ICTS
INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES
NEWS
VOLUME VII
ISSUE 2 2021
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

MY CLOSE FRIEND AND MENTOR

M. S. RAGHUNATHAN

The passing away of Professor Narasimhan is a great loss to mathematics, to the country and, of course, to the numerous people close to him. Narasimhan was a towering figure in contemporary mathematics. He is one of a small number of mathematicians who have made profound, beautiful and lasting contributions in several different (mathematical) fields: algebraic geometry, differential geometry, representation theory, analysis and mathematical physics. In this versatility he has no near rivals from our country. He remained active as a mathematician right

... continued on Page 9 ...

NARASIMHAN WAS OF A DIFFERENT MOULD

S. RAMANAN

The recent passing away of M.S. Narasimhan, known to friends as MSN, is a great loss to the mathematical world in general and to India in particular. His interests in mathematics were very broad. He was also generous with his ideas, which explains his proclivity to collaborate with many people, some of whom were top mathematicians and others young students.

... continued on Page 8 ...

THE NARASIMHAN–SESHADRI THEOREM AS (INDIRECT) INSPIRATION

CHANDRASHEKHAR KHARE

I met Professor M. S. Narasimhan just a few times spread over a couple of decades. These were chance encounters. I worked at TIFR for around a decade, after I finished my PhD from Caltech and UCLA in 1995, and came back to India with the goal of working there, and establishing myself as a research mathematician.

I first saw him at a banquet at a summer school I attended in Nice (probably around 1992-1993). I must have seen him a few times during the period I worked in TIFR. On these early occasions I never had a chance to speak to him or interact with him in any meaningful way. I knew him by his fame as a leading mathematician, who had developed a famous school in India based at TIFR, but he was a very distant figure to me.

The next time I remember seeing him was when he was in the audience when I once lectured in Bangalore. This must have been around 2008 or 2009, although it could have been a little earlier. The topic of my lecture was Serre's modularity conjecture, which I had just proved (or was about to complete proving) in work with my French colleague

... continued on Page 6 ...

ON THE WORK OF NARASIMHAN AND SESHADRI

EDWARD WITTEN

It was an honor to be invited to lecture in memory of M.S. Narasimhan and C. S. Seshadri and to write this article. At first I was not sure if I should accept, since I realized that my point of view would be somewhat limited. But I decided I could try to give a perspective on how parts of their work have influenced theoretical physics, including my own work.

The Narasimhan-Seshadri theorem, of which the first version was proved in 1964, is a comparison between flat vector bundles and holomorphic vector bundles over a Riemann surface Σ. Any flat bundle has a natural holomorphic structure. Narasimhan and Seshadri would have explained this as follows. Let H be the universal cover of Σ and let $H \times \mathbb{C}^N$ be a trivial rank N complex bundle over H. As a trivial bundle, it is a flat bundle over H and also a holomorphic bundle. Any flat bundle $E \to \Sigma$ is a quotient $E = (H \times \mathbb{C}^N)/\pi_1(\Sigma)$, with $\pi_1(\Sigma)$ acting on H in the natural way and on $\mathbb{C}^N$ via some homomorphism $\rho : \pi_1(\Sigma) \to GL(N)$. The quotient $E = (H \times \mathbb{C}^N)/\pi_1(\Sigma)$ comes with an obvious holomorphic structure: a local section s of E is holomorphic if it pulls back to a holomorphic section of the trivial holomorphic bundle $H \times \mathbb{C}^N$. In a gauge theory language that in this context was introduced later by M. F. Atiyah and R. Bott, one would say the following: a flat bundle E has a flat connection A; the corresponding gauge-covariant exterior derivative d_A has a $(0, 1)$ part $\bar\partial_A$ which defines a holomorphic structure of E.

The article is based on a lecture at a special session in memory of Narasimhan and Seshadri at the ICTS program QFT-GRT-21. The program was devoted to the geometric Langlands correspondence so in my lecture I made particular note of connections of the work of Narasimhan and Seshadri with that subject.

BETWEEN THE SCIENCE

ICTS faculty member **SAMRIDDHI SANKAR RAY** has been elected a member of the National Academy of Sciences, India.

Samriddhi works in the area of fluid dynamics and his research interests include fluid, magnetohydrodynamic, passive-scalar, and Burgers turbulence, inertial (finite-sized) particles in turbulent flows, truncated systems, thermalization, and statistical mechanics of turbulent flows and Singularities in the Euler equation.

WITTEN | *continued from Page 1 ...*

It was a classical result that if Σ is a Riemann surface and $\mathcal{L}\to\Sigma$ is a holomorphic line bundle, then $\mathcal{L}$ admits a unitary flat connection if and only if $\mathcal{L}$ has degree 0, or equivalently $c_1(\mathcal{L})=0$. Narasimhan and Seshadri became interested in the analogous question for the case of a holomorphic vector bundle $E\to\Sigma$ of rank $N>1$. Does E admit a unitary flat connection, which in this case would have structure group $U(N)$? There is an obvious necessary condition $c_1(E)=0$, but this is far from the whole story.

For example, suppose that

$$E=\mathcal{O}(p)\oplus\mathcal{O}(-p),$$

where p is a point in Σ. ($\mathcal{O}(p)$ is the line bundle whose sections are holomorphic functions that are allowed to have a simple pole at p). Then $c_1(E)=0$, but E does not admit any unitary flat connection. This can be proved as follows. $\mathcal{O}(p)$, and therefore E, has a holomorphic section s that vanishes at p. But if E has a flat unitary connection A, and $\bar\partial_A s=0$, then by integration by parts, after picking a Kahler metric on Σ, one can prove

$$\int_\Sigma d^2x\sqrt{g}\,|d_A s|^2=\int_\Sigma d^2x\sqrt{g}\,|\bar\partial_A s|^2=0,$$

and hence $d_A s=0$. Therefore, s (if not identically zero) cannot vanish anywhere. This explanation is in the spirit of gauge theory; it is probably not very close to what Narasimhan and Seshadri would have said in 1964.

Mumford had defined the notion of a *stable* holomorphic vector bundle over a Riemann surface. A bundle E, for simplicity with $c_1(E)=0$, is *unstable* if it has a holomorphic subbundle E' of positive degree:

$$0\to E'\to E\to E''\to 0.$$

For example, the bundle

$$E=\mathcal{O}(p)\oplus\mathcal{O}(-p)$$

is unstable because it has the subbundle $\mathcal{O}(p)$ of positive degree. Mumford also defined E to be semistable if it has a proper subbundle E' of degree 0 but none of positive degree. If every proper subbundle of E has negative degree, then in Mumford's terminology, E is stable.

