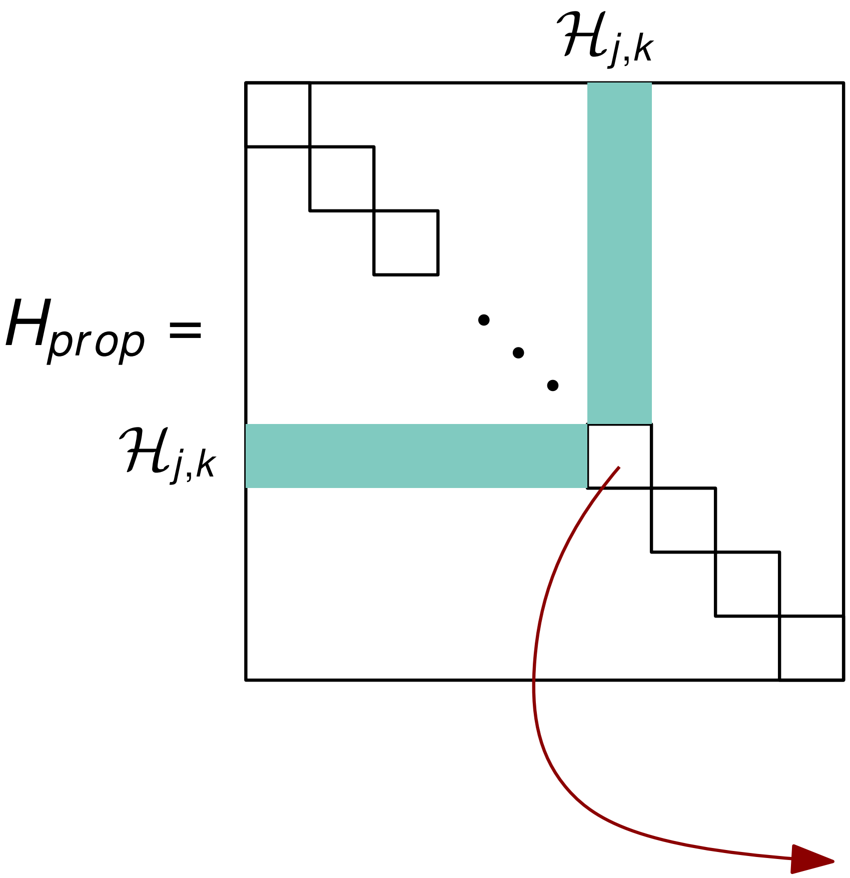


Null space of H_{prop} spanned by:

$$|\Psi_{j,k}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t U_{t-1} \cdots U_2 U_1 |a_j\rangle |\xi_k\rangle \otimes |t\rangle$$

$$\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & & & & & & \\
-\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\
 & & & \cdot & & & & \\
 & & & & \cdot & & & \\
 & & & & & \cdot & & \\
 & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
 & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\
 & & & & & & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

Propogation Matrix H_{prop}, cont.



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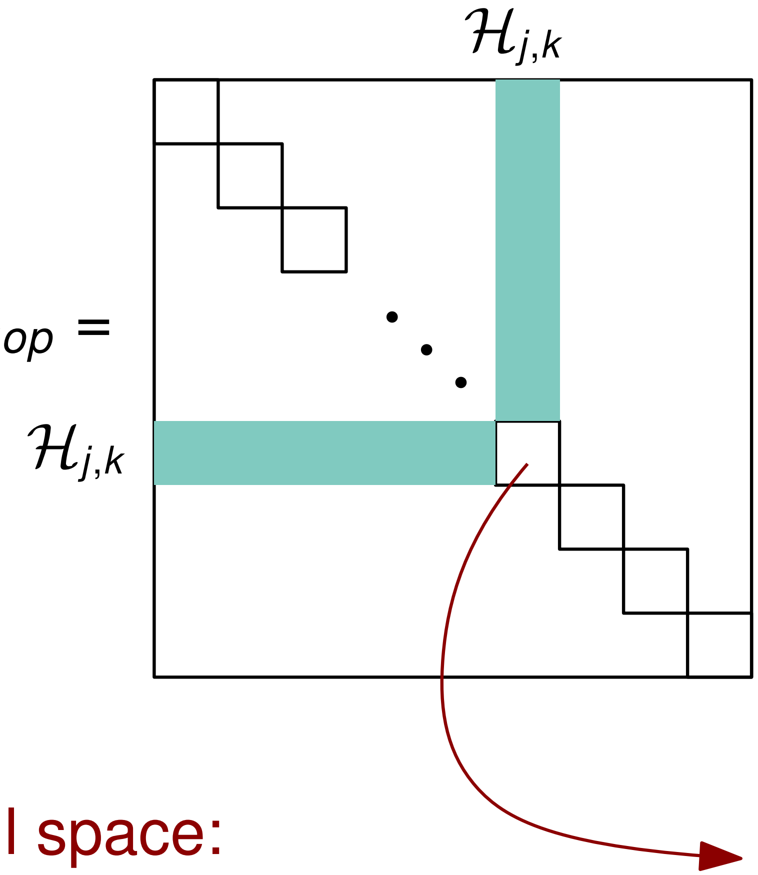
$$\sum_{j,k} \alpha_{j,k} |\Psi_{j,k}\rangle$$

$$\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t \cdots U_1 \left(\sum_{j,k} \alpha_{j,k} |a_j\rangle |\xi_k\rangle \right) \otimes |t\rangle$$

Propogation Matrix H_{prop}, cont.

H_{prop} =



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$$|\Psi_{j,k}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t U_{t-1} \cdots U_2 U_1 |a_j\rangle |\xi_k\rangle \otimes |t\rangle$$

$$\sum_{j,k} \alpha_{j,k} |\Psi_{j,k}\rangle$$

$$\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t \cdots U_1 \left(\sum_{j,k} \alpha_{j,k} |a_j\rangle |\xi_k\rangle \right) \otimes |t\rangle$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & \ddots & & & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Null space:

All valid computation states.

Second smallest eigenvalue:

$\geq \frac{1}{2(T+1)^2}$

Two more terms....

Enforce that start state has the correct input *x* in the input register:

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Enforce that start state has the correct input x in the input register:

Input $x = x_1 x_2 \cdots x_n$

$$H_{init} = \sum_{j=1}^{n} |\overline{x_j}\rangle\langle\overline{x_j}|_j \otimes |0\rangle\langle 0|_{clock}$$

Applied to bit j of the input register.

