Intersecting Families

Imre Leader

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1 Harper's Theorem

"Given the size of a set, how small can its boundary be?" For example,

- in $\mathbb{R}^2$, circular discs are best;
- in $\mathbb{R}^3$, spherical balls are best;
- in $S^2 \subset \mathbb{R}^3$, 'circular caps' are best.

For a fixed graph G and any set $A \subset V(G)$, the *boundary* of A is the set

$$b(A) = \{x \in V(G) - A : x \sim y \text{ for some } y \in A\}.$$

Given $|A|$, how do we minimize $|b(A)|$? An *isoperimetric inequality* on G is an inequality of the form

$$A \subset V(G), |A| = m \implies |b(A)| \geq f(m)$$

for some function f. Equivalently, we wish to minimize the *neighbourhood* $N(A)$ of A, where $N(A) = A \cup b(A)$.

A good candidate for a set with small boundary is a *ball,* i.e. a set of the form $B(x, r) = \{y \in G : d(x, y) \leq r\}$ where $d(x, y)$ denotes the usual graph distance (the length of a shortest x-y path).

Let X be a set. A *set system* on X is a collection $\mathcal{A} \subset \mathcal{P}X$ of subsets of X. Usually we take $X = [n] = \{1, 2, \ldots, n\}$. An example of a set system on X is $X^{(r)} = \{A \subset X : |A| = r\}$.

Make $\mathcal{P}X$ into a graph by joining A to B if $B = A \cup \{i\}$ for some $i \notin A$ (or vice versa). This graph is the *discrete cube* Q_n.

If we identify each $A \in Q_n$ with a 01-sequence of length n (for example, in Q_3 we make the identification $\emptyset \leftrightarrow 000$, $\{1\} \leftrightarrow 100$, $\{2, 3\} \leftrightarrow 011$ etc.) then

Q_n is identified with the unit cube in $\mathbb{R}^n$. Then $X^{(r)}$ (the family of all r-sets) is just a 'slice' through Q_n.

Which sets in Q_n have the smallest boundaries? In general, it seems that balls $X^{(\leq r)} = B(\emptyset, r) = X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(r)}$ are best. But what if $|A|$ is not the exact size of a ball?

A little experimentation suggests that if $|X^{(<r)}| < |A| < |X^{(\leq r)}|$ then it is best to take A to be $X^{(<r)}$ together with an initial segment of the lex order on $X^{(r)}$. (The *lexicographic* or *lex* or *dictionary* order on $X^{(r)}$ is defined by: if $x = \{a_1, a_2, \ldots, a_r\}$ $(a_1 < a_2 < \cdots < a_r)$ and $y = \{b_1, b_2, \ldots, b_r\}$ $(b_1 < b_2 < \cdots < b_r)$ then $x < y$ if $a_1 < b_1$, or $a_1 = b_1$ and $a_2 < b_2$, or ... or $a_1 = b_1$, $a_2 = b_2$, ..., $a_{r-1} = b_{r-1}$ and $a_r < b_r$. Equivalently, $x < y$ if $a_s < b_s$ where $s = \min\{t : a_t \neq b_t\}$. For example, the lexicographic order on $[4]^{(2)}$ is 12, 13, 14, 23, 24, 34.)

The *simplicial ordering* on Q_n is defined by $x < y$ if $|x| < |y|$ or $|x| = |y|$ and $x < y$ in lex. For example,

- on Q_3: $\emptyset$, 1, 2, 3, 12, 13, 23, 123;
- on Q_5: $\emptyset$, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 123, 124, 125, 134, 135, 145, 234, 235, 245, 345, 1234, 1235, 1245, 1345, 2345, 12345.

Theorem 1 (Harper's theorem)**.** *Let $A \subset Q_n$ and let C be the first $|A|$ points of Q_n in the simplicial order. Then $|N(A)| \geq |N(C)|$. In particular, if $|A| \geq \sum_{i=0}^{r} \binom{n}{i}$ then $|N(A)| \geq \sum_{i=0}^{r+1} \binom{n}{i}$.*

Remarks. A *Hamming ball* is a set A with $X^{(<r)} \subset A \subset X^{(\leq r)}$ for some r. If we knew A was a Hamming ball then we would be done by Kruskal-Katona (which says that to minimize the upper shadow $\partial^+ A$ of a family $A \subset X^{(r)}$, where $\partial^+ A = \{y \in X^{(r+1)} : y \supset x \text{ for some } x \in A\}$, take A to be an initial segment of lex). And, conversely, Theorem 1 implies Kruskal-Katona: given $A \subset X^{(r)}$, apply the theorem to $X^{(<r)} \cup A$.

The main idea is that of 'compressions'. We try to transform $A \to A'$ such that

- $|A'| = |A|$;
- $|N(A')| \leq |N(A)|$; and
- A' looks more like C than A did.

Ideally, we transform repeatedly $A \to A' \to A'' \to \cdots$, ending up with a family B so similar to C that we can see directly that $|N(B)| \geq |N(C)|$.

For $A \subset Q_n$ and $1 \leq i \leq n$, the *i-sections* of A are the set-systems $A_+ = A_+^{(i)}$ and $A_- = A_-^{(i)}$ in $\mathcal{P}(X - i)$ given by

$$A_+ = \{x \in \mathcal{P}(X - i) : x \cup i \in A\}$$

and

$$A_- = \{x \in \mathcal{P}(X - i) : x \in A\}.$$

For example, in Q_4 the family $A = \{12, 13, 23, 124, 134\}$ has $A_-^{(3)} = \{12, 124\}$ and $A_+^{(3)} = \{1, 2, 14\}$.

The *i-compression* or *codimension-1 i-compression* of A is the system $C_i(A) \subset Q_n$ defined by $|C_i(A)_+| = |A_+|$, $|C_i(A)_-| = |A_-|$, and $C_i(A)_+$ and $C_i(A)_-$ are initial segments of the simplicial order on $\mathcal{P}(X - i)$. Note that $|C_i(A)| = |A|$. Say $A \subset Q_n$ is *i-compressed* if $C_i(A) = A$.

Proof (of Theorem 1). The proof is by induction on n; the case $n = 1$ is trivial.

Claim. If $A \subset Q_n$ and $1 \leq i \leq n$ then $|N(C_i(A))| \leq |N(A)|$.

