Hamiltonian Complexity, Part IV

the Commuting Local Hamiltonian Problem.

August 3, 2023

Outline

I: Intro, Motivation, Survey.

II: Statement of the Structural Lemma.

III: 2-Local CLH.

IV: 4-Local 2D CLH on qubits.

V: Structural Lemma Proof Sketch.

I) Commuting Local Hamiltonians

system of n d-dimensional particles.

$$H = \sum_a H_a \qquad H_a \text{ is } k\text{-local}$$

Local terms are pairwise commuting:

$$\forall\, a, b. \qquad H_a H_b = H_b H_a .$$

If H is a commuting LH (CLH) then all the H_a's can be diagonalized in a single basis.

$$H = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i| \quad \Rightarrow \quad H_a|\phi_i\rangle = \lambda_{a,i}|\phi_i\rangle$$

$$\lambda_i = \sum_a \lambda_{a,i}$$

For the purposes of NP + above, it suffices to consider the case where terms are projectors.

$\exists\ |\phi\rangle$ such that $\langle\phi|H|\phi\rangle \leq T$

$\iff \exists\ \lambda_1 \cdots \lambda_r$ and $|\phi\rangle$ such that $\sum_a \lambda_a \leq T$

and $H_a|\phi\rangle = \lambda_a|\phi\rangle\ \forall a.$

$\iff \exists\ \lambda_1 \cdots \lambda_r$ and $|\phi\rangle$ such that $\sum_a \lambda_a \leq T$

and $\Pi_a|\phi\rangle = 0$

where $\Pi_a = I - P_a$

P_a projector onto λ_a-eigenspace of H_a.

A "solution" is a frustration-free ground state..

Reasons to be interested in CLH:

- Intermediate class between class between classical and quantum.
    - Eigenstate (up to degeneracies) can be described by eigenvalues for each term.
    - Eigenstates can be highly entangled.
- Stabilizer codes are ground states of commuting Hams.
- Test ground for proving difficult claims (e.g. qPCP)
- Easier case for studying gapped Hamiltonians

  Can ground states be efficiently represented or constructed?

Special Cases Known to be in NP

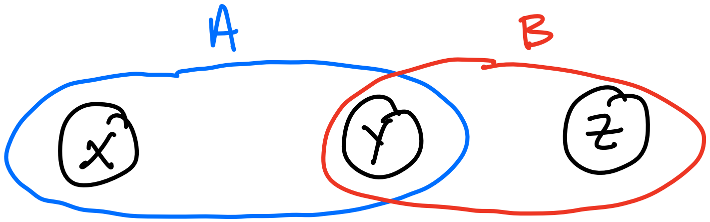
- 2 - Local [Bravyi - Vyalyi]
- 3 - Local, Qubits + Qutrits [Aharonov - Eldar]
  ↳ "Nearly Euclidean"
- 2D - qubits [Schuch] (non - constructive)
  [Aharonov, Kenneth, Vigdorovich] (constructive).
- 2D - qutrits [I., Jiang]
- Factorized - qubits [BV]
- Factorized - 2D [I., Jiang]

Factorized: every term is a product of operators on individual particles.

⇒ Is general CLH in NP? QCMA? or QMA - hard?

II) The Structural Lemma

[BV] arXiv: 0308021.



A acts on $\mathcal{H}_x \otimes \mathcal{H}_y$

B acts on $\mathcal{H}_y \otimes \mathcal{H}_z$.

A & B commute

then: $$\mathcal{H}_y = \bigoplus_\alpha \mathcal{H}_{y_\alpha}$$

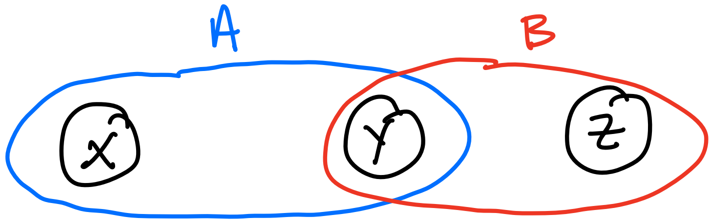
1. A & B are invariant on each $\mathcal{H}_{y_\alpha}$

2. $\mathcal{H}_{y_\alpha} = \mathcal{H}_{y_{\alpha,A}} \otimes \mathcal{H}_{y_{\alpha,B}}$

$A|_{y_\alpha}$ acts on $\mathcal{H}_x \otimes \mathcal{H}_{y_{\alpha,A}}$

$B|_{y_\alpha}$ acts on $\mathcal{H}_{y_{\alpha,B}} \otimes \mathcal{H}_z$

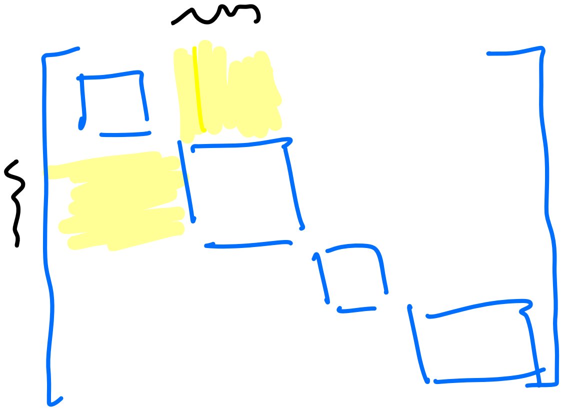
The Structural Lemma [BV]