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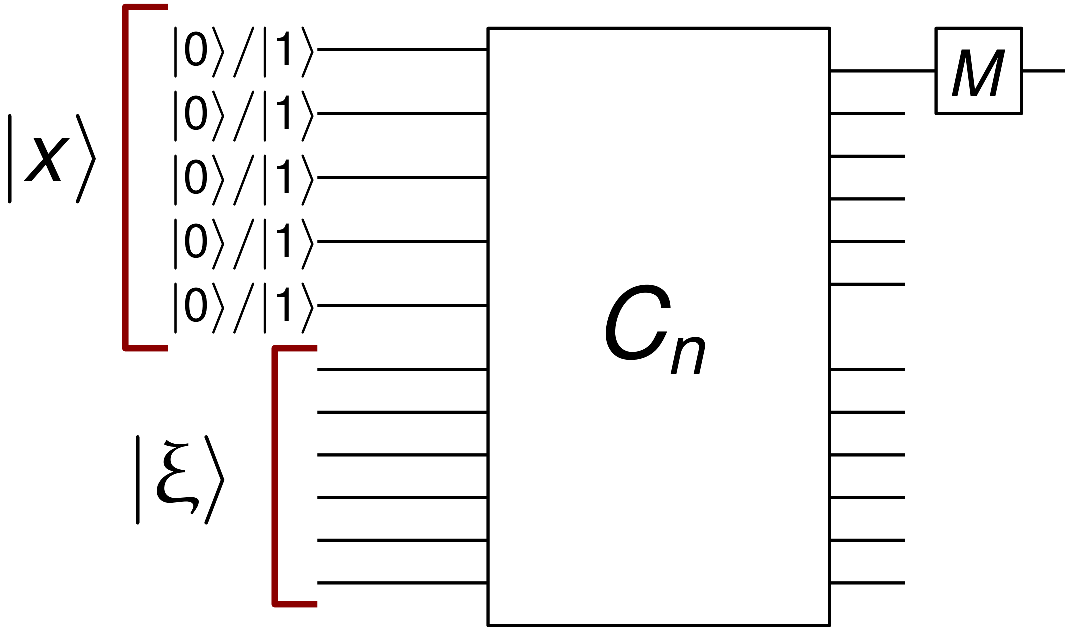
Enforce that the computation accepts:



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Input $x = x_1 x_2 \cdots x_n$

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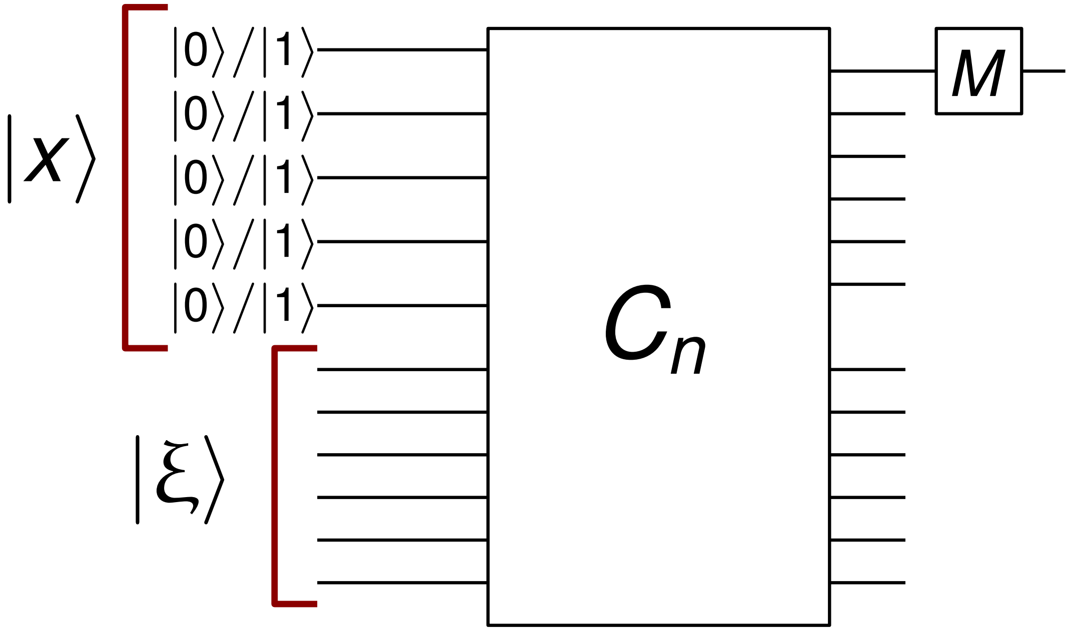

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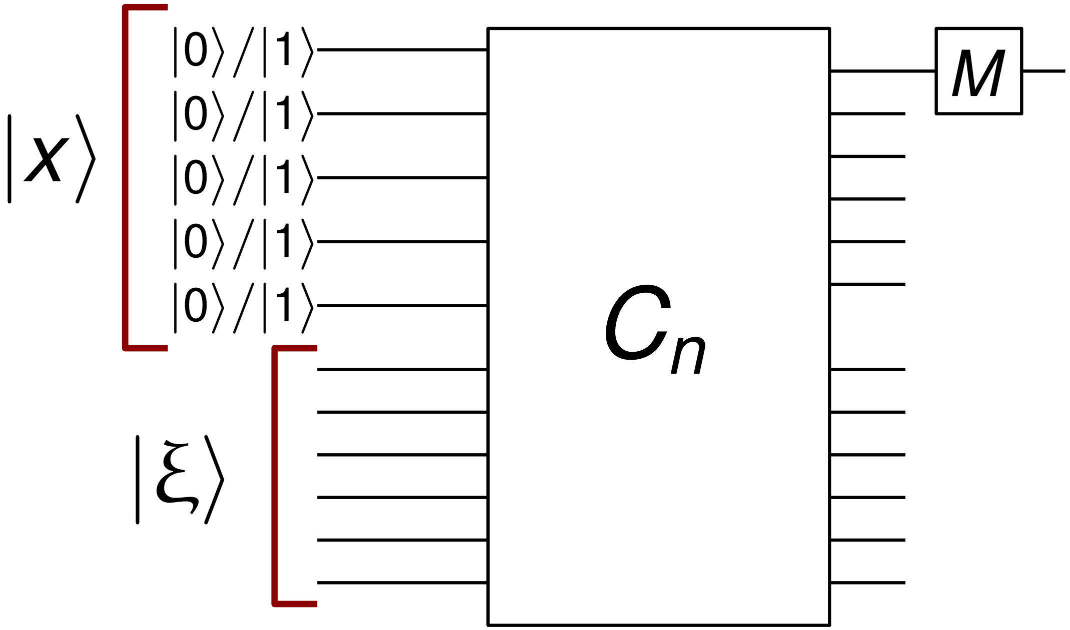
$$H = H_{prop} + H_{init} + H_{out}$$