Proof of claim. Write B for $C_i(A)$. We have

$$|N(A)| = |N(A_+) \cup A_-| + |N(A_-) \cup A_+|$$

and

$$|N(B)| = |N(B_+) \cup B_-| + |N(B_-) \cup B_+|.$$

Now $|B_-| = |A_-|$ and $|N(B_+)| \leq |N(A_+)|$ (by the induction hypothesis). Also, B_- is an initial segment of the simplicial order. And so is $N(B_+)$ (because the neighbourhood of an initial segment of the simplicial order is itself an initial segment of the simplicial order).

Hence B_- and $N(B_+)$ are nested (i.e. one is contained in the other), and so we have $|N(B_+) \cup B_-| \leq |N(A+) \cup A_-|$. Similarly, we also have $|N(B_-) \cup B_+| \leq |N(A_-) \cup A_+|$. This establishes the Claim.

Define a sequence $A_0, A_1, A_2, \ldots \subset Q_n$ as follows: set $A_0 = A$. Having defined $A_0, A_1, \ldots, A_k$, if A_k is i-compressed for all i then stop the sequence with A_k. Otherwise, there exists i with A_k not i-compressed; set $A_{k+1} = C_i(A_k)$ and continue. This process must terminate since $\sum_{x \in A_k} f(x)$ (where $f(x)$ denotes the position of x in the simplicial order on Q_n) is a decreasing function of k. Thus we have $B \subset Q_n$ such that

- $|B| = |A|$;
- $|N(B)| \leq |N(A)|$; and
- B is i-compressed for all i.

So, must a set that is i-compressed for all i be an initial segment of the simplicial order? (If so then we are done, as $B = C$.) Unfortunately, the answer is no; for example, take $\{\emptyset, 1, 2, 12\} \subset Q_3$. However, if $B \subset Q_n$ is i-compressed for all i and is not an initial segment of the simplicial order then *EITHER* n is odd, say $n = 2k+1$, and

$$B = X^{(\leq k)} \cup \{12\ldots(k+1)\} - \{(k+2)(k+3)\ldots(2k+1)\}$$

OR n is even, say $n = 2k$, and

$$B = X^{(<k)} \cup \{x \in X^{(k)} : 1 \in X\} \cup \{23\ldots(k+1)\} - \{1(k+2)(k+3)\ldots(2k)\}$$

(by Lemma 2 below).

Having proved Lemma 2, the proof of Theorem 1 will be complete as in each case it is clear that $|N(B)| \geq |N(C)|$. □

Lemma 2. *Let $B \subset Q_n$ be i-compressed for all i but not an initial segment of the simplicial order. Then* EITHER *n is odd, say $n = 2k+1$, and*

$$B = X^{(\leq k)} \cup \{12\ldots(k+1)\} - \{(k+2)(k+3)\ldots(2k+1)\}$$

OR *n is even, say $n = 2k$, and*

$$B = X^{(<k)} \cup \{x \in X^{(k)} : 1 \in X\} \cup \{23\ldots(k+1)\} - \{1(k+2)(k+3)\ldots(2k)\}.$$

Proof. As B is not an initial segment of the simplicial order, we have some $x < y$ with $x \notin B$ and $y \in B$. Fix $1 \leq i \leq n$: can we have $i \in x$ and $i \in y$? No, as B is i-compressed. Similarly, we cannot have $i \notin x$ and $i \notin y$. So $i \in x \triangle y$ for any i. Thus $y = x^c$.

So for each $y \in B$, at most one $x < y$ has $x \notin B$ (namely $x = y^c$); and for each $x \notin B$, at most one $y > x$ has $y \in B$ (namely $y = x^c$). Thus $B = \{z \in Q_n : z \leq y\} - \{x\}$ for some y, where x is the predecessor of y *and* $x = y^c$. Which $x \in Q_n$ have x^c the successor of x? If n is odd then x must be the last point of $X^{(\leq (n-1)/2)}$. If n is even then x must be the last point of $X^{(n/2)}$ containing a 1. □

Remark. This proof also proves the Kruskal-Katona theorem directly (if desired).

For $A \subset Q_n$ and $t = 0, 1, 2, \ldots$, the *t-neighbourhood* of A is the set $A_{(t)} = \{x \in Q_n : d(x, A) \leq t\}$. So, for example, $A_{(1)}$ is just $N(A)$.

Corollary 3. *Let $A \subset Q_n$ with $|A| \geq \sum_{i=0}^{r} \binom{n}{i}$. Then for any $t = 0, 1, 2, \ldots$, we have $|A_{(t)}| \geq \sum_{i=0}^{r+t} \binom{n}{i}$.*

Proof. If $|A_{(t)}| \geq \sum_{i=0}^{r+t} \binom{n}{i}$ then $|A_{(t+1)}| \geq \sum_{i=0}^{r+t+1} \binom{n}{i}$ by Harper's Theorem, so we are done by induction. □

2 Intersecting Families

Say $A \subset \mathcal{P}X$ is *intersecting* if for all $x, y \in A$ we have $x \cap y \neq \emptyset$. How large can A be?

We could take $A = \{x \in \mathcal{P}X : 1 \in x\}$. This has $|A| = 2^{n-1}$. It is impossible to beat this:

Proposition 4. *Let $A \subset \mathcal{P}X$ be intersecting. Then $|A| \leq 2^{n-1}$.*

Proof. For each $x \in \mathcal{P}X$, we cannot have both x, $x^c \in A$. $\square$

Remark. The extremal system is certainly not unique—for example, if n is odd we can take $\{x \in \mathcal{P}X : |x| > n/2\}$.

A better question is: how large can an intersecting $A \subset X^{(r)}$ be? If $r > n/2$ we can take the whole of $X^{(r)}$. If $r = n/2$ we can take one of x, x^c for each x, giving $|A| = \frac{1}{2}\binom{n}{r}$. So we shall focus on $r < n/2$. One obvious candidate is $A = \{x \in X^{(r)} : 1 \in x\}$. For example, in $[8]^3$ this has order $\binom{7}{2} = 21$, while the family $\{x \in [8]^3 : |x \cap \{1,2,3\}| \geq 2\}$ has order $1 + \binom{3}{2}\binom{5}{1} = 16 < 21$.

