Hamiltonian Complexity, Part IV

due to Commuting Local Hamiltonian Problem.

August 3, 2023
Outline

I: Intro, Motivation, Survey.
II: Statement of the Structural Lemma.
III: 2- Local CLH.
IV: 4- Local 2D CLH on qubits.
V: Structural Lemma Proof Sketch.
Commuting local Hamiltonians system of $n$ $d$-dimensional particles.

$$H = \sum_a H_a \quad H_a \text{ is } k\text{-local}$$

Local terms are pairwise commuting:

$$\forall a, b. \quad H_a H_b = H_b H_a.$$  

If $H$ is a commuting LH (CLH) then all the $H_a$'s can be diagonalized in a single basis.

$$H = \sum \lambda_i |\phi_i\rangle \langle \phi_i| \quad \Rightarrow \quad H_a |\phi_i\rangle = \lambda_{a,i} |\phi_i\rangle$$

$$\lambda_i = \frac{1}{n} \lambda_{a,i}$$
For the purposes of NP + above, it suffices to consider the case where terms are projectors.

\[ \exists \phi \text{ such that } \langle \phi | H | \phi \rangle \leq T \]

\[ \iff \exists \lambda_1 \ldots \lambda_r \text{ and } | \phi \rangle \text{ such that } \sum_a \lambda_a \leq T \]

and \( H_a | \phi \rangle = \eta_a | \phi \rangle \forall a \).

\[ \iff \exists \lambda_1 \ldots \lambda_r \text{ and } | \phi \rangle \text{ such that } \sum_a \lambda_a \leq T \]

and \( T \eta_a | \phi \rangle = 0 \)

where \( T \eta_a = 1 - P_a \)

A "solution" is a frustration-free ground state.

\( P_a \) projector onto \( \eta_a \)-eigenstate of \( H_a \).
Reasons to be interested in CLH:

- Intermediate class between classical and quantum.
  - Eigenstates (up to degeneracies) can be described by eigenvalues for each term.
  - Eigenstates can be highly entangled.
- Stabilizer codes are ground states of commuting Hams.
- Test ground for proving difficult claims (e.g. qPCP)
- Easier case for studying gapped Hamiltonians
  - Can ground states be efficiently represented or constructed?
Special Cases Known to be in NP

- 2-Local  [Bravyi-Vyalyi]
- 3-Local, Qubits + Qubits  [Aharonov-Eldar]  \( \xrightarrow{\text{"Nearly Euclidean"}} \)
- 2D - qubits  [Schnh] (non-constructive)
  [Aharonov, Kenneth, Vigdorovich] (constructive).
- 2D - qubits  [I., Jiang]  Factorized: every term is a product of operators on individual particles.
- Factorized - qubits  [BV]
- Factorized - 2D  [I., Jiang]

Is general CHSH in NP? QCHA? or QHA-hard?
II) The Structural Lemma

$A$ acts on $H_x \otimes H_y$

$B$ acts on $H_y \otimes H_z$.

$A$ and $B$ commute.

then:

$H_y = \bigoplus_{\alpha} H_{y \alpha}$

1. $A + B$ are invariant on each $H_{y \alpha}$

2. $H_{y \alpha} = H_{y \alpha}^A \otimes H_{y \alpha}^B$

$A|_{H_{y \alpha}}$ acts on $H_x \otimes H_{y \alpha}$, $A$

$B|_{H_{y \alpha}}$ acts on $H_{y \alpha}^B \otimes H_z$
The Structural Lemma

\[ H_y = \bigoplus \alpha H_{y\alpha} \]

1. \( A + B \) are invariant on each \( H_{y\alpha} \)

\[ A = \frac{1}{\alpha} P_{y,\alpha} A P_{y,\alpha} \quad \Rightarrow \text{Projector onto } H_{y,\alpha}. \]

\[ \Rightarrow \text{If a solution exists, then there is a solution entirely within one } H_{y,\alpha}. \]
The Structural Lemma

$A \Rightarrow B$

2. $H_x = H_y \otimes H_z$

$A|y_x$ acts on $H_x \otimes H_y, A$

$B|y_x$ acts on $H_y, B \otimes H_z$

$A|y_x^A$

$B|y_x^B$
Structural Lemma Holds for more than 2

Commuting terms:

\[ y = \bigoplus_{\alpha} Y_\alpha. \]

- \( Y, B, C \) all invariant on \( Y_\alpha \)
- Within \( Y_\alpha \) : tensor product structure.
Structural Lemma Implications

For a 2-local commuting Hamiltonian

NP witness consists of the description of a "slice" of each particle.