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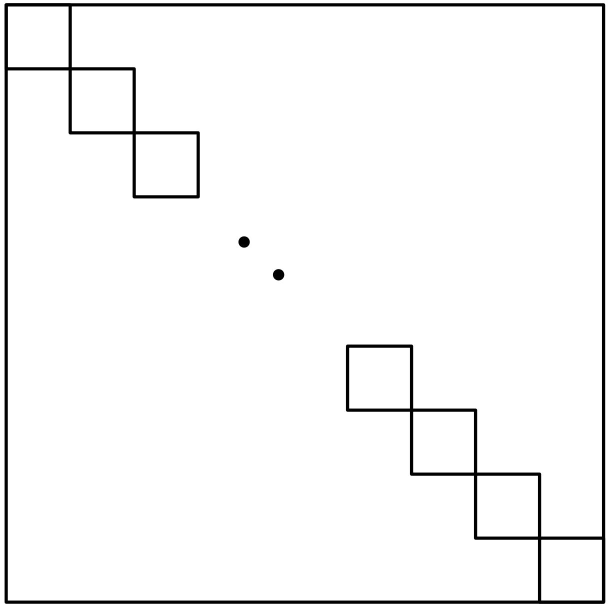

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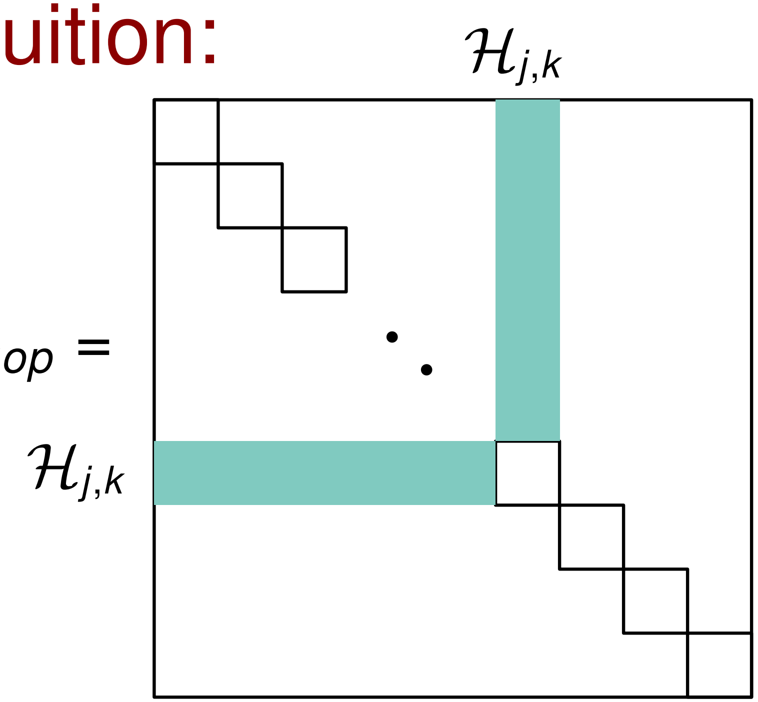

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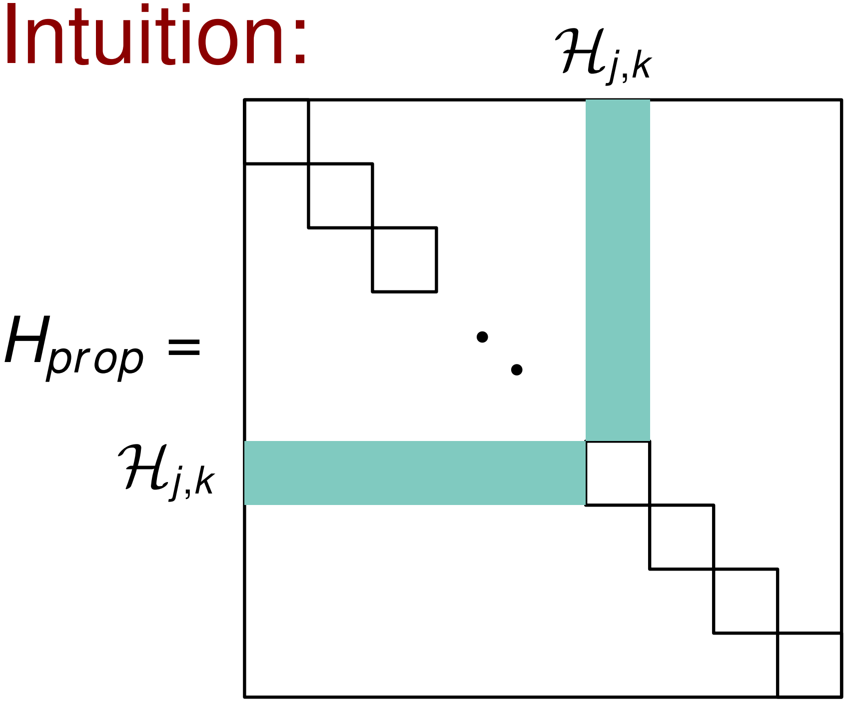

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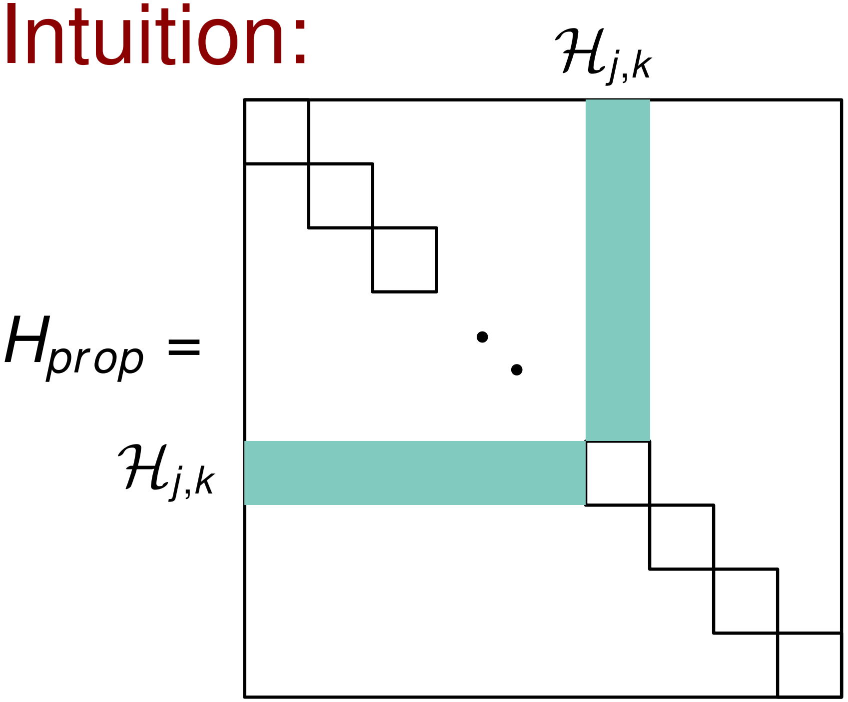