Theorem 5 (Erdős-Ko-Rado theorem)**.** *If $A \subset X^{(r)}$ ($r < n/2$) is intersecting then $|A| \leq \binom{n-1}{r-1}$.*

Proof. The condition $x \cap y \neq \emptyset$ is equivalent to $x \not\subset y^c$. So writing $\bar{A}$ for the family $\{x^c : x \in A\}$, we have $\partial^{+(n-2r)}A$ disjoint from $\bar{A}$. Suppose $|A| > \binom{n-1}{r-1}$, so also $|\bar{A}| > \binom{n-1}{r-1}$. We have $|A| \geq |\{x \in X^{(r)} : 1 \in X\}|$, so $|\partial^+ A| \geq |\{x \in X^{(r+1)} : 1 \in X\}|$ (by the Kruskal-Katona theorem), and so, inductively, we get $|\partial^{+(n-2r)}A| \geq |\{x \in X^{(n-r)} : 1 \in x\}| = \binom{n-1}{n-r-1}$. Thus inside $X^{(n-r)}$, which has size $\binom{n}{r}$, we have disjoint sets of sizes at least $\binom{n-1}{n-r-1}$ and greater than $\binom{n-1}{r-1}$. But $\binom{n-1}{n-r-1} + \binom{n-1}{r-1} = \binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$, a contradiction. $\square$

Remarks. 1. There are many other nice proofs.

2. The largest intersecting family has size $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$; the chance that a random r-set contains 1 is $\frac{r}{n}$.

We say that $A \subset \mathcal{P}X$ is *t-intersecting* if $|x \cap y| \geq t$ for all $x, y \in A$. How large can A be? For example, for $t = 2$ we could take $\{x \in \mathcal{P}X : 1, 2 \in x\}$ or $\{x \in \mathcal{P}X : |x| \geq n/2 + 1\}$.

Theorem 6 (Katona's t-intersecting theorem)**.** *Let $A \subset \mathcal{P}X$ be t-intersecting, with $n + t$ even. Then $|A| \leq |X^{(\geq (n+t)/2)}|$.*

Proof. If $|x \cap y| \geq t$ then $d(x, y^c) \geq t$, as there are at least t points which are in x but not in y^c. So letting $\bar{A} = \{x^c : x \in A\}$, we have that $A_{(t-1)}$ and $\bar{A}$ are disjoint.

Now, supppose $|A| > |X^{(\geq (n+t)/2)}| = |X^{(\leq (n-t)/2)}|$. Then, by Harper's theorem, we have $|A_{(t-1)}| \geq |X^{(\leq (n+t)/2-1)}|$. But then $|A^c_{(t-1)}| \geq |X^{(\leq (n+t)/2-1)}|$, a contradiction. $\square$

Remark. The same proof gives that if $n + t$ is odd then

$$|A| \leq |X^{(\geq (n+t+1)/2)} \cup \{x \in X^{((n+t-1)/2)} : n \notin x\}|.$$

What happens for r-sets, i.e. for $A \subset X^{(r)}$? In $[8]^{(4)}$, for $t = 2$, the family $A = \{x \in [8]^{(4)} : 1, 2 \in x\}$ has $|A| = \binom{6}{2} = 15$; but the family $B = \{x \in [8]^{(4)} : |x \cap \{1, 2, 3, 4\}| \geq 3|\}$ has $|B| = 1 + \binom{4}{3}\binom{4}{1} = 17 > 15$.

What is sometimes called the Second Erdős-Ko-Rado theorem states that, for given r and t, if n is sufficiently large then the largest t-intersecting family in $[n]^{(r)}$ is $\{x \in [n]^{(r)} : [t] \subset x\}$, which has size $\binom{n-t}{r-t}$.

Write $A_\alpha = \{x \in [n]^{(r)} : |x \cap [t + 2\alpha]| \geq t + \alpha\}$ for $\alpha = 0, 1, 2, 3, \ldots$. The *Frankl conjecture* was that if $A \subset X^{(r)}$ is t-intersecting then $|A| \leq \max\{|A_\alpha| : \alpha = 0, 1, 2, \ldots\}$. This was eventually proved by Ahlswede and Khachatrian.

3 Covering by Intersecting Families

How many intersecting families do we need to cover $\mathcal{P}X - \{\emptyset\}$? In other words, if $\mathcal{P}X - \{\emptyset\} = A_1 \cup A_2 \cup \cdots \cup A_s$ with each A_i an intersecting family, how small can s be?

We clearly need at least n families (one for each singleton); and, equally clearly, n families will suffice—for example, take $A_i = \{x \in \mathcal{P}X : i \in x\}$.

What happens for r-sets? How many intersecting families do we need to cover $X^{(r)}$?

If $r > n/2$ then $X^{(r)}$ itself is intersecting. If $r = n/2$ then we can cover by two intersecting families: for each x, select one of x and x^c for A_1 and the other for A_2. So we may assume that $r < n/2$.

We clearly need at least $\lfloor n/r \rfloor$ as there exist $\lfloor n/r \rfloor$ disjoint r-sets. Moreover, we need at least $\lceil n/r \rceil$ families, as each intersecting family has at most r/n of all r-sets.

Can we achieve this? Well, we can achieve $n - 2r + 2$ as follows: put $A_i = \{x \in X^{(r)} : i \in x\}$ for $1 \leq i \leq n-2r+1$, and $A_{n-2r+2} = [n-2r+2, n]^{(r)}$.

Our aim is to prove *Kneser's conjecture*, that we need at least $n-2r+2$ intersecting families to cover $X^{(r)}$. It turns out that the key tool will be the Borsuk-Ulam Theorem:

Theorem 7 (Borsuk-Ulam theorem)**.** *Let $f\colon S^n \to \mathbb{R}^n$ be continuous. Then there exists $x \in S^n$ with $f(x) = f(-x)$.*

For example, in the case $n = 1$, suppose we have a continuous $f\colon S^1 \to \mathbb{R}$. Put $g(x) = f(x) - f(-x)$. If $g(x) > 0$ then $g(-x) < 0$. So if g is not identically zero then there is some x with $g(x) > 0$ and then by the Intermediate Value theorem there is some y with $g(y) = 0$.

The result for the case $n = 2$ is not quite intuitively obvious.

Remark. The Borsuk-Ulam theorem trivially implies that there is no continuous injection from S^n to $\mathbb{R}^n$, and so in particular $\mathbb{R}^{n+1}$ is not homeomorphic to $\mathbb{R}^n$—this is the "Brouwer invariance of domain theorem" and is hard to prove. (For example, why are $\mathbb{R}^3$ and $\mathbb{R}^4$ not homeomorphic?)

We say that $f\colon S^n \to \mathbb{R}^n$ is *antipodal* if $f(-x) = -f(x)$ for all x.