The Narasimhan-Seshadri theorem, as originally formulated in 1964, says that *irreducible* unitary flat bundles of rank N are in one-to-one correspondence with stable bundles. The easier direction is to show that for a holomorphic bundle E to have a flat unitary structure, it is necessary for it to be stable. This can be proved by generalizing what I said about the case $\mathcal{O}(p)\oplus\mathcal{O}(-p)$. (Again, the approach of Narasimhan and Seshadri would have been somewhat different.) The opposite direction is more difficult: stability is sufficient for existence of a flat unitary structure. I am not the right one to explain the proof of this.

I have stated the Narasimhan-Seshadri result for bundles of degree 0, but they actually had a more general statement about bundles of rank N and degree d, for any d. In gauge theory language, for $d\neq 0$, instead of flat unitary bundles, one can talk about unitary bundles whose curvature is central. (Their formulation was slightly different.) The resulting moduli space is smooth and compact if and only if N and d are relatively prime, i.e. $(N,d)=1$. From a modern point of view of "D-branes" in string theory, this has the following interpretation. N and d are D-brane charges (twobrane and zerobrane charge, for instance), and the condition $(N,d)=1$ means that a D-brane configuration with charges N, d cannot separate into two subsystems in a supersymmetric fashion.

Seshadri went on to extend the Narasimhan-Seshadri theorem in two directions. The moduli space of *irreducible* unitary flat bundles is not compact if d and N are not relatively prime (for example in the most basic case $d=0$), because a reducible flat bundle $E_1\oplus E_2$ can be slightly perturbed to make it irreducible (i.e. it can arise as a limit of irreducible flat bundles). Likewise, in view of the Narasimhan-Seshadri theorem, the moduli space of stable bundles is not compact. However, Seshadri extended the Narasimhan-Seshadri theorem to an equivalence between compact moduli spaces: the moduli space of unitary flat bundles (not necessarily irreducible) and the moduli space of holomorphic bundles that are stable and/or semistable (this is usually called for short the moduli space of semistable bundles).

A primary subtlety here is that Seshadri had to understand and incorporate an equivalence relation for semistable bundles. There is not a sensible (Hausdorff) moduli space that parametrizes all equivalence classes of semistable bundles. I will explain this in gauge theory language. Consider a rank 2 bundle E that is a direct sum

$$E=\mathcal{L}\oplus\mathcal{L}^{-1}$$

of line bundles $\mathcal{L}$ of degree 0. Such an E is semistable (it would be unstable if $\mathcal{L}$ has nonzero degree since then either $\mathcal{L}$ or $\mathcal{L}^{-1}$ has positive degree).

The direct sum $\mathcal{L}\oplus\mathcal{L}^{-1}$ has a $\mathbb{C}^*$ group of automorphisms acting as diag (t,t^{-1}) on the two summands $\mathcal{L},\mathcal{L}^{-1}$. Now consider a triangular deformation of E, by adding an upper triangular perturbation to its $\bar\partial$ operator

$$\bar\partial+\begin{pmatrix}a&0\\0&-a\end{pmatrix}\to\bar\partial+\begin{pmatrix}a&c\\0&-a\end{pmatrix},$$

for some c that represents a nonzero class in $H^1(\Sigma,\mathrm{Hom}(\mathcal{L}^{-1},\mathcal{L}))$, and therefore cannot be gauged away. In general, such a c exists. By the action of $\mathbb{C}^*$ we can replace this with

$$\bar\partial+\begin{pmatrix}a&0\\0&-a\end{pmatrix}\to\bar\partial+\begin{pmatrix}a&\lambda c\\0&-a\end{pmatrix},$$

for any $\lambda\neq 0$. So the bundle E'_λ defined by the $\bar\partial$ operator on the right hand side has a holomorphic type that is independent of λ as long as $\lambda\neq 0$; on the other hand, at $\lambda=0$, E' reduces to E and its holomorphic type is different. Since E' is thus "infinitesimally close" to E, if we are going to define a sensible moduli space of semistable bundles, we have to treat E and E' as equivalent.

So Seshadri had to develop an equivalence relation, S-equivalence, on semistable bundles, basically considering any triangular deformation of $\mathcal{L}\oplus\mathcal{L}^{-1}$ (where $\mathcal{L}$ has degree 0) to be equivalent to the direct sum. He was able to define a compact moduli space of equivalence classes of semistable (or stable) bundles, usually called in brief the moduli space of semistable bundles, and to prove that it is equivalent to the moduli space of unitary flat bundles, not necessarily irreducible. (This way to state the result is for the basic case of degree 0.) Apart from the importance of this particular problem, the ideas in Seshadri's construction were important in many later constructions of moduli spaces in algebraic geometry.

Seshadri's other generalization of the Narasimhan-Seshadri theorem, in the 1970's, with V. B. Mehta, was to consider flat unitary bundles on a surface with punctures (Fig. 1). On one side of the correspondence, they consider flat bundles with prescribed conjugacy classes of the monodromy around the punctures. On the other side, they consider holomorphic bundles with "parabolic structure" at the punctures (meaning a reduction of the structure group of a bundle from $GL(N,\mathbb{C})$ to a parabolic subgroup).

Here it is necessary to generalize the ideas of stability, semi-stability, and S-equivalence to parabolic bundles. This is more complicated than the previous case because the stability condition depends on some parameters (which correspond to the conjugacy class of the monodromy in the other description). After finding the right definitions, Seshadri and Mehta proved a correspondence between flat bundles with prescribed monodromy and semistable parabolic bundles. I will mention two applications later.

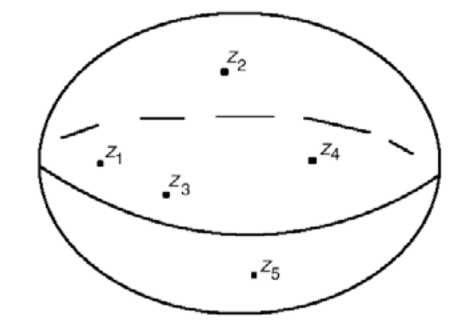


Fig. 1. A Riemann surface of genus 0 with 5 marked points or punctures

I will also mention two contributions by Narasimhan from the 1970's. First, with Ramanan, he introduced a geometric analog of the Hecke transformations of number theory. In number theory, a Hecke operator acts in a sense only at one prime. The geometric analog of a Hecke transformation by Narasimhan and Ramanan modifies the holomorphic structure of a holomorphic vector bundle $E\to\Sigma$ at just one point in Σ. It is impossible to modify a flat vector bundle over a Riemann surface at just one point, but a holomorphic vector bundle over a Riemann surface can be modified at just one point, and this is what Narasimhan and Ramanan did. Geometric Hecke transformations played a central role when the geometric Langlands program was developed, starting around 1990, by Beilinson and Drinfeld and then many others. In the gauge theory approach to geometric Langlands, geometric Hecke transformations have a natural interpretation in terms of what physicists know as 't Hooft operators. This is actually one of the most important points in the whole story.