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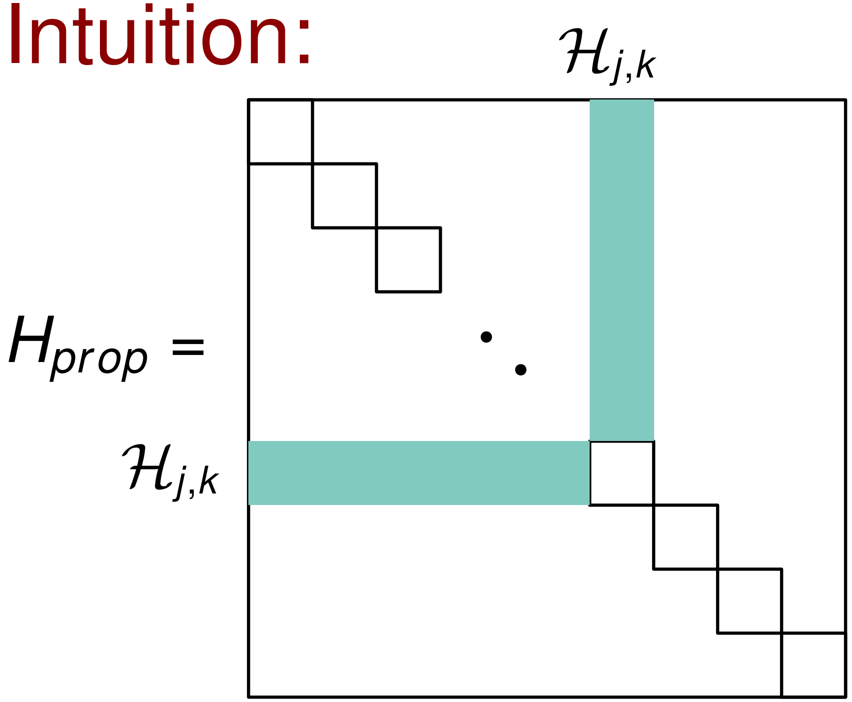

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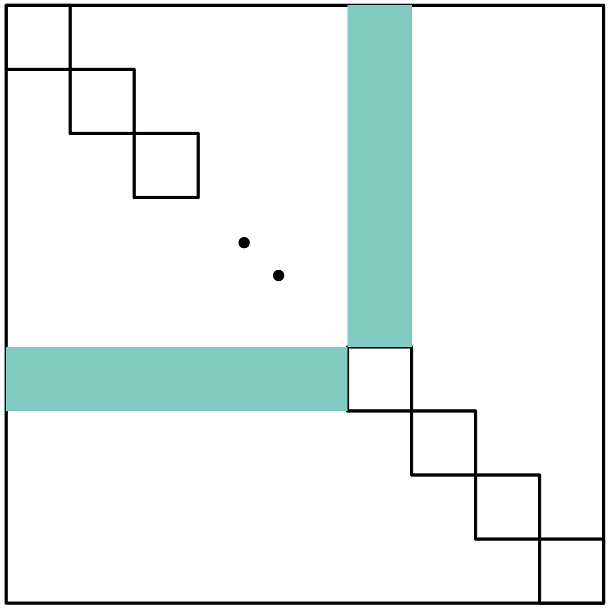

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$\Rightarrow$

lower bound for
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eigenvalue of H

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H_1 and H_2 are Hermitian positive semi-definite matrices.
Let N_1 and N_2 be the null spaces for H_1 and H_2.

If the second eigenvalue of H_1 and H_2 is $\geq \lambda$
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Apply with
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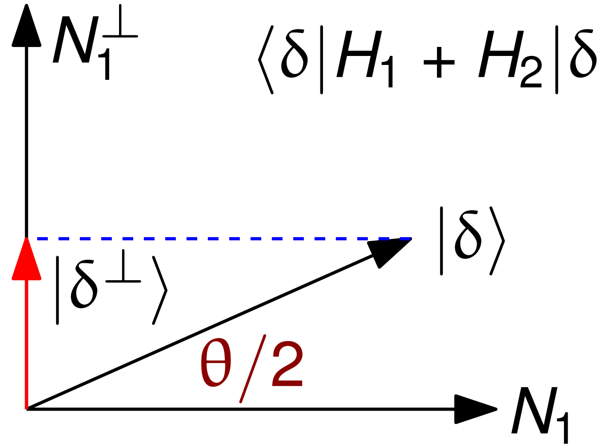

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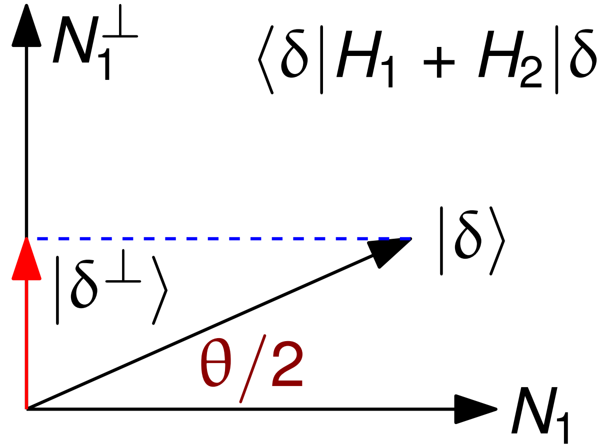

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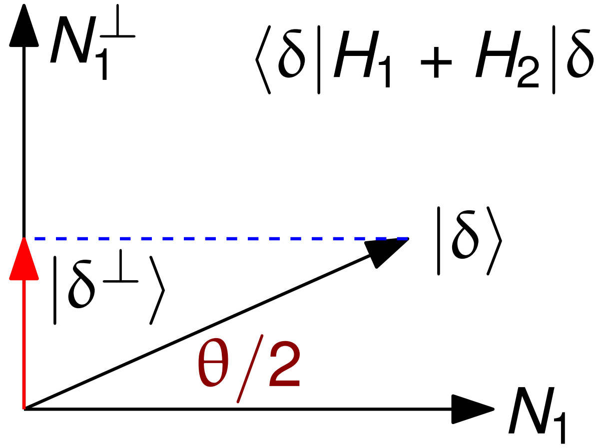

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$H_{init} + H_{out}$ is diagonal in the standard basis with integer entries.