Theorem 8. *The following are equivalent:*

1. *(i) The Borsuk-Ulam theorem;*
2. *(ii) If $f\colon S^n \to \mathbb{R}^n$ is an antipodal map then there is some $x \in S^n$ with $f(x) = 0$;*
3. *(iii) There is no continuous antipodal map $f : S^n \to S^{n-1}$.*

Proof. **(i) $\implies$ (ii).** If $f\colon S^n \to \mathbb{R}^n$ is antipodal then, by (i), we have $f(x) = f(-x)$ for some x, whence $f(x) = 0$ (as $f(-x) = -f(x)$).

(ii) $\implies$ (i). Given a continuous $f\colon S^n \to \mathbb{R}^n$, define $g\colon S^n \to \mathbb{R}^n$ by $g(x) = f(x) - f(-x)$. Then g is antipodal, so $g(x) = 0$ for some x, i.e. $f(x) = f(-x)$.

(ii) $\implies$ (iii). If $f\colon S^n \to S^{n-1}$ then $f(x) \neq 0$ for all $x \in S^n$.

(iii) $\implies$ (ii). Suppose $f\colon S^n \to \mathbb{R}^n$ is antipodal and continuous with $f(x) \neq 0$ for all $x \in S^n$. Define $g\colon S^n \to S^{n-1}$ by $g(x) = f(x)/\|f(x)\|$ (where $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$). Then g is continuous and antipodal, a contradiction. □

Suppose $A_1, A_2, \ldots, A_k \subset S^n$ are closed sets that cover S^n with no A_i containing an antipodal pair $\{x, -x\}$. How small can k be?

It is easy to obtain $k = n+2$: take $A_i = \{x \in S^n : x_i \geq \varepsilon\}$ for $1 \leq i \leq n+1$, and $A_{n+2} = \{x \in S^n : x_i \leq \varepsilon \text{ for all } i\}$. This works if $\varepsilon < 1/\sqrt{n}$.

Theorem 9. *The following are equivalent:*

1. *(i) The Borsuk-Ulam theorem;*
2. *(ii) If $A_1, A_2, \ldots, A_{n+1} \subset S^n$ are closed sets covering S^n then some A_i contains an antipodal pair $\{x, -x\}$;*
3. *(iii) If $A_1, A_2, \ldots, A_{n+1} \subset S^n$ cover S^n with each A_i open or closed then some A_i contains an antipodal pair.*

Proof. **(i) $\Longrightarrow$ (ii).** Define $f\colon S^n \to \mathbb{R}^n$ by

$$f(x) = (d(x, A_1), d(x, A_2), \ldots, d(x, A_n)).$$

Then f is continuous so, by (i), there exists $x \in S^n$ with $d(x, A_i) = d(-x, A_i)$ for all i with $1 \le i \le n$. If $x, -x \in A_{n+1}$ then we are done. If not, we may assume without loss of generality that $x \in A_i$ for some i with $1 \le i \le n$, so $d(x, A_i) = 0$ whence $d(-x, A_i) = 0$ whence $-x \in A_i$ (as A_i closed).

(ii) $\Longrightarrow$ (i). Suppose $f\colon S^n \to S^{n-1}$ is continuous and antipodal. Let A_1, $A_2, \ldots, A_{n+1}$ be closed sets covering S^{n-1} with no A_i containing an antipodal pair. Then $f^{-1}(A_1)$, $f^{-1}(A_2)$, $\ldots$, $f^{-1}(A_{n+1})$ would be closed sets covering S^n with none containing an antipodal pair, a contradiction.

(iii) $\Longrightarrow$ (ii). Trivial.

(i) $\Longrightarrow$ (iii). As for (i) $\Longrightarrow$ (ii), we get $x \in S^n$ with $d(x, A_i) = d(-x, A_i)$ for all i with $1 \le i \le n$. If $x, -x \in A_{n+1}$ then we are done. If not, we may assume without loss of generality that $x \in A_i$ for some i with $1 \le i \le n$, so $d(x, A_i) = 0$ whence $d(-x, A_i) = 0$.

If A_i is *closed* then $-x \in A_i$.

If A_i is *open* then we have $\{y \in S^n : d(x, y) < \varepsilon\} \subset A_i$ for some $\varepsilon > 0$. But some z with $d(z, -x) < \varepsilon$ belongs to A_i (as $d(-x, A_i) = 0$). □

Remarks. 1. The result of (ii) in Theorem 9 is sometimes called the *Lusternik-Schnirelmann theorem.*

2. The result of (iii) looks rather weird and pointless. But in fact it will be the key for our proof of Kneser's conjecture. This form is due to Greene, who discovered the proof of Kneser we give below.

Theorem 10 (Kneser's conjecture, proved by Lovász)**.** *Let $r < n/2$ and let $A_1, A_2, \ldots, A_d$ be a collection of intersecting families covering $[n]^{(r)}$. Then $d \ge n - 2r + 2$.*

Proof. Suppose $d = n - 2r + 1$. Let x_1, x_2, $\ldots$, x_n be points in general position in $S^d \subset \mathbb{R}^{d+1}$ (i.e. no d-dimensional subspace through the origin contains $d+1$ of the x_i). Identify $[n]$ with $\{x_1, x_2, \ldots, x_n\}$. For $x \in S^d$, write $H_x = \{y \in S^d : \langle x, y \rangle > 0\}$. For $1 \leq i \leq d$, let C_i be the set of $x \in S^n$ with H_x containing an r-set from A_i. Let $C_{d+1} = S^d - (C_1 \cup C_2 \cup \cdots \cup C_d)$, so that C_{d+1} is the set of $x \in S^d$ with H_x containing at most $r-1$ of x_1, x_2,..., x_n. Then C_1, C_2, $\ldots$, C_d are open and C_{d+1} is closed, so some C_i contains an antipodal pair $\{x, -x\}$. We cannot have $1 \leq i \leq d$ since H_x and H_{-x} are disjoint whence A_i would contain two disjoint r-sets. Thus $i = d+1$, so $H_x \cup H_{-x}$ contains at most $2(r-1)$ of x_1, x_2, $\ldots$, x_n, whence $\{y \in S^d : \langle x, y \rangle = 0\}$ contains at least $n - 2(r-1) = d+1$ of x_1, x_2, $\ldots$, x_n, a contradiction. □

The *Kneser graph* $K(n, r)$ ($r < n/2$) is the graph on vertex set $[n]^{(r)}$ with x joined to y if $x \cap y = \emptyset$. For example $K(5, 2)$ is the Petersen graph. So an intersecting family in $[n]^{(r)}$ is an independent set in $K(n, r)$. And, for any graph G, colouring G with k colours is equivalent to partitioning G into k independent sets. So Theorem 10 can be rephrased as:

Theorem 11. $\chi(K(n, r)) = n - 2r + 2$.