Also in the 1970's, Narasimhan and Harder computed the Betti numbers of the moduli space of stable bundles (in the case $(N,d)=1$ where this is smooth and compact). They did this in a very surprising way by comparing to characteristic p and using the Weil conjectures (which were proved by Deligne at just about this time). They introduced a stratification on the set of all bundles over a curve, basically according to how unstable a bundle is. Then, in the case of a curve over a finite field of characteristic p, they counted the stable bundles. The unstable bundles are relatively easy to count by induction on the rank. Knowing that the "Tamagawa number" of SL_N is 1 gave a "sum rule" which then gave a count of stable bundles. The form of the answer was such that, with the help of the Weil conjectures, the Betti numbers of the moduli space of stable bundles could be determined.

This helped set the stage for an important contribution by Atiyah and Bott. In 1982, Atiyah and Bott introduced a gauge theory perspective on the Narasimhan-Seshadri theorem and the Narasimhan-Harder results about the topology of the moduli space of stable bundles. They considered gauge theory of a compact group G on a Riemann surface Σ, with gauge field A and curvature $F=dA+A\wedge A$. The Yang-Mills action is

$$I=\int_\Sigma \mathrm{Tr}\,F\wedge\star F.$$

Their basic idea was to consider I as an equivariant Morse function on the space of all connections. For this, one has to consider the critical points of I. Unlike the situation in higher dimensions, in two dimensions it is possible to explicitly describe all of the critical points of the Yang-Mills action. The critical points with $I=0$ are the unitary flat connections. According to the Narasimhan-Seshadri theorem, these correspond to stable holomorphic bundles. Atiyah and Bott showed that the other components of critical points of I are in one-to-one correspondence with the strata of the Narasimhan-Harder stratification.

Because the Morse indices of the critical point sets are all even, Atiyah and Bott observed that *if* Morse theory applies to this situation, as they conjectured, then I is an equivariantly perfect Morse function on the space of all connections (equivariant with respect to the action of the gauge group). They showed that this would imply formulas for the Betti numbers of the moduli space of stable bundles that agreed with those of Narasimhan and Harder. Here they used the fact that the space of all connections (ignoring the action of the gauge group) is contractible, a fact that for them played the role of "Tamagawa number equals 1" for Narasimhan and Harder. A decade later it was proved independently by G. Daskapouos and J. Rade that Morse theory does apply in this problem. (This involves showing that the gradient flow equation associated to the Yang-Mills action, which is a parabolic partial differential equation in three dimensions, contracts the space of all connections onto the set of critical points.) In 1982, however, Atiyah and Bott avoided the analytical difficulties by considering instead the action on the space of connections of the complexification of the gauge group. This sufficed for proving the formulas for the Betti numbers.

The Atiyah-Bott paper inspired a new gauge theory proof of the Narasimhan-Seshadri theorem by S. Donaldson, in 1983. This in turn inspired many generalizations. One that is of particular importance for geometric Langlands was due to N. Hitchin. Narasimhan and Seshadri had interpreted *unitary* representations of the fundamental group of a Riemann surface in terms of algebraic geometry. What about representations that might be *nonunitary*? Hitchin proved that (semistable and not necessarily unitary) representations of the fundamental group of a surface correspond to stable Higgs bundles (E,φ), where E is a holomorphic bundle and φ is a holomorphic map $\varphi: E\to E\otimes K$. Extending this, he proved that the moduli space of either of those two types of object has a complete *hyper-Kahler* structure. This was the generalization of the Narasimhan-Seshadri theorem for non-unitary flat bundles.

Also in the 1980's, using the gauge theory formulation, the Narasimhan-Seshadri theorem was generalized to complex manifolds of dimension greater than 1 by Donaldson as well as by K. Uhlenbeck and S.-T. Yau. In higher dimensions, the theorem says that a semistable holomorphic bundle admits a hermitian metric such that the corresponding connection satisfies the Yang-Mills equations; conversely, a holomorphic bundle with such a hermitian metric is semistable.

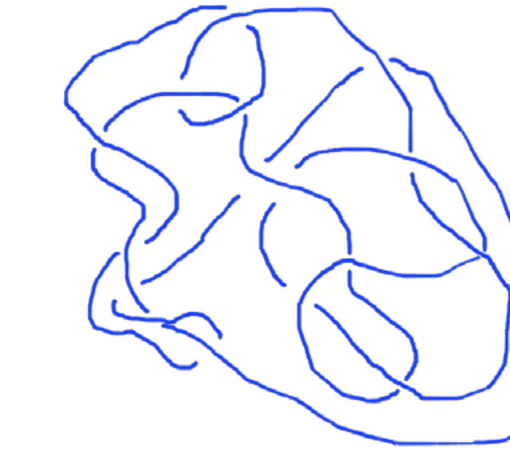


Fig. 2. A knot in $\mathbb{R}^3$

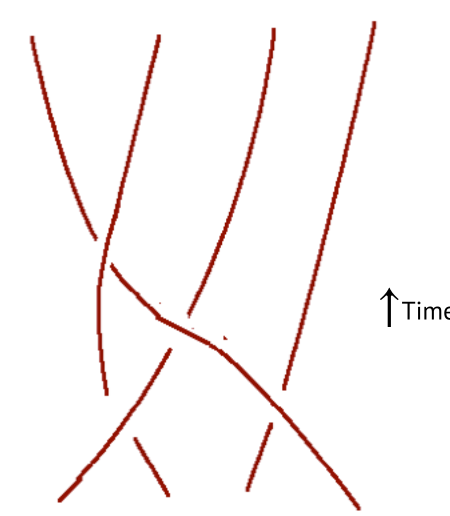


Fig. 3. A braid on four strands

I well remember how obscure these things seemed to me when I heard about them from Atiyah and Bott in the early 1980's. However, physicists became familiar with such matters starting in the mid-1980's when Calabi-Yau threefolds and holomorphic vector bundles over them were used to make models of particle physics in the context of string theory. This has actually become a very important direction. In this context, the existence of hermitian Yang-Mills metrics – in other words, the generalization by Donaldson and by Uhlenbeck and Yau of the Narasimhan-Seshadri theorem – is completely crucial. The subtleties of stable, unstable, and semistable bundles appear in this context as properties of the low energy effective action of the model.