$\Rightarrow$ second largest eigenvalue $\geq 1 \geq \lambda$

The Angle between N_{prop} and N_{IO}

Take generic ground state of H_{prop}:

$$|\eta\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t \cdots U_1 \underbrace{\left(\sum_{j,k} \alpha_{j,k} |a_j\rangle |\xi_k\rangle \right)} \otimes |t\rangle$$

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$\max_\eta \| \Pi |\eta\rangle \|^2 = cos^2 \theta$

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$(I - \Pi)|x\rangle|y\rangle|t\rangle$: $= 1$ if

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Penalty for rejecting computation (H_{out})

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The Angle between N_{prop} and N_{IO}

Take generic ground state of H_{prop}:

correct input

incorrect input

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The Angle between N_{prop} and N_{IO}

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Let Π be the projector onto the ground space of $H_{init} + H_{out}$.

$\max_\eta \| \Pi |\eta\rangle \|^2 = cos^2\theta$

$\sin^2 \frac{\theta}{2} \geq \frac{1}{8(T+1)}$

$\min_\eta \| (I - \Pi) |\eta\rangle \|^2 = sin^2\theta$ $\geq \frac{1}{T+1} \left(1 - \frac{1}{2^n}\right) \geq \frac{1}{2(T+1)}$

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If $x \neq L$, lowest eigenvalue of $H_{prop} + H_{init} + H_{out} \geq \lambda sin^2 \frac{\theta}{2}$

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Recall $E = \frac{1}{2^n(T+1)}$

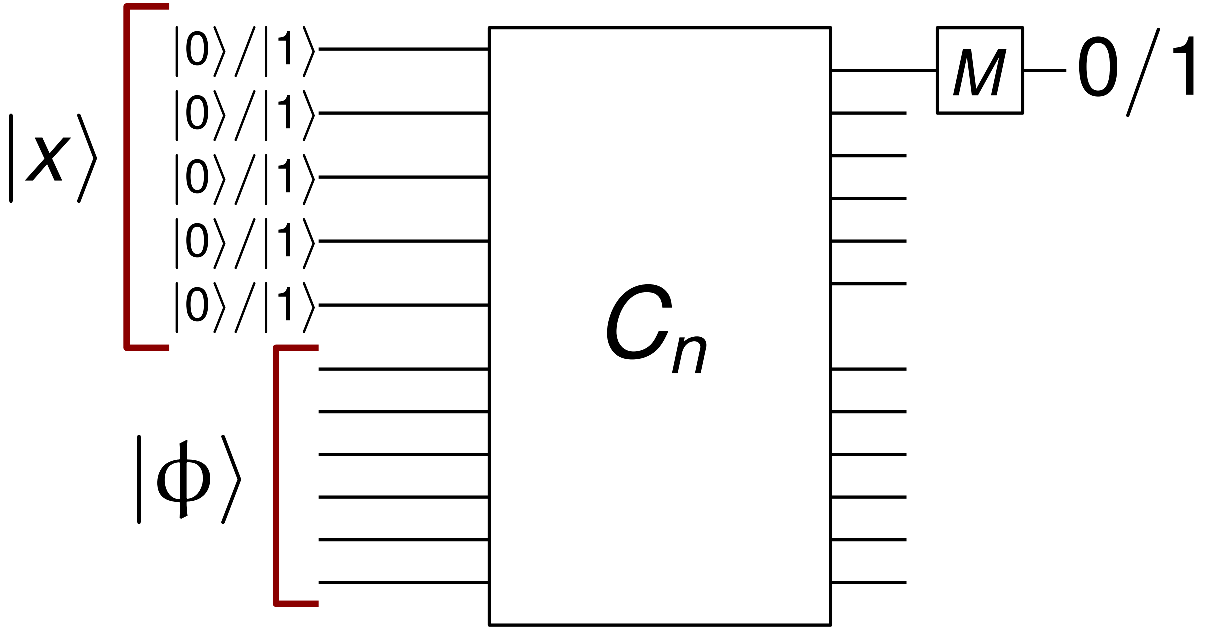
Choose

$E + \Delta = \frac{1}{16(T+1)^3}$

Recap: Local Hamiltonian is QMA-hard

Start with a generic language L in QMA

Is $x \in L$?



Is there a quantum state $\phi\rangle$
that causes this quantum circuit
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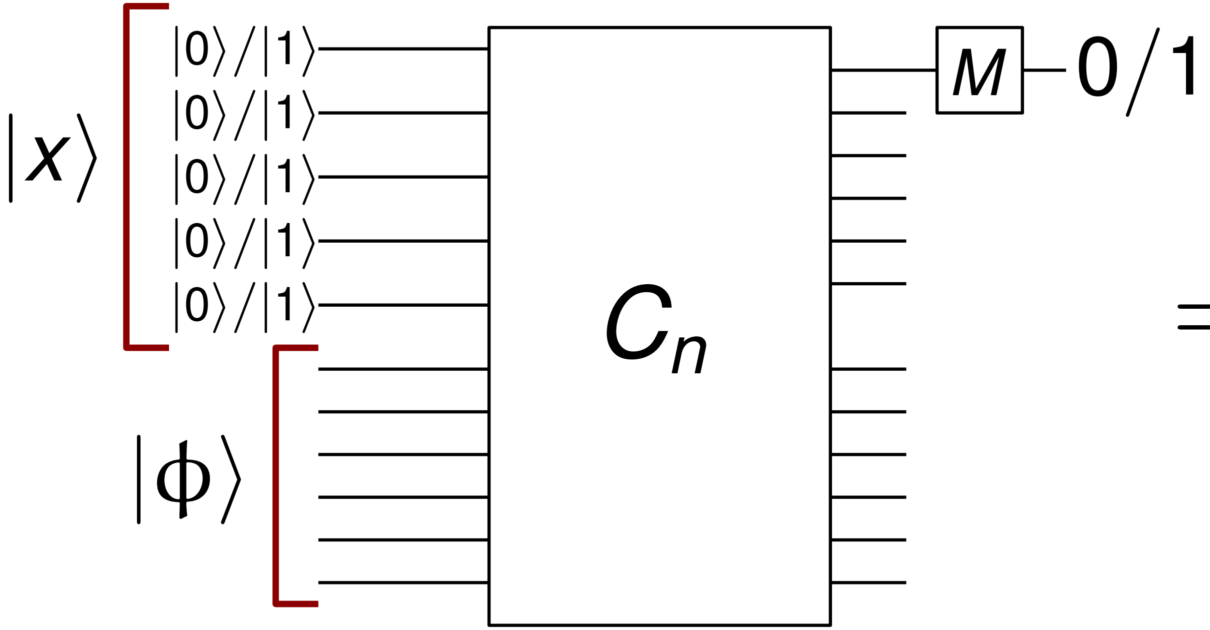