$$\mathcal{H}_y = \bigoplus_{\alpha} \mathcal{H}_{y\alpha}$$

(1) A & B are invariant on each $\mathcal{H}_{y\alpha}$

$\mathcal{H}_{y\alpha} \otimes \mathcal{H}_x$

$A =$ 

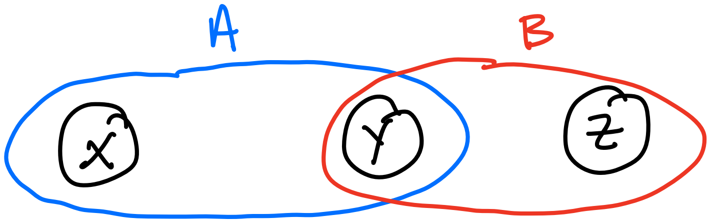
$$A = \sum_{\alpha} P_{y,\alpha} \, A \, P_{y,\alpha}$$

↳ Projector onto $\mathcal{H}_{y,\alpha}$.

(same for B).

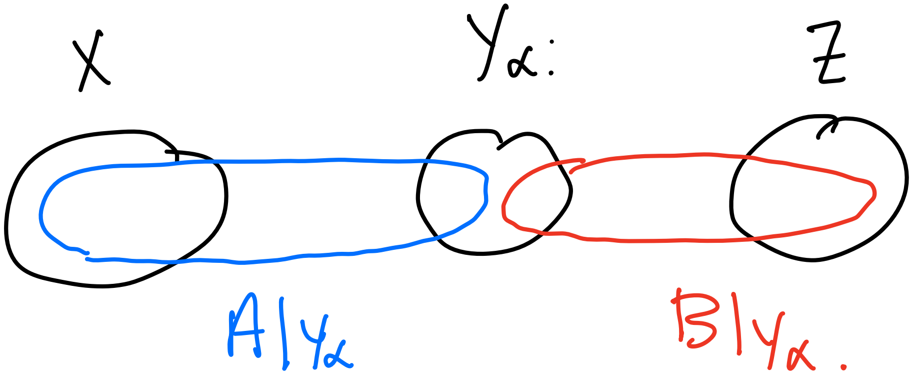
$\Rightarrow$ If a solution exists, then there is a solution entirely within one $\mathcal{H}_{y,\alpha}$.

The Structural Lemma [BV]



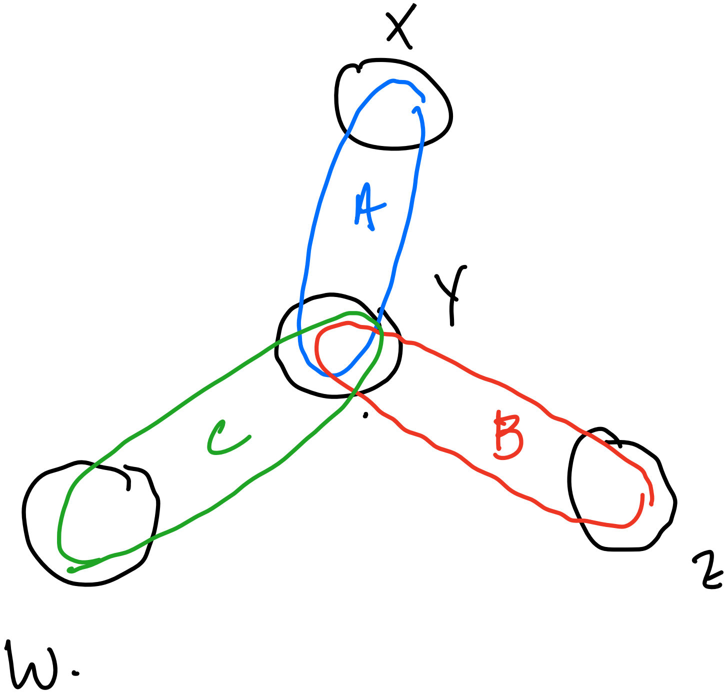
(2) $\mathcal{H}_{Y_\alpha} = \mathcal{H}_{Y_\alpha,A} \otimes \mathcal{H}_{Y_\alpha,B}$

$A|_{Y_\alpha}$ acts on $\mathcal{H}_X \otimes \mathcal{H}_{Y_\alpha,A}$ $\quad$ $B|_{Y_\alpha}$ acts on $\mathcal{H}_{Y_\alpha,B} \otimes \mathcal{H}_Z$

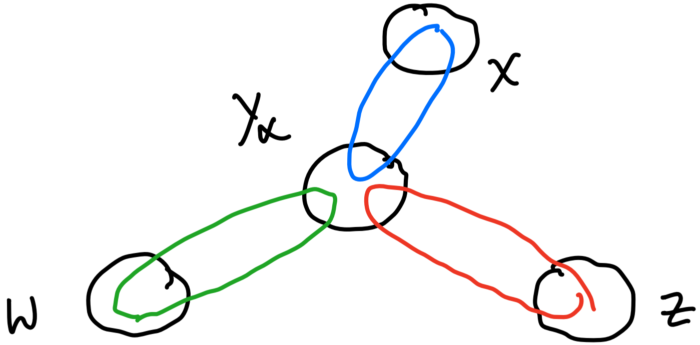


Structural Lemma Holds for more than 2 commuting terms:

$$Y = \bigoplus_{\alpha} Y_\alpha .$$



- A, B, C all invariant on Y_α
- Within Y_α: tensor product structure.