Note. The chromatic number χ is large even though there are huge independent sets (containing n/r of all vertices).

4 Modular Intersection Theorems

If $A \subset [n]^{(r)}$ is intersecting, i.e. $|x \cap y| \neq 0$ for x, $y \in A$, we know that $|A| \leq \binom{n-1}{r-1}$. What if, instead, we do not allow $|x \cap y| \equiv 0$ modulo some number?

Say, for example, r is odd and $A \subset [n]^{(r)}$ has $|x \cap y|$ odd for all x, $y \in A$. We can achieve $|A| = \binom{\lfloor (n-1)/2 \rfloor}{(r-1)/2}$ by taking A to consist of all sets containing 1 and $(r-1)/2$ of the pairs 23, 45, ... (finishing at $(n-1)n$ if n is odd and $(n-2)(n-1)$ if n is even).

How about r odd, $A \subset [n]^{(r)}$ such that $|x \cap y|$ is even for all x, $y \in A$ with $x \neq y$? We could take $\{x \in [n]^{(r)} : 1, 2, \ldots, r-1 \in x\}$, which has $|A| = n - r + 1$. Amazingly:

Theorem 12. *Let r be odd, and let $A \subset [n]^{(r)}$ have $|x \cap y|$ even for all x, $y \in A$ with $x \neq y$. Then $|A| \leq n$.*

Proof. Our main idea is to write down $|A|$ linearly independent points in an n-dimensional vector space.

View Q_n as $\mathbb{Z}_2^n$ by identifying $x \in \mathcal{P}[n]$ with $\bar{x} \in \mathbb{Z}_2^n$ where

$$\bar{x}_i = \begin{cases} 1 & \text{if } i \in x \\ 0 & \text{if } i \notin x \end{cases} .$$

For example, if $x = \{1, 3, 5\}$ then $\bar{x} = (1, 0, 1, 0, 1, 0, 0, \dots)$; this is simply the usual identification.

For $x \in A$, we have $\langle \bar{x}, \bar{x} \rangle = 1$ (as $|x|$ is odd). For x, $y \in A$ with $x \neq y$, we have $\langle \bar{x}, \bar{y} \rangle = 0$ (as $|x \cap y|$ is even). So the set $\{\bar{x} : x \in A\}$ is linearly independent over $\mathbb{Z}^2$: if $\sum_{x \in A} \lambda_x \bar{x} = 0$ then, by taking the inner product with $\bar{x}$, we see that $\lambda_x = 0$ for each $x \in A$. □

What happens if r is even?

For $A \subset [n]^{(r)}$ with $|x \cap y|$ even for all x, $y \in A$, we can get A large, for example $|A| = \binom{\lfloor n/2 \rfloor}{r/2}$. For $A \subset [n]^{(r)}$ with $|x \cap y|$ odd for all x, $y \in A$ with $x \neq y$, we must have $|A| \leq n+1$, because we may set $A' \subset [n+1]^{(r+1)}$ to be $\{x \cup \{n+1\} : x \in A\}$ and apply Theorem 12.

So our conclusion is that to get very small bounds on $|A|$ for $A \subset [n]^{(r)}$ we should forbid $|x \cap y| \equiv r \pmod 2$ for x, $y \in A$ with $x \neq y$. Does this generalize?

We shall now show that 's allowed values for $|x \cap y|$ modulo p implies $|A| \leq \binom{n}{s}$'.

Theorem 13 (Frankl, Wilson)**.** *Let p be a prime. Let $A \subset [n]^{(r)}$ be such that there exist integers λ_1, λ_2, …, λ_s (for some $s \leq r$), with no $\lambda_i \equiv r \pmod p$, for which given any x, $y \in A$ with $x \neq y$, we have $|x \cap y| \equiv \lambda_i \pmod p$ for some i. Then $|A| \leq \binom{n}{s}$. In particular, if $A \subset [n]^{(r)}$ satisfies $|x \cap y| \not\equiv r \pmod p$ for all distinct x, $y \in A$, then $|A| \leq \binom{n}{p-1}$.*

Remarks. 1. $\binom{n}{s}$ is a polynomial independent of r.

2. In general, we cannot improve on $\binom{n}{s}$; for example, we can take $A = [n]^{(s)}$ if $r = s$. If $r > s$, we can take $A = \{x \in [n]^{(r)} : 1, 2, \dots, r-s \in x\}$; this gives $|A| = \binom{n-r+s}{s}$, which is very close to $\binom{n}{s}$ (for fixed r).

3. If we allow $|x \cap y| \equiv r \pmod p$ then there is no polynomial bound: taking $r = a + \lambda p$ ($0 \leq a < p$), we can obtain $|A| = \binom{\lfloor (n-a)/p \rfloor}{\lambda}$ (by taking A to consist of all sets containing the points 1, 2, …, a together with λ of the blocks $[a+1, a+p]$, $[a+p+1, a+2p]$, …, $[a+(\lambda-1)p+1, a+\lambda p]$—this grows with r.

Proof. We seek a vector space V of dimension at most $\binom{n}{s}$ and $|A|$ linearly independent vectors in V.

For $i \le j$, let $N(i,j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix, with rows indexed by $[n]^{(i)}$ and columns indexed by $[n]^{(j)}$, given by

$$N(i,j)_{xy} = \begin{cases} 1 & \text{if } x \subset y \\ 0 & \text{otherwise} \end{cases}.$$

So $N(s,r)$ has $\binom{n}{s}$ rows. Let V be their linear span over $\mathbb{R}$. Then we have $\dim V \le \binom{n}{s}$.

Consider $N(i,s)N(s,r)$ for any $0 \le i \le s$. Its rows belong to V. Also,

$$(N(i,s)N(s,r))_{xy} = \begin{cases} \binom{r-i}{s-i} & x \subset y \\ 0 & \text{otherwise} \end{cases}$$

(as $N(i,s)N(s,r)$ is simply the number of s-sets z with $x \subset z \subset y$). So $N(i,s)N(s,r) = \binom{r-i}{s-i}N(i,r)$, whence $N(i,r)$ has rows in V.