But instead of explaining that, I will explain the role of the Narasimhan-Seshadri theorem in my own work a few years later on rational conformal field theory and the Jones polynomial. Vaughn Jones had discovered the Jones polynomial in 1983. It is a subtle invariant of a knot in $\mathbb{R}^3$ (Fig. 2). From the beginning, it was related to mathematical physics in a bewildering variety of ways.

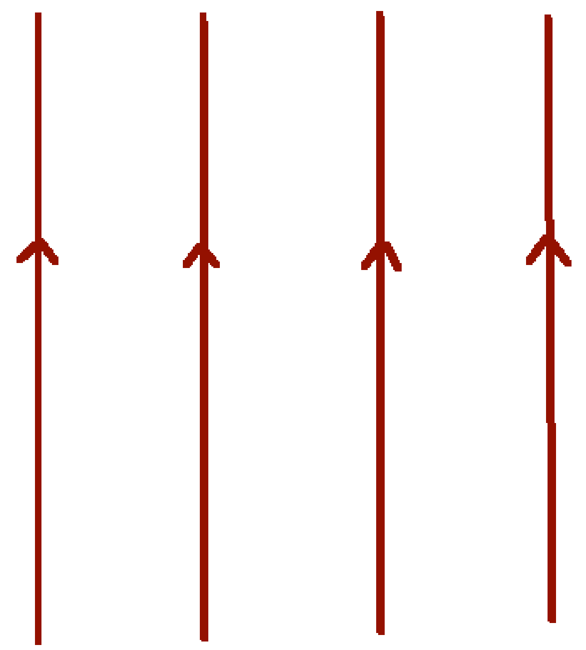


Fig. 4. To construct the Jones representations of the braid group from gauge theory, one first constructs a Hilbert space in the presence of n parallel vertical strands. Then, letting the points vary, one constructs a flat connection over the configuration space of n distinct points in the plane; the monodromy of this flat connection gives the Jones representations

Jones's original definition involved the Jones representations of the braid group. A braid in the mathematical sense is a picture like the one sketched in Fig. 3. Points move around in the plane, then return to their starting positions. In this example, I have drawn a braid with 4 strands. Braids form a group because they can be composed by gluing one braid on top of another.

Jones's original definition of the Jones polynomial of a knot was as follows. We can build a knot by gluing together the top and bottom ends of a braid. This gluing is a little bit like taking a trace. Let B be a braid. If R is a representation of the braid group, let us write $\mathcal{R}(B)$ for the matrix that represents the braid B in this representation. In his original work, Jones constructed representations $R_i(q)$ of the braid group, where q is a complex parameter, and defined the Jones polynomial as a linear combination of the corresponding traces:

$$J(q)=\sum_i c_i(q)\,\mathrm{Tr}_{R_i(q)}\,\mathcal{R}_{i,q}(B)$$

It worked, but it was unclear why, since there was no manifest three-dimensional symmetry in the construction. There are many different ways to construct the same knot as the "trace" of a braid. Why would this construction always give the same result? What is special about the particular representations $R_i(q)$ and the coefficients $c_i(q)$? Atiyah recommended these questions for physicists.

A very important step was taken by A. Tsuchiya and Y. Kanie. They showed that the Jones representations of the braid group are the ones that arise when one decomposes the correlation functions of the two-dimensional WZW model in conformal blocks, as originally analyzed by V. Knizhnik and A. Zamolodchikov. The background to this statement is as follows. Consider the correlation functions of a primary field Φ in this theory, say in genus 0:

$$G(z_1,\bar z_1;z_2,\bar z_2;\cdots;z_n,\bar z_n)=\langle\Phi(z_1,\bar z_1)\Phi(z_2,\bar z_2)\Phi(z_3,\bar z_3)\cdots\Phi(z_n,\bar z_n)\rangle.$$

These functions are neither holomorphic nor antiholomorphic, and they cannot be factored as the product of a holomorphic and an antiholomorphic function. But Knizhnik and Zamolodchikov had shown that they are *finite* sums of products of holomorphic and antiholomorphic functions:

$$G(z_1,\bar z_1;z_2,\bar z_2;\cdots;z_n,\bar z_n)=\sum_\alpha f_\alpha(z_1,z_2,\dots z_n)\,\bar f_\alpha(\bar z_1,\bar z_2,\dots\bar z_n)$$

Here the functions $f_\alpha(z_1,z_2,\dots z_n)$ are multivalued holomorphic functions. For each n, we can define a vector bundle V_n over the configuration space of n distinct points $z_1, z_2, \dots z_n \in \mathbb{C}\cup\infty$ with a basis given by the f_α. These are automatically flat vector bundles (with the f_α understood as covariantly constant sections), and their *monodromies* when the points move around give representations of the braid group. Single-valuedness of the correlation functions of the WZW model implies that these representations are unitary. The observation of Tsuchiya and Kanie was essentially that, for symmetry group $G = SU(2)$ and a primary field in the 2-dimensional representation, these are the Jones representations.

The holomorphic functions f_α are called conformal blocks. E. Verlinde had defined operators that act on the space of conformal blocks, and intuitively it seemed that he was treating the space of conformal blocks of the WZW model as the Hilbert space of some quantum system. But what was that system? At some point, I had the idea that the appropriate quantum system was simply Chern-Simons gauge theory, with the action

$$\mathrm{I}=\frac{k}{4\pi}\int_w d^3x\,\epsilon^{ijk}\,\mathrm{Tr}\left(A_i\,\partial_j\,A_k+\frac{2}{3}\,A_iA_jA_k\right)$$

where A is a connection on a G-bundle over a three-manifold W and k is the level of the WZW model. Because the only structure of W used in defining this action is an orientation, one can hope that a quantum theory derived from this action will be topologically invariant and will give three-manifold invariants.

We can include a knot or embedded circle $K\subset W$ by including a Wilson loop operator

$$W_R(K)=\mathrm{Tr}_R P\exp\oint_K A,$$

where R is a representation of G and the expression on the right hand side is just a way to describe the trace, in the representation R, of the holonomy of a connection A around K. Now the path integral

$$\int DA\,\exp(iI)\,W_R(K)$$

depends on K (and R) as well as on W, but nothing else. So (taking $W=\mathbb{R}^3$) this will potentially give an invariant of a knot. (It turns out that these statements need slight modification: because of an anomaly, both W and K must be "framed.")