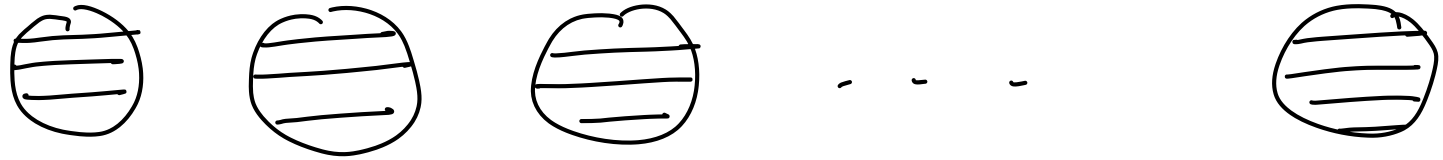


Structural Lemma Implications

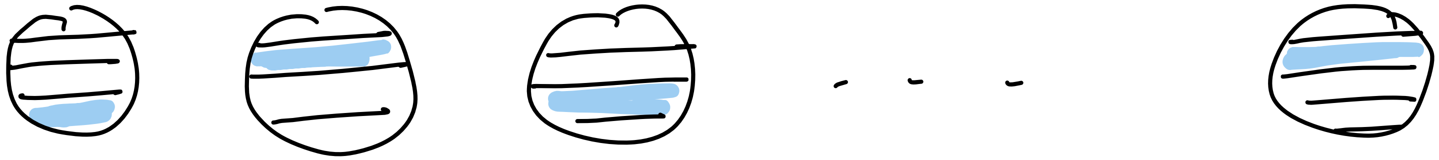
For a 2-local commuting Hamiltonian

NP witness consists of the description of a "slice" of each particle

Solution within the slices has a tensor-product structure

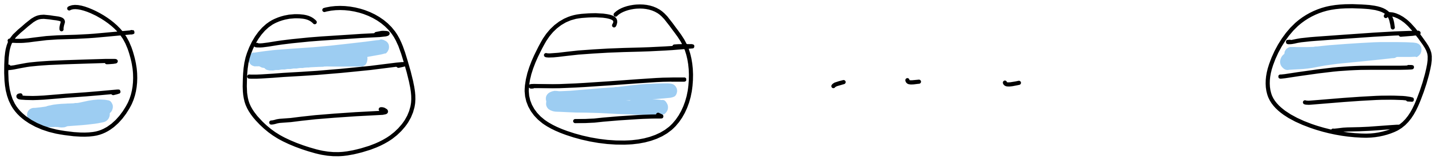


Particles



Particles

Witness: which slice to take for each particle

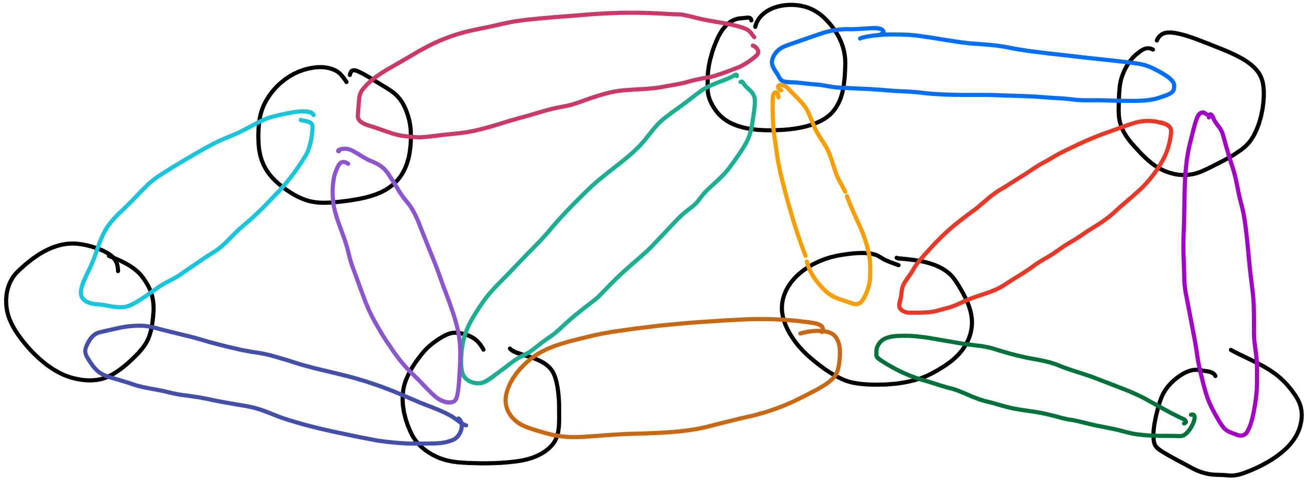


Particles

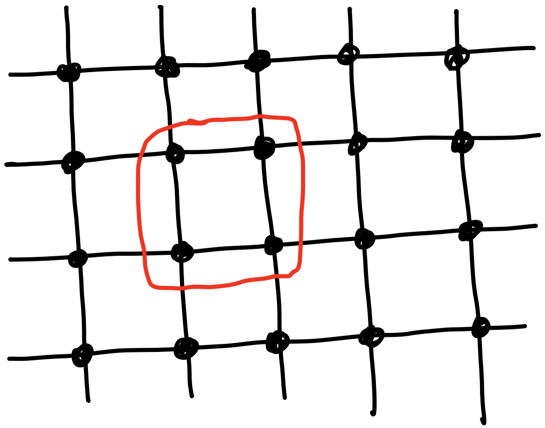
Witness: which slice to take for each particle

Looking at the chosen slices:

Solution is tensor product of states that span pairs of particles.