Solution within the slices has a tensor-product structure.
Particles
Particles

Witness: which slice to take for each particle
Particles

Witness: which slice to take for each particle

Looking at the chosen slices:

Solution is tensor product of states that span pairs of particles.
CLH on 2D lattice (4-local)

Particles at grid vertices
Terms are vertices on a face.

Particles on edges
Star & plaquette terms.
CLH on 2D lattice (4-local)

Particles at grid vertices
Terms are vertices on a face.

Particles on edges
Star & plaquette terms.
CLH on 2D lattice ($4$-local)

Particles at grid vertices
Terms are vertices on a face.

Particles on edges
Star & plaquette terms.
CLH on a 2D lattice

The ground states of these Hamiltonians will not have a "local" structure as in the 2-local case.

\[ \Rightarrow \text{Toric code is a special case.} \]

Red terms: \(XXX\)
CLH on a 2D lattice

The ground states of these Hamiltonians will not have a "local" structure as in the 2-local case.

⇒ Toric code is a special case.

Red terms: \(XXX\)  
Blue terms: \(ZZZZ\)

\(\Rightarrow\) ground state will have global entanglement.
we will consider general commuting 4-local Hamiltonians on a 2D lattice of qubits.
CLH on a 2D lattice

we will consider general commuting 4-local Hamiltonians on a 2D lattice of qubits.

Checkerboard Pattern:
- Blue faces
- Red faces

Will show CLH for this case is in NP [Schuch]

arXiv: 1105.2843
CLH on a 2D lattice of Qubits

\[ H = \sum_i \lambda_i \ell_i = \sum_i (\ell_i - \Pi_i) \]

\( \ell \) projector onto ground space for term \( \ell_i \).

Let

- \( B = \) set of blue faces.
- \( R = \) set of red faces.

\[ P_B = \prod_{i \in B} \Pi_i \quad P_R = \prod_{i \in R} \Pi_i \]

This will be non-constructive!

Want to show: \( \text{Tr} (P_B P_R) > 0 \)
Blue faces overlap on a single qubit:

Can apply structural lemma

\[ H_q = \bigoplus H_{q,i} \]

\( P_i + P_j \) are invariant on each \( H_{q,i} \)

Let \( P_{q,i} \) be the projector onto \( H_{q,i} \)

\[ P_i = \sum_{\alpha} P_{q,i} \alpha P_i P_{q,i} \alpha \quad (\text{same for } P_j) \]
Let \( \vec{\alpha} \) = vector of indices for all the qubits.

\[ P_{\vec{\alpha}}^{i} = \prod_{k} P_{\alpha_{k}} \text{if } \vec{\alpha} \text{ is a } \vec{\beta} \]

\[ P_{\text{Blue}} = \prod_{i \in B} P_{i}^{\vec{\alpha}} \quad P_{\text{Blue}} = \sum_{\vec{\alpha}} P_{\vec{\alpha}}^{\frac{1}{2}} P_{\text{Blue}}^{\frac{1}{2}} \]

Same will hold for the red terms:
except that it will be a different direct sum for each qubit: \( \vec{\beta} \)
Want to show

$$\sum_{\alpha, \beta} \text{Tr} \left[ \left( \prod_{i \in B} \hat{P}_i^{\alpha} \right) \left( \prod_{i \in R} \hat{P}_i^{\beta} \right) \right] > 0$$

each individual term is $\geq 0$

NP prover gives $\hat{x}$ and $\hat{p}$ for which trace $> 0$

This witness doesn't necessarily say much about the ground state.
Example: Toric Code

Blue terms: XXXX
Red terms: ZZZZ

\[ \mathcal{L} = \zeta +, - 3^N \]
\[ s \beta = 3^0, 1 3^N \]
\[ \hat{\mathcal{L}} = (+, +, \ldots, +) \hat{\beta} = (0, 0, \ldots, 0) \]

\[ \text{Tr} \left[ 1^N \langle + | \otimes N | 0^N \rangle \otimes N \right] = 2^{-N} > 0 \]

\[ \alpha_q = \pm \alpha \]

\[ P_i + P_j \text{ invariant on } \mathcal{H}_q^+ \]
\[ \mathcal{H}_q^- \]

Similarly: for red terms
\[ f_q = 0 \text{ or } \pm 1. \]
Back to Geneve: $4$-local qubit CTH in 2D:

There are two ways for $H_q$ to be divided:

1. $(1,1)$ - way:

$H_q = H_{q,1} \oplus H_{q,2}$

$\dim 1 \quad \dim 1$

$\hat{2}$ will project on to $1$-dim space

2. $(2)$ - way:

Trivial partition.