$\Rightarrow$

k-Local Hamiltonian: (H_x, E, Δ)

$H_x = H_{init} + H_{prop} + H_{out}$

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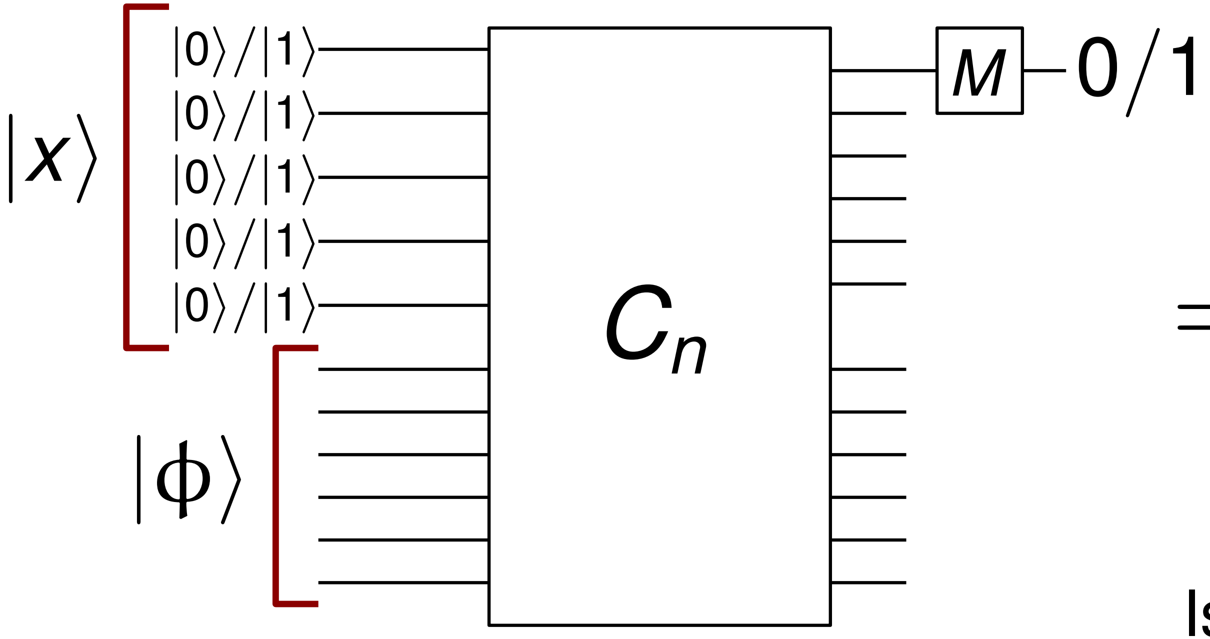

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$\Leftrightarrow$

Is the ground energy of H_x $\leq E$ or $\geq E + \Delta$?

$$E = \frac{1}{2^n}$$

$$\Delta = \frac{1}{16(T+1)^3} - \frac{1}{2^n}$$

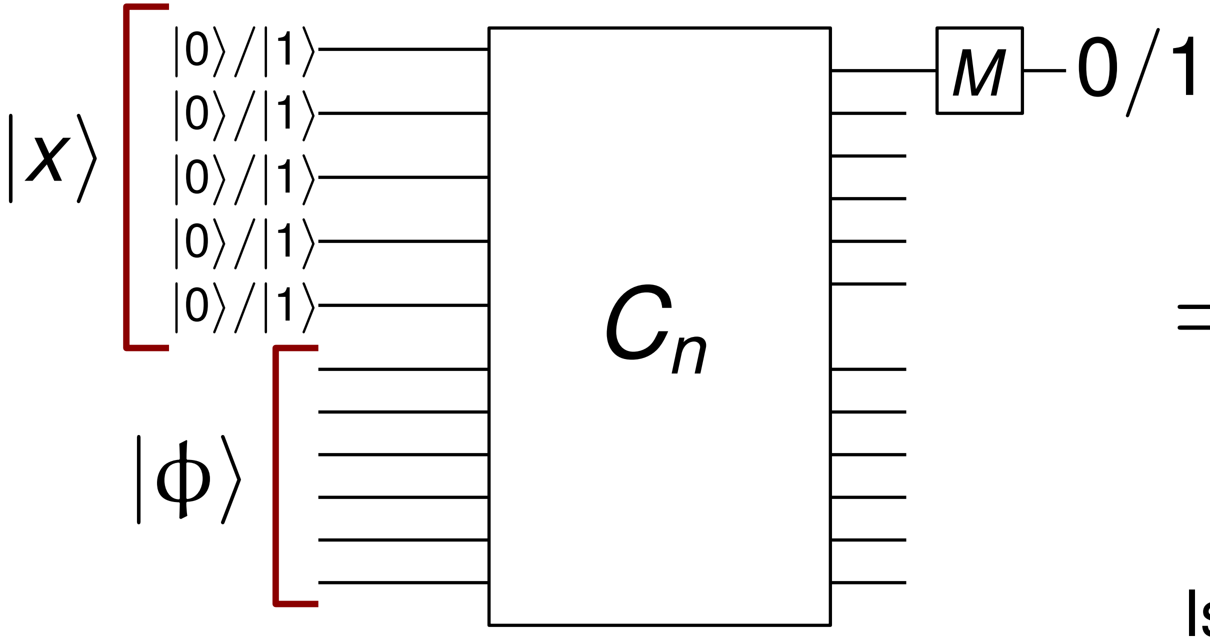
Recap: Local Hamiltonian is QMA-hard [Kitaev 1995]

Start with a generic language L in QMA

Is $x \in L$?