CLH on 2D lattice (4-local)



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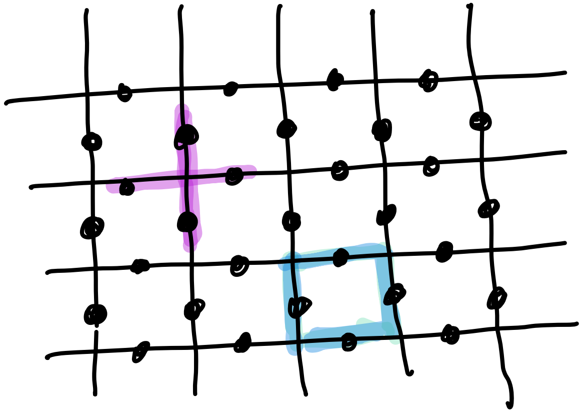


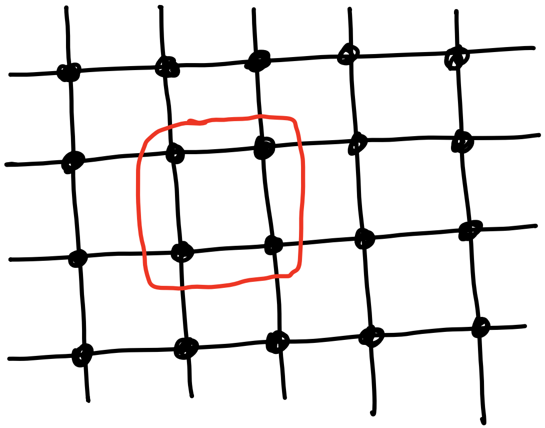
particles at grid vertices

terms are vertices on a face.

particles on edges

star & plaquette terms.

CLH on 2D lattice (4-local)



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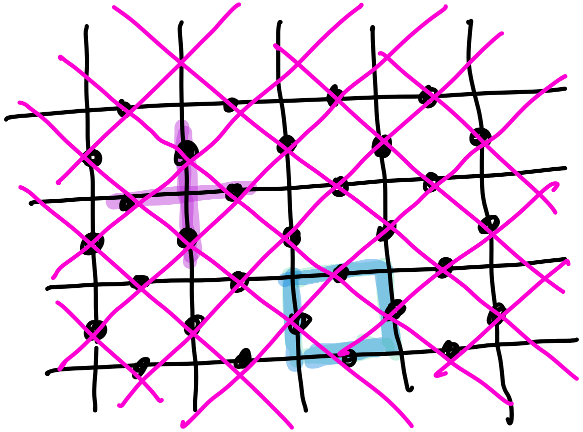

particles at grid vertices

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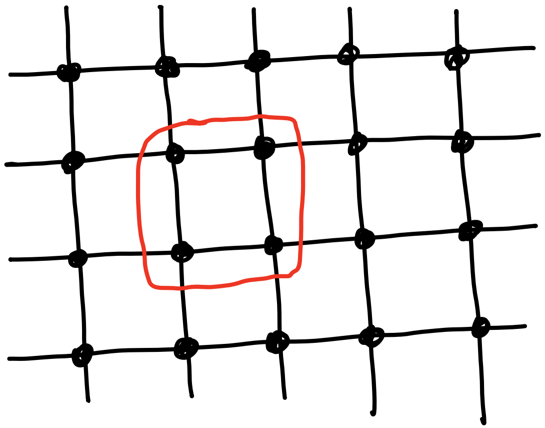
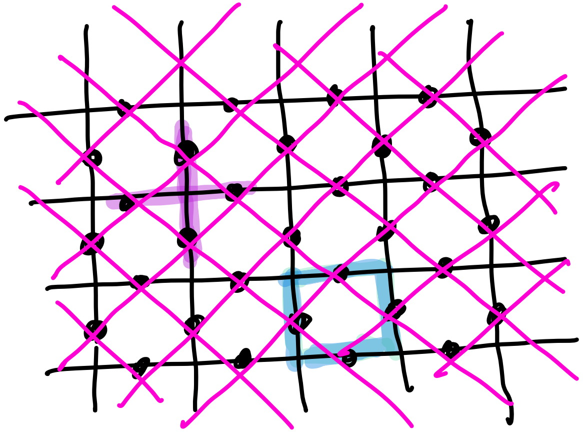
CLH on 2D lattice (4-local)



$\equiv$



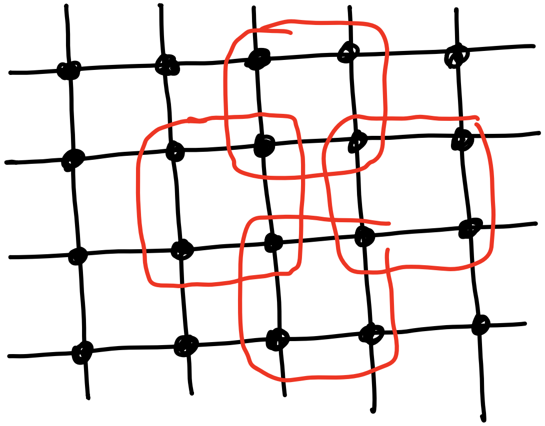
particles at grid vertices

terms are vertices on a face.

particles on edges

star & plaquette terms.