But to get anywhere, we need to know that the quantum Hilbert space $\mathcal{H}_\Sigma$ of Chern Simons theory, on a Riemann surface Σ, is the same as the space of conformal blocks of the WZW model on Σ (with the same symmetry group G and the same "level" k). That is where the Narasimhan-Seshadri theorem came in. First of all, to construct $\mathcal{H}$ we are supposed to "quantize" the classical phase space. The Euler-Lagrange equations of the Chern-Simons action just say that $F = dA+A\wedge A$, the curvature of a connection A, should vanish. The phase space that we have to quantize to construct $\mathcal{H}_\Sigma$ is thus the moduli space M of flat connections on $\mathbb{R}\times\Sigma$ (here $\mathbb{R}$ parametrizes the "time") which is the same as the moduli space of homomorphisms $\rho:\pi_1(\Sigma)\to G$. Somehow we have to quantize M and show that the result is the same as the space of conformal blocks of the WZW model. M should be quantized with symplectic structure $k\omega_0$ where $\omega_0/2\pi$ generates $H^2(M,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{R}$.

There is not any known way to quantize M without picking some additional structure, and I expect that there is no way to do so. Suppose we choose the additional structure to be a complex structure on Σ. Then we can invoke the Narasimhan-Seshadri theorem, which says that M is the same as the moduli space $\mathcal{M}$ of semistable holomorphic G-bundles over Σ. Knowing this, we can quantize $\mathcal{M}$ using geometric quantization. There is a fundamental holomorphic line bundle $\mathcal{L}\to\mathcal{M}$, whose first Chern class is represented in de Rham cohomology by ω_0 (for $G = SU(N)$, $\mathcal{L}$ is the determinant line bundle of the Dirac operator on Σ, coupled to a rank N holomorphic vector bundle $E\to\Sigma$). Then $\mathcal{L}^k$ is a prequantum line bundle for this problem in the language of geometric quantization, and geometric quantization tells us that we should define $\mathcal{H}=H^0(\mathcal{M},\mathcal{L}^k)$ as a quantization of $\mathcal{M}$.

At this point we get "lucky." The answer $H^0(\mathcal{M}_\Sigma,\mathcal{L}^k)$ coincides with a known and in a sense standard – though rather abstract – description of the space of WZW model conformal blocks on a genus g surface. (I learned this description from G. Segal and I was fortunate here as this description was certainly not well-known among physicists at the time and possibly also not today. It is widely used – in a more general form – in research on the geometric Langlands program.) This is the basic link between 2 and 3 dimensions, and as I explained it depends on the Narasimhan-Seshadri theorem.

Actually, I explained this in the absence of knots. To incorporate knots, we need to use the later refinement of the Narasimhan-Seshadri theorem by Seshadri to include parabolic structure. To get the Jones representations of the braid group, we pick some points $p_i\in\Sigma$ and place knots, in some representations R_i, on $p_i\times\mathbb{R}\subset\Sigma\times\mathbb{R}$. See Fig. 4 for a local picture. Now one can show that the phase space that has to be quantized is a moduli space M' of flat bundles on $\Sigma\backslash\{p_1,\cdots,p_s\}$ with a prescribed conjugacy class of the monodromy around the p_i. So now we have to quantize M'. To do this, we again pick a complex structure on Σ, and now, invoking Seshadri's extension of the Narasimhan-Seshadri theorem, we can identify M' with the moduli space $\mathcal{M}'$ of semistable holomorphic $G_\mathbb{C}$ bundles over Σ with parabolic structure at the points p_i.

A key ingredient is missing in what I have said so far. We picked a complex structure J on Σ in order to quantize the moduli space M of flat bundles or its counterpart M' of flat bundles with point singularities. Of course, the choice of complex structure violated topological invariance. So now we have to show that the Hilbert space $\mathcal{H}_J$ that we construct in complex structure J really does not depend on J. This is proved by constructing a (projectively) flat connection that lets us compare the $\mathcal{H}_J$'s for different (nearby) J's. For the case of punctures on the plane, as in Fig. 4, the positions of the punctures are the complex moduli, and the monodromy of the flat connection gives the Jones representations of the braid group. Construction of the flat connection is a very important part of the story, which can be understood by adapting to infinite dimensions some simple facts about quantization in finite dimensions. But as this would take us rather far afield from the Narasimhan-Seshadri theorem, I will not explain this here.

However, I want to draw attention to a superficially similar problem that has *not* been solved. We do not have a good understanding of Chern-Simons gauge theory with a noncompact real semi-simple gauge group G such as $SL(N,\mathbb{R})$

$$\mathrm{I}=\frac{k}{4\pi}\int_W d^3x\,\epsilon^{ijk}\,\mathrm{Tr}\left(A_i\,\partial_j\,A_k+\frac{2}{3}\,A_iA_jA_k\right),$$

because the Narasimhan-Seshadri theorem does not apply to flat bundles with non-compact gauge group. The closest analog is Hitchin's theorem about Higgs bundles. After picking a complex structure J on Σ, we can indeed use that theorem, as was shown by Hitchin, to construct a Kahler polarization of the appropriate phase space M. But we do not understand in what sense the resulting Hilbert space $\mathcal{H}_J$ is independent of J, so we do not know how to exhibit topological invariance. (This is a broad brush statement; there are some important special cases in which the problem has been solved or circumvented.)

There is one much more recent contribution by Seshadri that I wish to mention. Seshadri and V. Balaji (2010) developed a theory of holomorphic vector bundles over a Riemann surface with "parahoric" structure, which is more general than parabolic structure. (There is also a gauge theory approach by P. Boalch, and there have been other more recent papers.) I only learned about this work very recently and will not attempt to describe it here. I will just make the following remark. In 2006-8, Sergei Gukov and I wrote two papers on "surface operators" in the context of the gauge theory approach to geometric Langlands. In the first paper, we used Seshadri's theory of parabolic structure, or more precisely the extension of this to Higgs bundles by C. Simpson, to describe in gauge theory the "ramified" case of geometric Langlands. But then we realized that there were more general singularities to consider; we tried to analyze these in the second paper but were not able to get a systematic picture. Since learning about parahoric structure, I have suspected that this might provide a better framework for at least a large part of what we were trying to do.