Now consider $M(i) = N(i,r)^T N(i,r)$. It has rows in V. But $M(i)_{xy}$ is the number of i-sets z with $z \subset x$ and $z \subset y$, i.e. $M(i)_{xy} = \binom{|x\cap y|}{i}$. 'So we can get any polynomial in $|x \cap y|$.'

Write the polynomial $(X-\lambda_1)(X-\lambda_2)\cdots(X-\lambda_s)$ as $\sum_{i=0}^{s} a_i \binom{X}{i}$, where $a_0, a_1, \ldots, a_s \in \mathbb{Z}$; this is possible as, for each i, $i!\binom{X}{i}$ is monic. Let $M = \sum_{i=0}^{s} a_i M(i)$. All its rows are in V. Then

$$M_{xy} \text{ is } \begin{cases} 0 \pmod p & \text{when } |x\cap y| \equiv \lambda_i \pmod p \text{ for some } i = 1, 2, \ldots, s \\ \not\equiv 0 \pmod p & \text{otherwise} \end{cases}.$$

Consider the submatrix whose rows and columns are indexed by A. This submatrix has $|A|$ rows, which are linearly independent over $\mathbb{Z}_p$ and so are certainly linearly independent over $\mathbb{R}$. Hence we have $|A|$ linearly independent rows of M and so $|A| \le \binom{n}{s}$. □

Remark. The theorem fails if p is not prime. Grolmusz constructed, for each n, a value $r \equiv 0 \pmod 6$ and a set system $|A| \subset [n]^{(r)}$ such that for any distinct $x, y \in A$, we have $|x \cap y| \not\equiv 0 \pmod 6$, but with $|A| \ge n^{c \log n / \log\log n}$ (for some c). There is a similar construction for any non-prime modulus.

If we have some half-size sets, we expect the intersections to have size around $n/4$, but they are very unlikely to have size exactly $n/4$. Nevertheless:

Corollary 14. *Let p be prime and let $A \subset [4p]^{(2p)}$ with $|x \cap y| \ne p$ for any distinct $x, y \in A$. Then $|A| \le 2\binom{4p}{p-1}$.*

Remark. Note that this bound is *very* small: $\binom{n}{n/4} \le 4e^{-n/32} \cdot 2^n$ (whereas $\binom{n}{n/2} \sim (c/\sqrt{n}) \cdot 2^n$). These estimates follow easily from Stirling's formula, for example.

Proof. By halving the size of A if necessary, we may assume that there is no pair $\{x, x^c\} \subset A$. Then if x, $y \in A$ with $x \neq y$ we have $|x \cap y| \neq 0$, p, so $|x \cap y| \not\equiv 0 \pmod{p}$, and so $|A| \leq \binom{4p}{p-1}$. $\square$

5 Borsuk's Conjecture

Suppose we have $S \subset \mathbb{R}^n$ of diameter d. How many pieces do we need to break S into so that each piece has diameter strictly less than d?

For example, in $\mathbb{R}^2$, taking the vertices of an equilateral triangle shows that we need at least 3 pieces. Similarly, in $\mathbb{R}^n$, a regular n-simplex shows that we need at least $n + 1$ pieces.

Borsuk conjectured that $n + 1$ pieces suffice.

Borsuk's conjecture is true for $n = 1$, 2, 3, and for S smooth, and for S symmetric. However, it is massively false.

Theorem 15 (Kahn, Kalai)**.** *For any n, there is a set $S \subset \mathbb{R}^n$ such that to partition S into pieces of smaller diameter requires at least $c^{\sqrt{n}}$ pieces (for some constant $c > 1$).*

Notes. 1. Our proof will show that Borsuk's conjecture is false for n around 2000.

2. We shall prove Theorem 15 for n of the form $\binom{4p}{2}$ for p prime. We are then done as, for example, for all n there is a prime p with $n/2 \leq p \leq n$.

Proof. We shall construct $S \subset Q_n \subset \mathbb{R}^n$ with $S \subset [n]^{(r)}$ for some r.

For x, $y \in [n]^{(r)}$, we have $d(x,y)^2 = 2(r - |x \cap y|)$. So $d(x,y)$ increases as $|x \cap y|$ decreases. So we seek $S \subset [n]^{(r)}$, say with minimum intersection size k, but such that any subset of S with minimum intersection size greater than k is *much* smaller than S.

Identify $[n]$ with $[4p]^{(2)}$—the edges of K_{4p}, the complete graph on $[4p]$. For each $x \in [4p]^{(2p)}$, let G_x be the complete bipartite graph on vertex-classes x, x^c. Let $S = \{G_x : x \in [4p]^{(2p)}\} \subset [n]^{(4p^2)}$. Then $|S| = \frac{1}{2}\binom{4p}{2p}$.

Now, $|G_x \cap G_y| = k^2 + (2p-k)^2$, where $k = |x \cap y|$, which is minimized at $k = p$. Thus if we have a piece of S, say $\{G_x : x \in A\}$, of diameter smaller than the diameter of S, then we *cannot* have $|x \cap y| = p$ for any x, $y \in A$. So $|A| \leq \binom{4p}{2p-1}$ by Corollary 14. Thus the number of pieces needed is at least

$$\begin{aligned} \frac{\frac{1}{2}\binom{4p}{2p}}{\binom{4p}{p-1}} &\geq \frac{c \cdot 2^{4p}/\sqrt{p}}{4 \cdot e^{-p/8} \cdot 2^{4p}} \quad \text{(for some constant } c\text{)} \\ &\geq c'^p \quad \text{(for some constant } c' > 1\text{)} \\ &\geq c''^{\sqrt{n}} \quad \text{(for some constant } c'' > 1\text{)}, \end{aligned}$$

as required. $\square$

6 Projections

Let $A \subset \mathcal{P}X$ and let $Y \subset X$. The *projection* or *trace* of A on Y is $A|Y = \{x \cap Y : x \in A\}$; thus $A|Y \subset \mathcal{P}Y$—'project A onto the coordinates corresponding to Y'.

In general, if we have upper bounds on some projections $|A|Y_i|$, do we get upper bounds on $|A|$?

A *brick* or *box* in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ where $a_i \leq b_i$ for all i. A *body* $S \subset \mathbb{R}^n$ is a finite union of bricks. The volume of S is written $|S|$ or $m(S)$.

Remarks. 1. In fact, everything will go through for a general compact $S \subset \mathbb{R}^n$.

2. A set system $A \subset Q_n$ gives a body

$$S = \bigcup_{x \in A} [x_1, x_1 + 1] \times [x_2, x_2 + 1] \times \cdots \times [x_n, x_n + 1]$$

with $|A| = m(S)$.