CLH on a 2D lattice

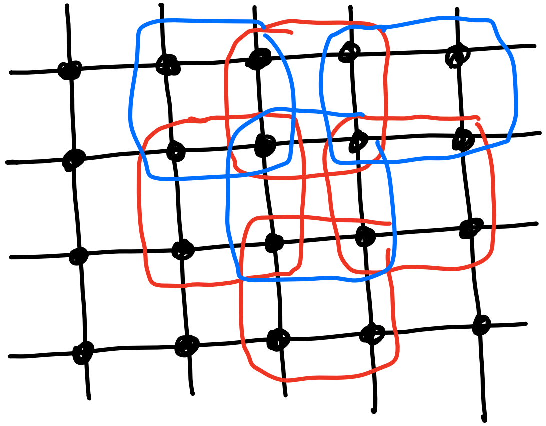


Red terms: $X\,X\,X\,X$

The ground states of these Hamiltonians will not have a "local" structure as in the 2-local case

$\Rightarrow$ Toric code is a special case.

CLH on a 2D lattice



The ground states of these Hamiltonians will not have a "local" structure as in the 2-local case

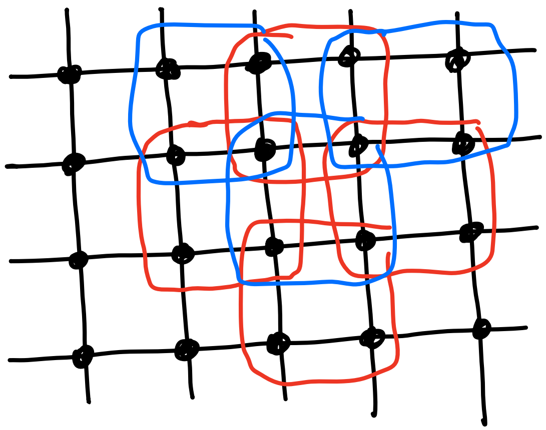
⇒ Toric code is a special case.

Red terms: X X X X

Blue terms: Z Z Z Z

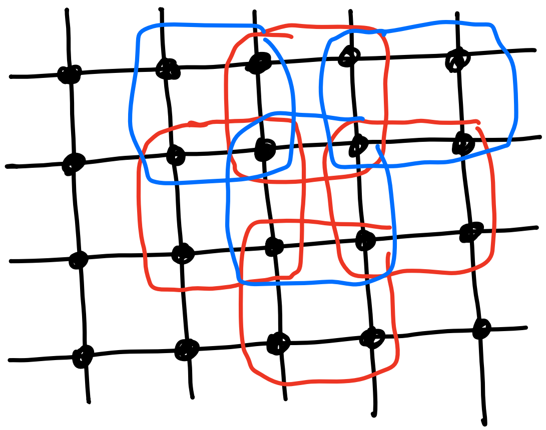
} ⇒ ground state will have global entanglement.

CLH on a 2D lattice



We will consider general commuting 4-local Hamiltonians on a 2D lattice of qubits.

CLH on a 2D lattice



We will consider general commuting 4-local Hamiltonians on a 2D lattice of qubits.

Checkerboard Pattern:

Blue faces

Red faces.

Will show CLH for this case is in NP [Schuch]

arXiv: 1105.2843

CLH on a 2D lattice of Qubits

$$H = \sum_i h_i = \sum_i (I - P_i)$$

projector onto ground space for term h_i.

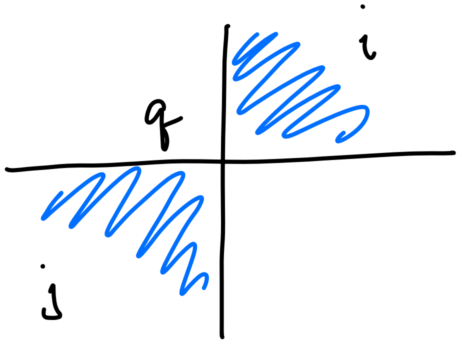
Let B = set of blue faces.
R = set of red faces.

$$P_B = \prod_{i \in B} P_i \qquad P_R = \prod_{i \in R} P_i$$

This will be a non-constructive!

Want to show: $\mathrm{Tr}(P_B P_R) > 0$

Blue faces overlap on a single qubit:



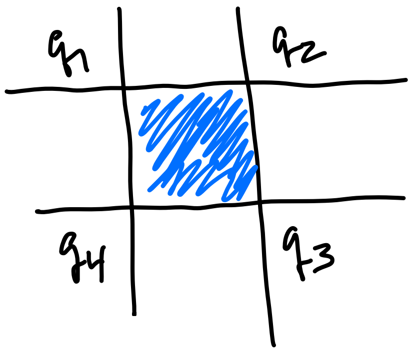
Can apply structural lemma

$$\mathcal{H}_q = \bigoplus_{\alpha} \mathcal{H}_{q,\alpha}$$

P_i + P_j are invariant on each $\mathcal{H}_{q,\alpha}$

Let $P_{q,\alpha}$ be the projector onto $\mathcal{H}_{q,\alpha}$

$$P_i = \sum_{\alpha} P_{q,\alpha} P_i P_{q,\alpha} \quad (\text{same for } P_j)$$



Let $\vec{\alpha}$ = vector of indices for all the qubits.