Start with $k = O(\log n)$

Then improve to $k = 5$



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The Clock Register

$$H = \sum_{j=0}^{T-1} \frac{1}{2} \left(|t\rangle\langle t| + |t+1\rangle\langle t+1| - |t\rangle\langle t+1| - |t+1\rangle\langle t| \right)$$

The Clock Register

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Binary Clock:

$$|t\rangle\langle t+1| \quad \Leftrightarrow |01111111\rangle\langle 10000000|$$

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$$|t\rangle\langle t+1| \quad \Leftrightarrow |100\rangle\langle 110|_{t-1,t,t+1}$$

Applied to qubits $t-1$, t, and $t+1$
of the clock register

5-local Hamiltonian

$$H_t = \frac{1}{2}\left[I \otimes |t\rangle\langle t| + I \otimes |t-1\rangle\langle t-1| + \underbrace{U_t}_{\text{2 qubits}} \otimes \underbrace{|t\rangle\langle t-1|}_{\text{3 qubits}} - U_t^\dagger \otimes |t-1\rangle\langle t|\right]$$

$$H \sum_{t=1}^{T} H_t$$

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$$H_{init} = \sum_{j=1}^{n} |\overline{x_j}\rangle\langle\overline{x_j}|_j \otimes |0\rangle\langle 0|_{clock}$$

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2 qubits

3 qubits

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$$H_{out} = |0\rangle\langle 0|_1 \otimes \underbrace{|T\rangle\langle T|_{clock}}$$

$$|1\rangle\langle 1|_{T-clock}$$

Applied to qubit *T* of clock

One More Set of Terms

Valid Clock State:

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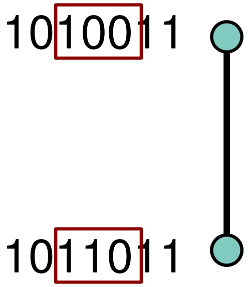
Forbidden States: In general $|ab\rangle\langle ab|_{i,j}$ gives an energy penalty to any state with qubits i and j in state $|ab\rangle$.

Clock Configuration Graph

Vertices: $\{0,1\}^T$ (Standard basis of all clock states)

Edge (x, y) if a propogation term converts x to y

$$(|100\rangle\langle 100| + |110\rangle\langle 110| - |100\rangle\langle 110| - |110\rangle\langle 100|)_{3,4,5}$$



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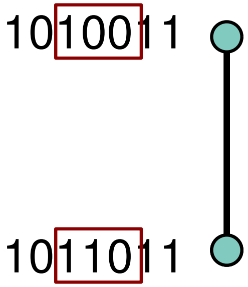

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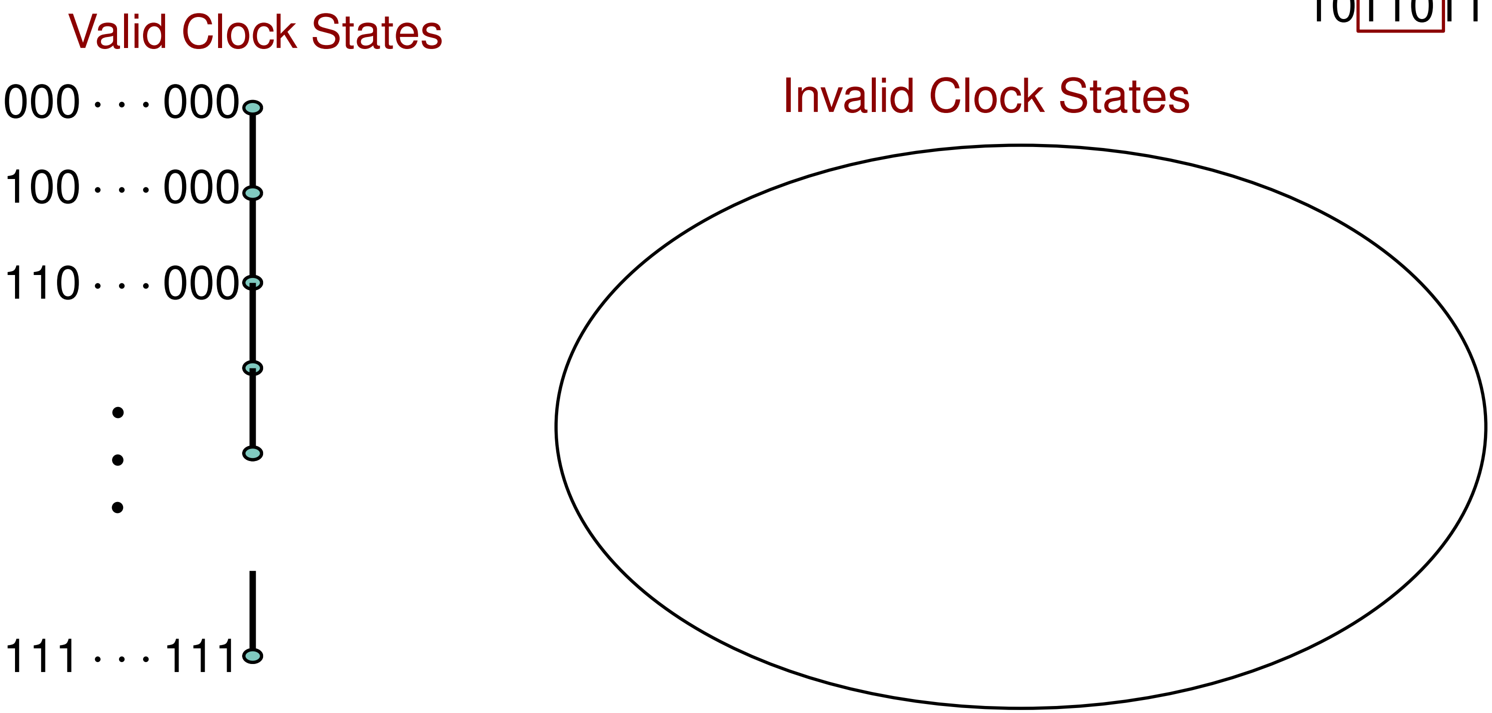
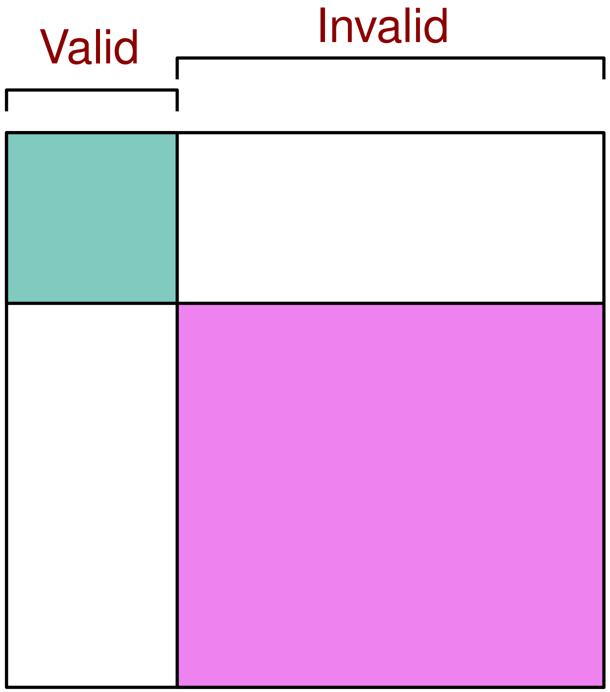