In conclusion, in this article I have described some of the celebrated contributions of Narasimhan and Seshadri and hopefully I have given at least a few hints of the role that their work has played in theoretical physics. □

Edward Witten is the Charles Simonyi Professor at the Institute for Advanced Study, Princeton.

MY DEAR FRIEND, NARASIMHAN

GEORGE THOMPSON

I first met Narasimhan in Paris in the late 80's. I was there for a year and Ramadas was also there for an extended stay. One day, in the then library of the LPTHE of Paris VI/VII, Ramadas introduced me to a handsome, elegant, though very quiet man. The meeting was bit awkward and we did not speak much at all. Later Ramadas was shocked to learn that I did not know who Narasimhan was. As fate would have it this ignorance was to be corrected very shortly afterwards.

In the early 90's I moved to the ICTP and a year or so later Narasimhan was appointed as Head of the Mathematics section here. By that time my research had wandered into the physics side of moduli spaces of bundles on Riemann Surfaces, I learnt to my embarrassment something I should have known, namely that Narasimhan was a father of the field. So for the following years you could find us animatedly discussing in the bar at the ICTP, at least that is, when he wasn't helping every young mathematician in sight. Our discussions were about mathematics (somewhat one sided) and about cricket (happily, at that time, other sided).

The mathematics discussions certainly helped me and my colleagues understand better what we were up to. I must say that Narasimhan was a staunch supporter of the idea that one could learn something from physicists and so he helped me and others along. He put in a large effort to understand us and our language. In his time at the ICTP he began to build a solid mathematics group, and he was always available to help young postdoctoral fellows (often hunting them down so they would not find themselves isolated while here). And it always amazed me that he always had something useful to impart to these young people regardless of their field.

When is the moment that you pass from being colleagues to being friends? I do not know, but we passed it and that friendship has been a joy for me. In those days you could also sometimes find us in our favourite restaurant in the evening drinking red wine (he very much appreciated the Shiraz wines of Australia) and he would tell me very funny anecdotes of his life and about other mathematicians that he knew. We would also discuss more serious matters such as the mission of the ICTP-dear to both of us- on which he often proclaimed that I was a bleeding heart liberal only to turn around and help everyone that he could.

That is how I think he should be remembered or at least how I will. A great mathematician, a wonderful human being but, even more importantly, a dear friend. □

George Thompson is Professor of Physics at ICTP, Trieste.

KHARE | *continued from Page 1 ...*

Jean-Pierre Wintenberger. Professor Narasimhan was enthusiastic about our work, in spite of it being far away from his mathematical interests, and this pleased me greatly.

Later I met him a few times at the prize ceremonies of the Infosys Science Foundation, and also at a Commonwealth Science Congress, in Bangalore where he then lived. He had a natural charisma, presence, intensity, an aura around him, and also an enthusiasm to converse and discuss ideas. We talked about the changing political climate of the country, the decline in the quality of public debate, a growing illiberalism and a decline in scientific temper and support for basic science. He talked about the quality of the political leadership in the past of the newly independent India. He lauded its empathy for open ended intellectual inquiry, and a thought-out rationale and vision that translated into a keenness to invest in basic sciences, even when the country had many more immediate demands on its scarce resources.

Much to my regret, I did not get to know him well, either personally or mathematically. In this tribute to him, I will focus on his role as an inspiring figure – because of the importance of the work he had done, where he had done it, and when he had done it – to a young person (like myself) trying to do research in pure mathematics in India in the 1990's.

TIFR had established itself in the world of pure mathematics through important theorems proved by mathematicians working there, through the decades from the 1950's onwards, and of these there was none more celebrated than the Narasimhan-Seshadri theorem. It was in an area of algebraic geometry and differential geometry that was far away from my area of work which was in number theory.

I had come back to India immediately after my thesis in 1995, and joined TIFR as a Visiting Fellow (the entry level postdoctoral position available to someone after finishing their PhD). I did not know the mathematical content of the theorem. Many of my senior colleagues at TIFR worked in different aspects of the mathematical specialty – vector bundles on curves – that had been decisively impacted by the Narasimhan-Seshadri theorem. It continued to be the focus of much of the research done at TIFR decades after the theorem had been proved.

For me the influence of the Narasimhan-Seshadri theorem was more indirect but still psychologically quite important. The discovery of such an influential theorem by two young brilliant Indian mathematicians, in their early thirties, working in Bombay at TIFR in the 1960's, led to putting TIFR on the world map of mathematics. It also made one feel that as someone working at the same Institute, one had to try and live up to the high repute of the place, live up to that theorem in a way. Further it gave a sense that it was possible to do first class mathematics working at a place somewhat distant from the traditional, mainly Western, centres of mathematical research, especially as their work had been done in Bombay in times when the world was far less connected than now.

I read later in an interview with Narasimhan that he felt it could be advantageous to work somewhere at a distance from the main mathematical centres, and follow the latest developments from this distance. One could then work on one's own ideas, partly inspired by the work happening at these centers, without being overwhelmed by the influence of the leaders in the subject. Such direct influence, while it could be greatly beneficial for some, might dissuade others from trying out ideas that could seem unpromising to the experts, but that one could not give up on internally.

Although I did not know Prof. Narasimhan's quote at that time, I found for myself that this awareness from a distance of important developments in one's area, while working independently on one's own, worked quite well for me in the first three to four years of my time at TIFR.

After arriving in India in 1995, I worked on trying to generalize the work I had done in my thesis, which used ingredients that overlapped with the astonishing work of Andrew Wiles on Fermat's Last Theorem. I arrived in a faltering way at a satisfactory generalization of my thesis work over a period of a few years after I returned to India.

Temperamentally I am drawn to tilting at windmills, and I started doing, more or less simultaneously, more open-ended work inspired by Serre's modularity conjecture. The very few known cases of the conjecture had provided Wiles a foothold to launch his audacious attack on the modularity conjecture for elliptic curves (motivated by the corollary that would follow, namely Fermat's Last Theorem). Serre's conjecture seemed wide open even after Wiles' work. It was a great prize sought after by many working in this area of number theory. Its allure was that it was a simple striking statement, a conjectured correspondence between two types of objects (modular forms and Galois representations) that were very different. It had a wealth of consequences. Its attraction was also that it was made by Jean-Pierre Serre, one of the important mathematicians of the 20th century, who worked in Paris and who could have influenced both Narasimhan and Seshadri by his personality, mathematical exposition, and his own research. They probably had interacted with him as young men when they went to Paris in the late 1950's, deputed there from TIFR to learn the latest mathematics at one of the great centers for pure math.