$P_i$ & $P_j$ operate on disjoint portions of the space

$\Rightarrow$ only one acts non-trivially on $q$. 
if either \((P_1, P_2)\) or \((P_1', P_2')\) commute in a \((1,1)\)-way then we can trace out qubit \(q\).

Why?

Need to show:

\[
\text{Tr} \left[ \left( \prod_{i \in B} P_i^{\alpha} \right) \left( \prod_{i \in R} P_i^{\beta} \right) \right] > 0
\]

\(P_1, P_2, P_1', P_2'\) only terms operating on \(q\).

Also: \(P_{q,\alpha}\) and \(P_{q,\beta}\).
Need to show:

\[
\text{Tr} \left[ \left( \prod_{i \in B} P_i^2 \right) \left( \prod_{i \in R} P_i^2 \right) \right] > 0
\]

\[P_1, P_2, P_1', P_2'\] only terms operating on \(\mathfrak{g}\).

Also: \(P_{q,1}\) and \(P_{q,1}'\).

Case 1: \((P_1, P_2)\) and \((P_1', P_2')\) are both \((1,1)\)

then \(P_{q,1} = |\phi\rangle \langle \phi|_{\mathfrak{g}}\), \(P_{q,1}' = |\psi\rangle \langle \psi|_{\mathfrak{g}}\).

\[P_{q,1} P_1 P_{q,1} = |\phi\rangle \langle \phi| \otimes |\phi\rangle \langle P_1 | \langle \phi\rangle \] (same for \(P_2\))

\[P_{q,1}' P_1' P_{q,1}' = |\psi\rangle \langle \psi| \otimes |\psi\rangle \langle P_1' | \langle \psi\rangle \] (same for \(P_2'\)).
Need to show:

\[
\text{Tr} \left[ \prod_{i \in B} P_i^{\alpha} \right] \left( \prod_{i \in R} P_i^{\beta} \right) > 0
\]

\( P_1, P_2, P_1', P_2' \) only terms operating on \( q \).

Also:

\( P_{q,1,\alpha} \) and \( P_{q,1,\beta} \).

Case 1:

\( (P_1, P_2) \) and \( (P_1', P_2') \) are both \((1,1)\)

then \( P_{q,1,\alpha} = 1\phi < \phi | q \) \hspace{1cm} P_{q,1,\beta} = 1\psi < \psi | q \).

\( P_{q,1,\alpha} \) and \( P_{q,1,\beta} = 1\phi < \phi | q \) \hspace{1cm} \phi | P_1 | \phi \) \hspace{1cm} (Same for \( P_2 \)).

\( P_{q,1,\alpha} \) and \( P_{q,1,\beta} = 1\psi < \psi | q \) \hspace{1cm} \psi | P_1' | \psi \) \hspace{1cm} (Same for \( P_2' \)).
Need to show:

\[ \text{Tr} \left[ \left( \prod_{i \in B} P_i^2 \right) \left( \prod_{i \in R} P_i^3 \right) \right] > 0 \]

\( P_1, P_2, P_1', P_2' \) only terms operating on \( q \).

Also: \( P_{q,1,\alpha} \text{ and } P_{q,1,\beta} \).

**Case 2:** \( (P_1, P_2) \text{ is } (1,1) \) \( (P_1', P_2') \text{ is } (2) \)

\[ \Rightarrow P_2' \text{ is identity on } q. \]

\[ \text{Tr} \left[ P_{q,1,\alpha} P_1 P_{q,1,\alpha} \cdots P_{q,1,\beta} P_2 P_{q,1,\beta} \cdots P_1' \cdots \right] \]

(all other terms Identity on \( q \))
Need to show:

\[
\text{Tr} \left[ \left( \prod_{i \in B} P_i^2 \right) \left( \prod_{i \in R} P_i^3 \right) \right] > 0
\]

\[ P_1, P_2, P_1', P_2' \text{ only terms operating on } g. \]

Also: \( P_{g,1,0} \) and \( P_{g,1,p} \).