$$P_i^{\vec{\alpha}} = \prod_k P_{\alpha_k, q_k} \; P_i \; P_{\alpha_k q_k}$$

$$P_{Blue}^{\vec{\alpha}} = \prod_{i \in B} P_i^{\vec{\alpha}} \qquad P_{Blue} = \sum_{\vec{\alpha}} P_{Blue}^{\vec{\alpha}}$$

Same will hold for the red terms: except that it will be a different direct sum for each qubit: $\vec{\beta}$

Want to show $$\sum_{\vec{\alpha}\,\vec{\beta}} \underbrace{\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right]}_{\text{each individual term is } \geq 0} > 0$$

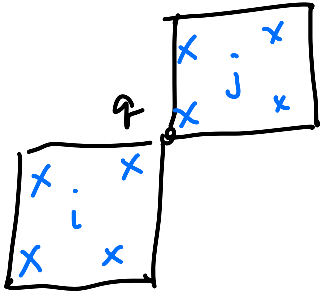
NP prover gives $\vec{\alpha}$ and $\vec{\beta}$ for which trace > 0

This witness doesn't necessarily say much about the ground state.

Example : Toric Code

Blue terms : XXXX

Red terms : ZZZZ



$q \; \frac{+}{-}$

$P_i + P_j$ invariant on $\mathcal{H}_{q+}$

$\mathcal{H}_{q-}$

$\alpha_q = +$ or $-$

Similarly: for red terms

$\beta_q = 0$ or 1.

$$\overset{s}{\alpha} = \{+,-\}^N$$

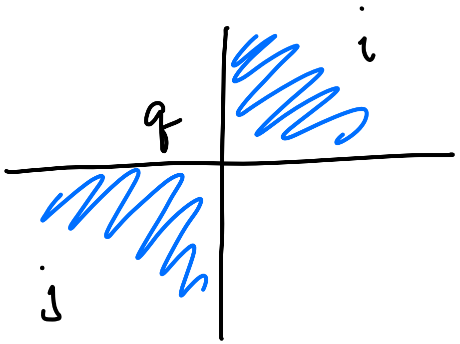
$$\overset{s}{\beta} = \{0,1\}^N$$

$$\overset{s}{\alpha} = (+,+,\dots,+) \qquad \overset{s}{\beta} = (0,0,\dots,0)$$

$$\mathrm{Tr}\left[\, |+\rangle\langle+|^{\otimes N} \; |0\rangle\langle 0|^{\otimes N} \right]$$

$$= 2^{-N} > 0$$

Back to General 4-local qubit CLH in 2D:



There are two ways for $\mathcal{H}_q$ to be divided:

(1, 1) - way:

$$\mathcal{H}_q = \underbrace{\mathcal{H}_{q,1}}_{\text{dim } 1} \oplus \underbrace{\mathcal{H}_{q,2}}_{\text{dim } 1}$$

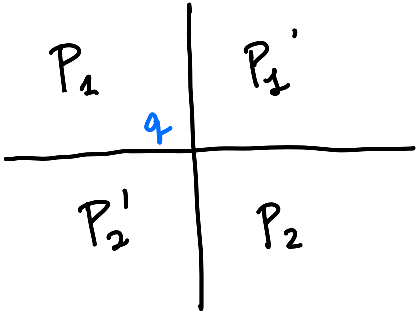
$\vec{\alpha}$ will project onto 1-dim space

(2) - way:

trivial partition.

P_i + P_j operate on disjoint portions of the space

$\Rightarrow$ only one acts non-trivially on q.



if either (P_1, P_2) or (P_1', P_2') commute in a $(1,1)$-way then we can trace out qubit q.

Why?

Need to show:

$$\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right] > 0$$

P_1 P_2 P_1' P_2' only terms operating on q.

Also: $P_{q,\alpha}$ and $P_{q,\beta}$.

Need to show:

$$\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right] > 0$$

P_1 P_2 P_1' P_2' only terms operating on q.

Also: $P_{q,\alpha}$ and $P_{q,\beta}$.

Case 1: $(P_1\ P_2)$ and (P_1', P_2') are both $(1,1)$

then $P_{q,\alpha} = |\phi\rangle\langle\phi|_q$. $\quad P_{q,\beta} = |\psi\rangle\langle\psi|_q$.

$$P_{q,\alpha} P_1 P_{q,\alpha} = |\phi\rangle\langle\phi| \otimes \langle\phi|P_1|\phi\rangle$$ (same for P_2)

$$P_{q,\beta} P_1' P_{q,\alpha} = |\psi\rangle\langle\psi| \otimes \langle\psi|\ P_1'\ |\psi\rangle$$ (same for P_2').

$\mathrm{Tr}\left[|\phi\rangle\langle\phi|\psi\rangle\langle\psi|\right] \mathrm{Tr}\left[\text{does not touch } q.\right]$

Need to show:

$$\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right] > 0$$

$P_1 \quad P_2 \quad P_1' \quad P_2'$ only terms operating on q.

Also: $\quad P_{q,\alpha}$ and $P_{q,\beta}$.

Case 1: $(P_1\ P_2)$ and (P_1', P_2') are both $(1,1)$

then $\quad P_{q,\alpha} = |\phi\rangle\langle\phi|_q. \qquad P_{q,\beta} = |\psi\rangle\langle\psi|_q.$

$P_{q,\alpha}\, P_1\, P_{q,\alpha} = |\phi\rangle\langle\phi| \otimes \langle\phi|P_1|\phi\rangle$ (same for P_2)

$P_{q,\beta}\, P_1'\, P_{q,\alpha} = |\psi\rangle\langle\psi| \otimes \langle\psi|\, P_1'\, |\psi\rangle$ (same for P_2').