I kept musing about questions suggested by Serre's conjecture, carrying them in my mind, experiencing mainly frustration, but also small eureka moments, making small observations that I wrote up as short papers. I sent some of these observations to Serre himself, and a couple of times surprised him with my remarks.

It was a little like kicking the ball around on a field, with the goalposts obscured by a thick fog. As there was no way of reasonably aiming to kick at the goal which was smothered in the fog, one just kicked the ball around and chased after it in bursts of somewhat random, sporadic, but still intense, activity.

I read later of another piece of Prof. Narasimhan's advice to young people to work "off the top": work on something without necessarily knowing precisely all the background required, getting by on a sense of the subject, impressionistic to begin with, which could be deepened as one continued thinking (continually!) about the subject. This way one would not get bogged down and overwhelmed by the myriad technical details right at the beginning, which could have a paralyzing effect on a novice, and instead learn them as one needed to.

In the interview he said:
"....one should learn any subject from as sophisticated a point of view as one is capable of; so that in some sense, from then on, you can move down into the details.

I don't prescribe going canonically from definitions, theorems and so on to start with. You have to eventually do it anyway, at some stage. But then the problem is that you can get lost in the minutiae without any idea of what it is all adding up to. As Eilenberg once said, everybody in mathematics has a "natural boundary". *What he meant was that up to a certain level, you can understand things. Beyond that, it becomes very difficult to do so, even in a particular field. When you have to cross this boundary, how do you do it? Well, you could just get discouraged and simply give up, which is not at all what I am talking about. You could also say* "I'll go to some classic textbook and read linearly word by word, the definitions, and so on." *On the other hand, as in a metaphor, suppose that you can see and identify a few small holes in a large and otherwise mostly opaque structure. And through which some scattered bits of understanding are filtering through. This affords you some view into the world beyond, though only slightly and incrementally better than before. Now, you try to get ahead by using these little toeholds of incremental understanding that has filtered through the small openings. It is essentially that sort of a thing. For this, you clearly need some sophistication... Finally, of course, you have to sit down and get all the minute details thrashed out. No escaping that."*

On reading Prof. Narasimhan's interview, I realized that I have been unconsciously following his advice of working "off the top" all along. In my work I reach my "natural boundary" (as defined by Eilenberg) very quickly, and can go past it only by obsessing about a piece of mathematics, living with it in my mind. The fact that the Narasimhan-Seshadri theorem had been proved when India was still young as an independent country (not yet twenty years old) was also fascinating. TIFR was founded by Homi Bhabha who had filled it with paintings by contemporary Indian artists (through the 1950's and 1960's), many of them working in Bombay, not so far away from Navy Nagar where TIFR is located. The works of the members of the Bombay Progressive Artists' Group were amply present on the walls of the Institute. The modernity the paintings represented, made in a newly independent country, which had its own ancient culture and tradition of art, architecture, music, dance, seemed to marry these civilizational influences with what was happening in the contemporary world of art then. Many of the artists whose paintings Bhabha collected (with discernment and a remarkable sense or intuition for what was vital in the art made in India then) had spent time in Paris and returned to produce work which was influenced by what they had absorbed in their time there. Narasimhan and Seshadri also had been deputed to Paris, from 1957 to 1960 as I learnt from interviews of Prof. Narasimhan, absorbed new ideas and influences there, and after returning to India proved their landmark theorem. A four month long visit in 1997 that I made to Paris, made possible by an Indo-French scientific exchange program that owed its existence to the relationship between the Indian and French mathematical communities that went back a few decades, played an important role in my own development as well.

It was a different India that I lived in during my years working at TIFR (post the economic liberalization of 1991), but the earlier example of Prof. Narasimhan, Prof. Seshadri and their colleagues, who had worked at TIFR and proved path breaking theorems decades earlier, lived on as an inspiration, present in the air, setting a certain tone, holding me and my colleagues accountable, pushing us to try and live up to their formidable legacy. □

Chandrashekhar Khare is Professor and David Saxon Presidential Term Chair in Mathematics at University of California, Los Angeles.

MEMORIAL EVENT

On July 12, 2021, a special online event was organised by ICTS-TIFR to celebrate the work of M. S. Narasimhan (1932-2021) and C. S. Seshadri (1932-2020). This event was part of the program *Quantum Fields, Geometry and Representation Theory 2021.* Narasimhan and Seshadri, the doyens of Indian science, put TIFR on the mathematical map of the world in the early days of independent India. The speakers were Vikraman Balaji (Chennai Mathematical Institute, India), Jacques Hurtubise (McGill University, Canada), Shrawan Kumar (University of North Carolina at Chapel Hill, USA) and Edward Witten (Institute for Advanced Study, Princeton, USA).

A special virtual memorial meeting for M.S. Narasimhan was held at TIFR, Mumbai, on 4 June 2021.

Stable and unitary vector bundles on a compact Riemann surface

By M. S. Narasimhan and C. S. Seshadri

1. Introduction

D. Mumford has defined the notion of a stable vector bundle on a compact Riemann surface X and proved that the set of equivalence classes of stable bundles (of fixed rank and degree) has a natural structure of a non-singular, quasi-projective, algebraic variety [13]. We prove in this paper that, if X has genus $\geqq 2$, the stable vector bundles are precisely the holomorphic vector bundles on X which arise from certain *irreducible unitary* representations of suitably defined fuchsian groups acting on the unit disc and having X as quotient (Theorem 2, § 12). We also prove that, if $\{W(t)\}$ is an algebraic (resp. holomorphic) family of vector bundles on X parametrised by an algebraic (resp. complex) space T, the set of points $t \in T$ for which $W(t)$ is stable is a Zariski open subset of T (resp. an open subset whose complement is an analytic subset).

It follows from our results that the space of (equivalence classes of) stable vector bundles of rank n and degree q on X is *compact* if n and q are coprime.

A particular case of our result is that a holomorphic vector bundle of degree zero on X is stable if and only if it arises from an irreducible unitary representation of the fundamental group of X. As a consequence one sees that a holomorphic vector bundle on X arises from a unitary representation of the fundamental group of X if and only if each of its indecomposable components is of degree zero and stable.

In this paper an essential role is played by the relationship between holomorphic vector bundles on X (not necessarily of degree zero) and representations of certain fuchsian groups (which may have fixed points). This relationship (the functor p_*^π of § 5) is already implicit in the classical paper of A. Weil [19].

For an outline of the proofs of the main results, one may refer to our note [15].