Case 2: \( (P_1, P_2) \) is \((1,1)\) \hspace{1cm} \( (P_1', P_2') \) is \((2)\)

\[ \Rightarrow P_2' \text{ is identity on } g. \]

\[
\text{Tr} \left[ P_{g,1a} P_1 P_{g,1a} \cdots P_{g,1a} P_2 P_{g,1a} \cdots P_{g,1a} P_1' P_{g,1a} \cdots \right]
\]

(all other terms Identity on \( g \))

\[ |\psi\rangle \langle \psi| \otimes [\text{Id on } g] \]

After tracing those out, only left with:

\[
\begin{array}{c|c}
P_1 & P_2' \\
\hline
P_1' & P_2 \\
\end{array}
\]

\[ (P_1, P_2) \xrightarrow{q} (P_1', P_2') \]

both pairs commute in \((2)\)-way.
Put a dot • in the corner if term acts non-trivially on that qubit. Otherwise put an ×

two terms "overlap" if they act non-trivially on the same qubit.

⇒ Overlapping terms form chains (no branching)
Cannot have structures like:

Case analysis, using the fact that the following two structures can't happen:

If two terms overlap on a single qubit, they cannot both have a dot at that qubit.
Now just need to determine trace of products of terms forming chains or cycles:

\[ \text{Tr}(P_1 P_2 \cdots P_8) \]
II) Structural Lemma Proof Sketch

A C*-Algebra is a Banach algebra with *-op.

For us: $A \subseteq \mathcal{L}(\mathcal{H})$

closed under, $+, \cdot, \ast$, scalar mult.
contains $I$.

The \textbf{center of $A$, $C(A)$}, is the set of all $x \in A$ that
with everything in $A$.

If $C(A) = \emptyset$ i.e. $A$ has a "trivial" center
then $A = \mathcal{L}(\mathcal{H}_a) \otimes \mathcal{L}(\mathcal{H}_b)$ $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. 
Structural Lemma Proof Sketch

Lemma: If \( \exists M \in C(A) \) such that \( M \neq I \) then \( M = \sum \lambda_i T_i \) where \( \lambda_i \) projects on to \( T_i \). Projector onto eigenspaces of \( M \).

and for \( N \in A \) \( N \) is invariant on \( T_i \).

Proof idea: need to show \( T_i \in C(A) \).

If \( M \in C(A) \) then \( p(M) \in C(A) \) for polynomial \( p \).

Find \( p_i(x) \) such that \( p_i(\lambda_i) = 1 \).

\( p_i(x_j) = 0 \quad j \neq i \) \( \Rightarrow p_i(M) = T_i \)
Structural Lemma Proof Sketch

Idea: Find $M \in C(A)$ $M \neq I$

use $M$ to divide up $\mathcal{H} = \bigoplus \mathcal{H}_i$.

if $A|\mathcal{H}_i$ does not have a trivial center, repeat on $A|\mathcal{H}_i \leq \mathcal{A}(\mathcal{H}_i)$.

$\Rightarrow$ end up with: $\mathcal{H} = \bigoplus \mathcal{H}_i$.

$A|\mathcal{H}_i = \mathcal{L}(\mathcal{H}_i) \otimes \mathcal{H}_i^b$.

$\mathcal{H}_i = \mathcal{H}_i^a \otimes \mathcal{H}_i^b$.

every $N+1$ in variant on $\mathcal{H}_i$.

$A|\mathcal{H}_i$ has a trivial center.
Structural Lemma Proof Sketch

If $A + B$ commute then $\hat{A}$ and $\hat{B}$ commute.
Structural Lemma Proof

Sketch

Use $\hat{A}$ to divide $Y$:

$$Y = \oplus Y_i$$

$\hat{A}$ invariant on $Y_i$.

$\hat{A}|_{Y_i}$ is $\text{L}(Yia) \otimes I_{V_Y}$.

C*-algebras: $\hat{A} = \sum A_{\alpha\beta}$

Since $B$ commutes with $A$:

- $B$ is invariant on each $Y_i$.
- $B|_{Y_i} \leq I_{V_{Y_i}} \otimes \text{L}(Yib)$. 

$$A = \sum_{\alpha\beta} |x\rangle \langle y| \otimes A_{\alpha\beta} \otimes I$$

$$A = \chi A_{\alpha\beta}$$

$B$ commutes with $A_\alpha$. 

$$A = \sum_{\alpha\beta} |x\rangle \langle y| \otimes A_{\alpha\beta} \otimes I$$

$B$ is invariant on each $Y_i$. 

$$B|_{Y_i} \leq I_{V_{Y_i}} \otimes \text{L}(Yib)$$