For a body $S \subset \mathbb{R}^n$ and $Y \subset [n]$, the projection of S onto the span of $\{e_i : i \in Y\}$ is denoted by S_Y. For example, if $S \subset \mathbb{R}^3$ then S_1 is the projection of S onto the x-axis:

$$S_1 = \{x_1 \in \mathbb{R} : (x_1, x_2, x_3) \in S \text{ for some } x_2, x_3 \in \mathbb{R}\};$$

and S_{12} is the projection of S onto the xy-plane:

$$S_{12} = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, x_3) \in S \text{ for some } x_3 \in \mathbb{R}\}.$$

We have that $S_Y \subset \mathbb{R}^{|Y|}$.

What bounds on $|S|$ do we get given bounds on some S_Y?

For example, let S be a body in $\mathbb{R}^3$. Then trivially $|S| \leq |S_1||S_2||S_3|$ as $S \subset S_1 \times S_2 \times S_3$. Similarly, $|S| \leq |S_{12}||S_3|$ as $S \subset S_{12} \times S_3$.

What if $|S_{12}|$ and $|S_{13}|$ are known? This tells us nothing—for example, consider $S = [0, 1/n] \times [0, n] \times [0, n]$.

What if $|S_{12}|$, $|S_{13}|$ and $|S_{23}|$ are known?

Proposition 16. *Let S be a body in $\mathbb{R}^3$. Then $|S|^2 \leq |S_{12}||S_{13}||S_{23}|$.*

Remark. We have equality if S is a brick.

For $S \subset \mathbb{R}^n$, the *n-sections* are the sets $S(x) \subset \mathbb{R}^{n-1}$ for each $x \in \mathbb{R}$ defined by

$$S(X) = \{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, x_2, \dots, x_{n-1}, x) \in S\}.$$

Proof (of Proposition 16). Consider first the case when each 3-section is a square, i.e. when $S(x) = [0, f(x)] \times [0, f(x)]$. Then $|S_{12}| = M^2$, where $M = \max_{x\in\mathbb{R}} f(x)$. Also, $|S_{13}| = |S_{23}| = \int f(x)\,dx$, and $|S| = \int f(x)^2\,dx$. Thus we want:

$$\left(\int f(x)^2\,dx\right)^2 \leq M^2 \left(\int f(x)\,dx\right)^2.$$

But $\int f(x)^2\,dx \leq M \int f(x)\,dx$ as $f(x) \leq M$ for all x, so this indeed holds.

For the general case, define a body $T \subset \mathbb{R}^3$ by

$$T(x) = [0, \sqrt{|S(x)|}] \times [0, \sqrt{|S(x)|}].$$

Then $|T| = |S|$ and $|T_{12}| \leq |S_{12}|$ (as $|T_{12}| = \max_{x\in\mathbb{R}} |T(x)|$).

Let $f(x) = |S(x)_1|$ and $g(x) = |S(x)_2|$. Then

$$|T_{23}| = |T_{13}| = \int \sqrt{|S(x)|}\,dx \leq \int \sqrt{f(x)g(x)}\,dx.$$

Also, $|S_{13}| = \int f(x)\,dx$ and $|S_{23}| = \int g(x)\,dx$. So we need

$$\left(\int \sqrt{f(x)g(x)}\,dx\right)^2 \leq \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right),$$

i.e.

$$\int \sqrt{f(x)}\sqrt{g(x)}\,dx \leq \left(\int f(x)\,dx\right)^{1/2}\left(\int g(x)\,dx\right)^{1/2},$$

which is just the Cauchy-Schwarz inequality. □

We say that sets Y_1, Y_2, …, Y_r *cover* $[n]$ if $\bigcup_{j=1}^r Y_j = [n]$. They are a *k-uniform cover* if each $i \in [n]$ belongs to exactly k of the Y_j. For example, for $n = 3$: $\{1\}$, $\{2\}$, $\{3\}$ is a 1-uniform cover, as is $\{1\}$, $\{2, 3\}$; $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ is a 2-uniform cover; $\{1, 2\}$, $\{1, 3\}$ is not uniform.

Our aim is to show that if Y_1, Y_2, …, Y_r form a k-uniform cover then $|S|^k \leq |S_{Y_1}||S_{Y_2}| \cdots |S_{Y_r}|$.

Let $\mathcal{C} = \{Y_1, Y_2, \dots, Y_r\}$ be a k-uniform cover of $[r]$. Note that $\mathcal{C}$ is a *multiset*, i.e. repetitions are allowed—for example, $\{12, 12, 3, 3\}$ is a 2-uniform cover of [3]. Put $\mathcal{C}_- = \{Y_i : n \notin Y_i\}$ and $C_+ = \{Y_i - n : n \in Y_i\}$ (as usual), so $\mathcal{C}_- \cup \mathcal{C}_+$ is a k-uniform cover of $[n-1]$.

Note that if $n \in Y$ then $|S_Y| = \int |S(x)_{Y-n}| dx$ (e.g. if $S \subset \mathbb{R}^3$ then $|S_{13}| = \int |S(x)_1| dx$), and this holds even if $Y = [n]$. Also, if $n \notin Y$ then $|S(x)_Y| \leq |S_Y|$ for all x (e.g. $|S_{12}| \geq |S(x)_{12}|$ for all x).

In the proof of Proposition 16 we used the Cauchy-Schwarz inequality:

$$\int fg \leq \left(\int f^2\right)^{1/2} \left(\int g^2\right)^{1/2}.$$

Here, we'll need Hölder's inequality:

$$\int fg \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

for $(1/p) + (1/q) = 1$, whence, iterating, we get

$$\int f_1 f_2 \cdots f_k \leq \left(\int |f_1|^k\right)^{1/k} \left(\int |f_2|^k\right)^{1/k} \cdots \left(\int |f_1|^k\right)^{1/k}.$$

Theorem 17 (Uniform covers theorem)**.** *Let S be a body in $\mathbb{R}^n$, and let $\mathcal{C}$ be a k-uniform cover of $[n]$. Then*

$$|S|^k \leq \prod_{Y \in \mathcal{C}} |S_Y|.$$

Proof. The proof is by induction on n; the case $n = 1$ is trivial.