The Propogation Term in Matrix Form

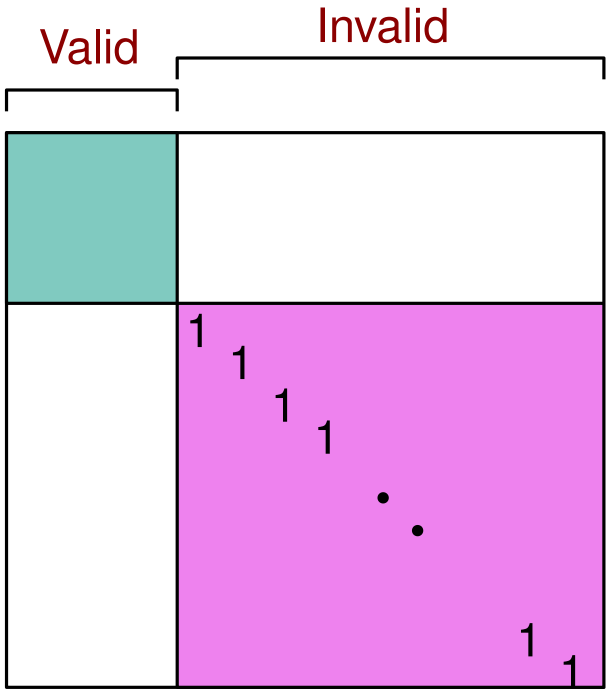
$H_{prop} =$



The Propogation Term in Matrix Form

$H_{prop} =$

$+H_{valid}$



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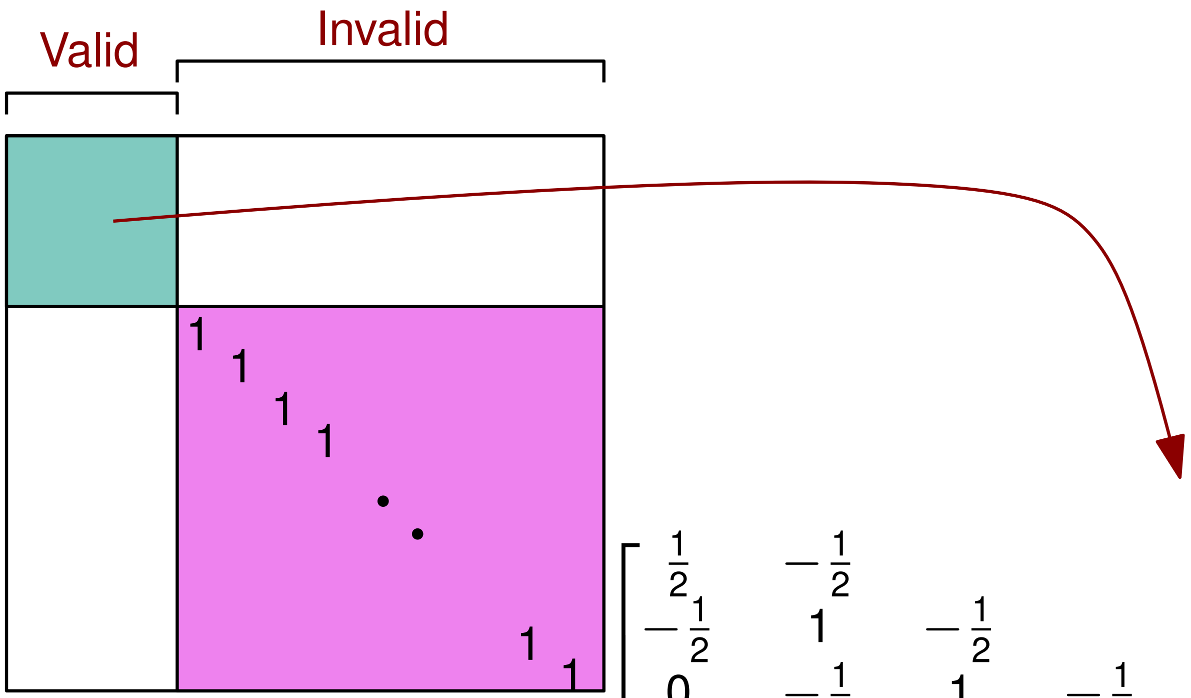

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & \ddots & & & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Why not a 2-qubit clock term?

$$(|10\rangle\langle 10| + |11\rangle\langle 11| - |10\rangle\langle 11| - |11\rangle\langle 10|)_{t,t+1}$$

1111100000 ●

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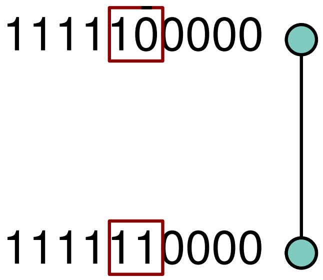

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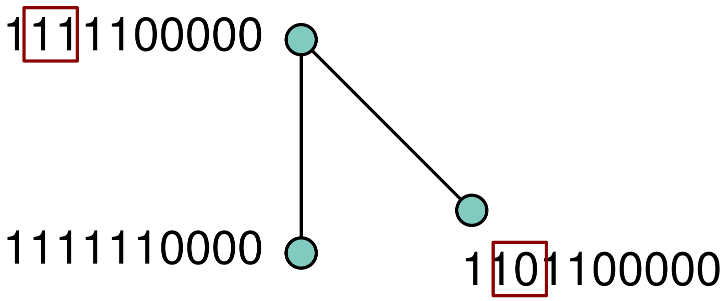

Thank You!