Need to show:

$$\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right] > 0$$

P_1 P_2 P_1' P_2' only terms operating on q.

Also: $P_{q,\alpha}$ and $P_{q,\beta}$.

Case 2: $(P_1\ P_2)$ is $(1,1)$ (P_1', P_2') is (2)

$\Rightarrow$ P_2' is identity on q.

$$\mathrm{Tr}\left[\quad P_{q,\alpha} P_1 P_{q,\alpha} \cdots P_{q,\alpha} P_2 P_{q,\alpha} \cdots P_1' \cdots \right]$$

(all other terms Identity on q)

Need to show:

$$\mathrm{Tr}\left[\left(\prod_{i\in B} P_i^{\vec{\alpha}}\right)\left(\prod_{i\in R} P_i^{\vec{\beta}}\right)\right] > 0$$

P_1 P_2 P_1' P_2' only terms operating on q.

Also: $P_{q,\alpha}$ and $P_{q,\beta}$.

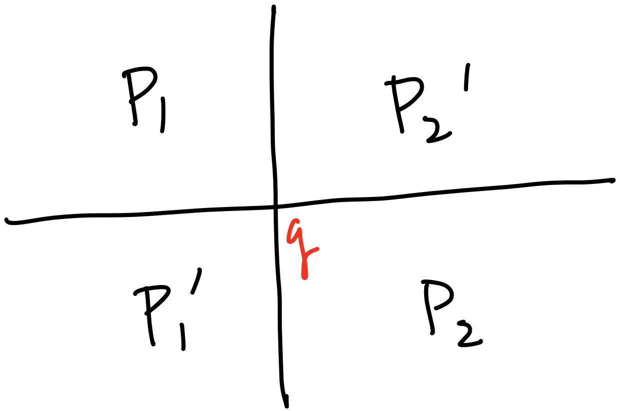
Case 2: $(P_1\ P_2)$ is $(1,1)$ (P_1', P_2') is (2)

$\Rightarrow$ P_2' is identity on q.

$$\mathrm{Tr}\left[\ P_{q,\alpha} P_1 P_{q,\alpha} \cdots P_{q,\alpha} P_2 P_{q,\alpha} \cdots P_{q,\alpha} P_1' P_{q,\alpha} \cdots\ \right]$$

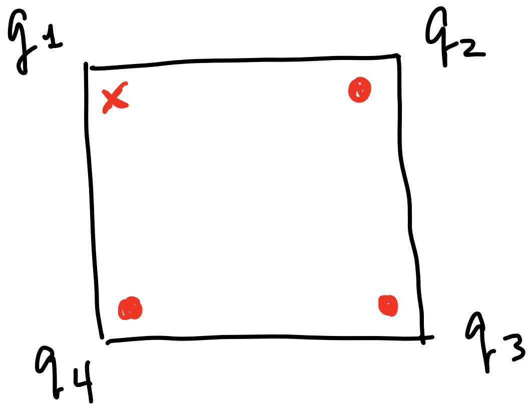
(all other terms Identity on q) $|\phi\rangle\langle\phi| \otimes [\mathrm{Id}$ on $q.\ ]$

After tracing those out, only left with:



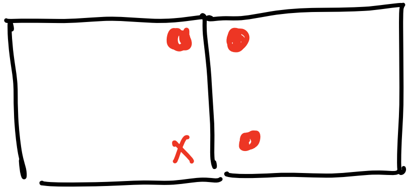
$$\left.\begin{array}{l} (P_1, P_2) \\ \\ (P_1', P_2') \end{array}\right\}$$

both pairs commute in (2)-way.



Put a dot ● in the corner if term acts non-trivially on that qubit.

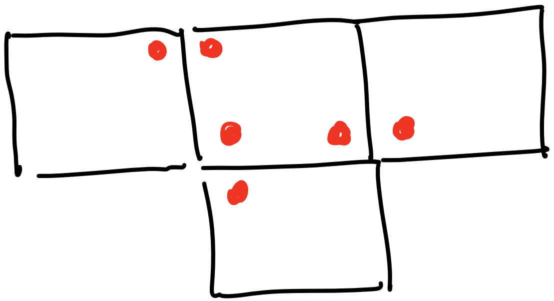
Otherwise put an ×



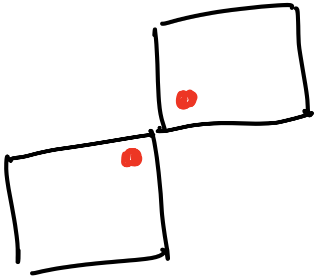
two terms "overlap" if they act non-trivially on the same qubit.

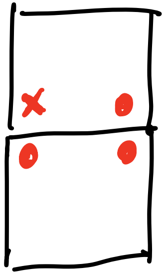
$\Rightarrow$ Overlapping terms form chains (no branching)

Can not have structures like:



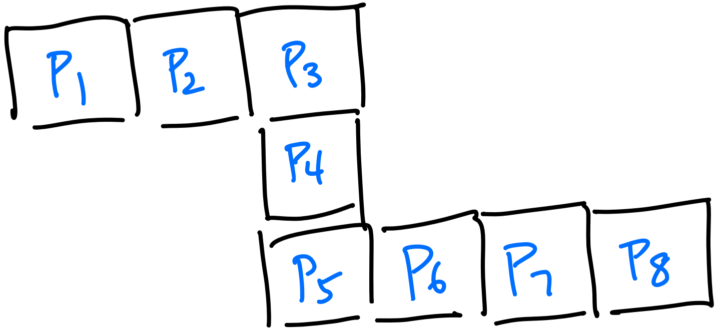
Case analysis, using the fact that the following two structures can't happen:





If two terms overlap on a single qubit, they cannot both have a dot at that qubit.