Given a body $S \subset \mathbb{R}^n$ for $n \geq 2$, we have

$$\begin{aligned}
|S| &= \int |S(x)|\, dx \\
&\leq \int \prod_{Y \in \mathcal{C}_+} |S(x)_Y|^{1/k} \prod_{Y \in \mathcal{C}_-} |S(x)_Y|^{1/k}\, dx \\
&\leq \prod_{Y \in \mathcal{C}_-} |S_Y|^{1/k} \int \prod_{Y \in \mathcal{C}_+} |S(x)_Y|^{1/k}\, dx \\
&\leq \prod_{Y \in \mathcal{C}_-} |S_Y|^{1/k} \prod_{Y \in \mathcal{C}_+} \left(\int |S(x)_Y|\, dx\right)^{1/k} \\
&= \prod_{Y \in \mathcal{C}_-} |S_Y|^{1/k} \prod_{Y \in \mathcal{C}_+} |S_{Y \cup n}|^{1/k} \\
&= \prod_{Y \in \mathcal{C}} |S_Y|^{1/k}.
\end{aligned}$$

□

Corollary 18 (Loomis-Whitney theorem)**.** *Let S be a body in $\mathbb{R}^n$. Then*

$$|S|^{n-1} \leq \prod_{i=1}^{n} |S_{[n]-i}|.$$

Proof. The family $[n]-1$, $[n]-2$, ..., $[n]-n$ is an $(n-1)$-uniform cover of $[n]$. □

Remark. The case $n = 3$ of the Loomis-Whitney theorem is Proposition 16.

Corollary 19. *Let $A \subset Q_n$, and let $\mathcal{C}$ be a k-uniform cover of $[n]$. Then*

$$|A|^k \leq \prod_{Y \in \mathcal{C}} |A|Y|.$$

In particular, if $\mathcal{C}$ is a uniform cover with $|A|Y| \leq 2^{c|Y|}$ for all $y \in \mathcal{C}$ then $|A| \leq 2^{cn}$.

Proof. For the first part, consider the body

$$S = \bigcup_{x \in A} [x_1, x_1+1] \times [x_2, x_2+1] \times \cdots \times [x_n, x_n+1].$$

Then $m(S) = |A|$ and $m(S|Y) = |A|Y|$ for all Y.

For the second part, suppose that $\mathcal{C}$ is a k-cover. Then

$$|A|^k \leq \prod_{Y \in \mathcal{C}} |A|Y| \leq \prod_{Y \in \mathcal{C}} 2^{c|Y|} = 2^{c \sum_{Y \in \mathcal{C}} |Y|} = 2^{ckn}.$$

□

There is a remarkable extesnsion of the uniform covers theorem, called the 'Bollobás-Thomason box theorem'. This states that for any body S there is a box B with $|B| = |S|$ and $|B_Y| \leq |S_Y|$ for all Y. This theorem has no right to be true. For example, we can then read off all possible projection theorems—just check them for boxes.

7 Intersecting Families of Graphs

What happens to intersecting families if we have more structure in our ground set?

One natural example is to take our ground set to be $[n]^{(2)}$, the edges of the complete graph on $[n]$. There are a total of $2^{\binom{n}{2}}$ graphs on $[n]$.

How many graphs can we find such that any two intersect in something containing P_2, the path of length 2? We want to find $\max|A|$ subject to $G, H \in A \implies G \cap H \supset P_2$. Clearly $|A| \le (1/2)2^{\binom{n}{2}}$ (as we cannot have both $G \in A$ and $G^c \in A$ for any graph G). We can get $|A| \sim (1/2)2^{\binom{n}{2}}$ by fixing $x \in [n]$ and taking

$$A = \left\{ G : d_G(x) \ge \frac{n}{2} + 1 \right\};$$

this has

$$|A| \sim \left(\frac{1}{2} - \frac{c}{\sqrt{n}} \right) 2^{\binom{n}{2}}.$$

Similarly, we can get $|A| \sim (1/2)2^{\binom{n}{2}}$ for $G \cap H$ containing a star.

Conjecture 20. *If $G, H \in A \implies G \cap H$ contains a triangle, then $|A| \le (1/8)2^{\binom{n}{2}}$.*

Note that we can obtain $|A| = (1/8)2^{\binom{n}{2}}$ by taking A to consist of all graphs G which contain some fixed triangle.

Theorem 21. *Let $A \subset \mathcal{P}\left([n]^{(2)}\right)$ be such that if $G,\ H \in A$ then $G \cap H$ contains a triangle. Then $|A| \le (1/4)2^{\binom{n}{2}}$*

Proof. We want $|A| \le 2^{\binom{n}{2}-2} = 2^{\binom{n}{2}(1-2/\binom{n}{2})}$, so it is enough to find a uniform cover $\mathcal{C}$ of $[n]^{(2)}$ such that for all $Y \in \mathcal{C}$ we have $|A \cap Y| \le 2^{c|Y|}$, where $c = 1 - 4/(n(n-1))$.

For n even, take all Y of the form $B^{(2)} \cup (B^c)^{(2)}$ with $|B| = |A|/2$. This is clearly a uniform cover. Now for any such Y, $G \cap H$ is not bipartite and so G and H meet on Y. Thus $A|Y$ is intersecting, whence

$$|A|Y| \le (1/2)2^{|Y|} = 2^{2\binom{n/2}{2}-1} = 2^{2\binom{n/2}{2}\left(1-1/\left(2\binom{n/2}{2}\right)\right)},$$

so we need

$$1 - \frac{1}{2\binom{n/2}{2}} \le 1 - \frac{4}{n(n-1)}.$$

For n odd, we do the same thing but with $|B| = (n-1)/2$. □

This conjecture was eventually proved by Ellis, Filmus and Friedgut.

The interest of the above result is that we bound the size of the family away from $(1/2)2^{\binom{n}{2}}$. Let us call a fixed graph F *common* if there is a family A of graphs on n vertices such that for any G and H in A the intersection

$G \cap H$ contains a copy of F, and the size of A is $(1/2 - o(1))2^{\binom{n}{2}}$. Thus the above result states that a triangle is not common.

It is not hard to see that any star is common (as mentioned above), and also that any disjoint union of stars is common. Alon's common graphs conjecture asserts that these are the only common graphs. Now, it is easy to check that any graph that is not a disjoint union of stars contains either a triangle or a path P_3 of length 3. Hence Alon's common graphs conjecture boils down exactly to the question of whether or not P_3 is common!

Interestingly, it is known that the greatest size of a family of graphs with any two having intersection containing a P_3 is not obtained by taking all graphs that contain a fixed copy of P_3. This would give density 1/8, but Christofides constructed an example of density 17/128.