Now just need to determine trace of products of terms forming chains or cycles:



$$\mathrm{Tr}(P_1 P_2 \cdots P_8)$$

V) Structural Lemma Proof Sketch

A C^*- Algebra is a Banach algebra with $*$ -op.

For us: $A \subseteq \mathcal{L}(\mathcal{H})$

closed under, $+$, $\cdot$, $*$, scalar mult.

contains I.

The center of A $C(A)$ is the set of all $X \in A$ that with everything in A.

If $C(A) = \{cI : c \in \mathbb{C}\}$ (i.e. A has a "trivial" center)

then $A = \mathcal{L}(\mathcal{H}_a) \otimes I_{\mathcal{H}_b}$ $\quad \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$.

Structural Lemma Proof Sketch

Lemma: If $\exists\, M \in C(A)$ such that $M \not\propto I$

then $M = \sum \lambda_i \Pi_i$ Π_i projects on to $\mathcal{H}_i$.

projector onto eigenspaces of M.

and for $N \in A$ N is invariant on Π_i.

Proof idea: need to show $\Pi_i \in C(A)$.

if $M \in C(A)$ then $p(M) \in C(A)$ for polynomial p.

find $P_i(x)$ such that $P_i(\lambda_i) = 1$.

$P_i(\lambda_j) = 0 \quad j \neq i \Rightarrow P_i(M) = \Pi_i$

Structural Lemma Proof Sketch

Idea: Find $M \in \mathcal{C}(A)$ $\quad M \cancel{\propto} I$

use M to divide up $\mathcal{H} = \bigoplus \mathcal{H}_i$.

if $A|_{\mathcal{H}_i}$ does not have a trivial center, repeat on $A|_{\mathcal{H}_i} \subseteq \mathcal{L}(\mathcal{H}_i)$.

$\Rightarrow$ end up with: $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

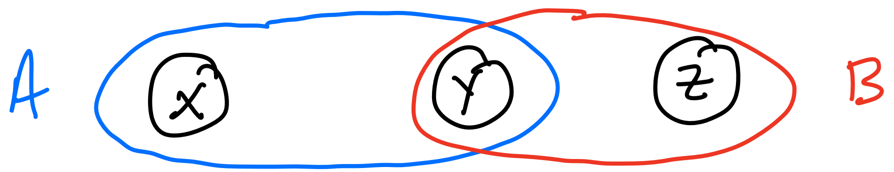
every $N \in A$ invariant on $\mathcal{H}_i$.

$A|_{\mathcal{H}_i}$ has a trivial center.

$A|_{\mathcal{H}_i} = \mathcal{L}(\mathcal{H}_{ia}) \otimes \mathcal{H}_{ib}$.

$\mathcal{H}_i \cong \mathcal{H}_{ia} \otimes \mathcal{H}_{ib}$.

Structural Lemma Proof Sketch



$$A = \sum_{\alpha\beta} \underbrace{|\alpha\rangle\langle\beta|}_{X} \otimes \underbrace{A_{\alpha\beta}}_{Y} \otimes \underbrace{I}_{Z}$$

$$\sum_{\gamma\delta} \underbrace{I}_{X} \otimes \underbrace{B_{\gamma\delta}}_{Y} \otimes \underbrace{|\gamma\rangle\langle\delta|}_{Z}.$$

C*-algebras: $\tilde{A} = \{ A_{\alpha\beta} \}$ $\qquad \tilde{B} = \{ B_{\gamma\delta} \}$

If A + B commute then $\tilde{A}$ and $\tilde{B}$ commute.

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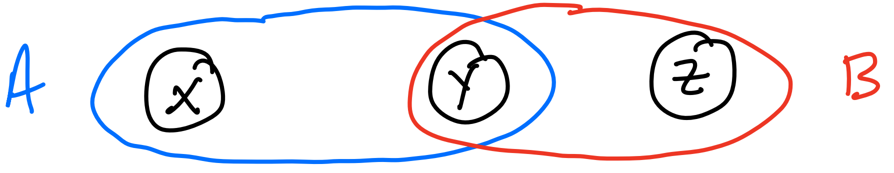

$$A = \sum_{\alpha\beta} \underbrace{|\alpha\rangle\langle\beta|}_{X} \otimes \underbrace{A_{\alpha\beta}}_{Y} \otimes \underbrace{I}_{Z}$$

C^*-algebras: $\tilde{A} = \{ A_{\alpha\beta} \}$

Use $\tilde{A}$ to divide Y:

$$Y = \bigoplus Y_i$$

$\tilde{A}$ invariant on Y_i.

$\tilde{A}|_{Y_i}$ is $\mathcal{L}(Y_{ia}) \otimes I_{Y_{ib}}$.

Since $\tilde{B}$ commutes with $\tilde{A}$:

- $\tilde{B}$ is invariant on each Y_i.
- $\tilde{B}|_{Y_i} \subseteq I_{Y_{ia}} \otimes \mathcal{L}(Y_{ib})$.