We are grateful to Professor D. Mumford for sending us a manuscript containing his results on geometric invariant theory. Although we do not use any of his results, his theory of stable and semi-stable points has been indirectly of great help to us.

The first page of Narasimhan-Seshadri's famous 1965 paper

RAMANAN | *continued from Page 1 ...*

Mathematicians usually acquire a deep understanding of a theory and try to solve re-lated problems using their expertise on the subject. Narasimhan was of a different mould. iEven if he was initially not conversant with the nitty-gritty of a branch, he had the capacity to come to grips with the essentials of a problem and bring to it an original way of thinking and solve it. He would fill in the details of the theory later, often holding a seminar on the subject. It was my good fortune to have collaborated with him for decades and, to a certain extent, to have imbibed this methodology. Our joint papers were published from time to time, between 1961 and 1989. He was my mentor, colleague, collaborator and a close friend.

He joined TIFR as a graduate student in 1953 and was deputed to France in 1957. There he came in contact with Takeshi Kotake, a student of the Fields medallist Lau-rent Schwartz, who was instrumental in facilitating Narasimhan's visit. Kotake and Narasimhan wrote a very interesting paper on the regularity behaviour of the solu-tions of linear elliptic equations with analytic coefficients.

When he returned to India in 1960, I was a graduate student looking for a problem to work on. Our first work was on universal connections, which proves that the bun-dle universal for principal unitary bundles has a connection that is universal for uni-tary bundles with connections. As soon as I told him that I felt this could be true, he got excited and began asking me questions, such as "did you check it for U(1)" and "did you check it for trivial bundles?" This helped me understand how one should go about solving problems: try and solve simpler cases and look for analogies. We solved this question within a couple of weeks, and it appeared in a prestigious jour-nal. It was later applied by Chern and Simons to define new invariants, and had also applications in theoretical physics.

Later he started to discuss a problem with C.S. Seshadri, another great mathemati-cian who passed away a few months ago. This related to an understanding of a ques-tion that arose out of the work of Andre Weil twenty-five years earlier. Weil had in-vestigated the analogue of the theory of divisors on a Riemann surface, due to Jacobi of the nineteenth century, and generalised it to what he called "matrix divisors". In the 1950's, this was subsumed in the notion of holomorphic vector bundles on alge-braic curves. Weil's approach was analytic, but it had given birth to algebraic vector bundles of a certain kind. Weil had wondered that "unitary bundles play an im-portant part, without doubt, in this theory but every indecomposable bundle is not unitary."

David Mumford had, around that time, introduced the concept of stable and semi-stable bundles. In a path-breaking paper, again written in a short time, Narasimhan and Seshadri settled this question. In these instances, the solutions came fast and were published soon. At the same time, the TIFR's School of Mathematics did not put undue pressure on its researchers to publish, which proved to be an advantage. For instance, in my next collaborative result with MSN, the moduli of vector bun-dles were determined in the first non-trivial case: rank 2, genus 2. After we had proved it, we worked on it for nearly a year to get a full understanding of this result in a satisfactory set-up and this enabled further work both by us and others.

MSN had the knack of choosing interesting problems that were within reach -- as he would say "abordable". As I mentioned above, his ability to absorb the essentials of a theory quickly and "think from the top", as he would call it, enabled him to work on a variety of subjects: differential equations, differential geometry, representations of Lie groups, gauge theory, etc. He could thus collaborate with mathematicians in different areas and with different methodologies of research. His work with Gunter Harder and, much later, with T.R. Ramadas are examples of his wide interests and quick adaptability. It is not an exaggeration to say that he was the most versatile In-dian mathematician.

The technical details of his work have been summarised by C.S. Seshadri in the first part of The Collected Papers of M.S. Narasimhan, published by the Hindustan Book Agency.

Rather than dwell more on his contributions, I shall recall an incident, illustrative of his passion for mathematics, since we collaborated for long years, both of us were often invited to the same conference. Once during such a visit to Madrid, we took a walk one evening, and had got into a small lonely lane near Plaza Major, discussing some mathematical question. He stopped for a minute, lost in thought, while I was going ahead. Perhaps assuming I was alone, a young Spaniard came and tried to snatch my wallet. I held it tight, clutching my pocket and shouted for Narasimhan. The fellow ran away, and at once, another guy rushed at me with the same intent. Narasimhan came running and I could breathe easy, although understandably nerv-ous. Narasimhan continued on the mathematical theme we were discussing as if nothing had happened!

We lived in the same housing colony, so we could work together day and night. The American mathematician Bertram Kostant, who was visiting TIFR, remarked that he had never seen two mathematicians working together all the time. Another wit quipped that people may think that 'Narasimhan Ramanan', is one mathematician.

It was my privilege and good luck to have had his company for so long. Even after he had moved to Bengaluru and I to Chennai, we visited each other often. Almost till his last days, we used to talk on the phone and email each other. He wrote to me last year saying, "You know, guys have now generalised our results and have defined something they call Narasimhan-Ramanan branes. We should try and understand what they are."

I will remember him and our time together to the very end. □

S. Ramanan was a student of M. S. Narasimhan and retired as Distinguished Professor from the School of Mathematics, Tata Institute of Fundamental Research. Currently he is Adjunct Professor of Mathematics at Chennai Mathematical Institute.

M. S. Narasimhan with C. S. Seshadri, S. Ramanan and M. S. Raghunathan. Credit: TIFR Archives

RAGHUNATHAN | *continued from Page 1 ...*

until the end. He was second to no Indian in terms of scientific achievements.

I was first his student and later his colleague at the Tata Institute of Fundamental Research (TIFR). Narasimhan was a great teacher as any of his students would attest. He was good, not spectacular, in the class-room, but he had few peers in informal one-on-one communication with students or fellow mathematicians. He had the gift of getting to the heart of the most abstruse problems and then conveying it to the student, side-stepping technicalities that could cloud the issue. I benefitted from this ability of his early on (towards the end of a seminar on Differential Geometry he had conducted with Ramanan): He explained to me all of the Kodaira-Spencer deformation theory (incidentally, one of his favourites, and a recently developed theory at that time) in about 3 hours during a few walks on the seashore at TIFR. I learnt later that Narasimhan had studied the Kodaira-Spencer theory while recuperating from an illness in a hospital in France. During one of those walks Narsimhan suggested a problem for my doctoral degree. The background material needed for the problem had been covered by the Differential Geometry seminar.

Narasimhan had an abiding interest in Tamil literature, especially contemporary writing. And in conversations with him I learnt a good deal about Tamil writers and their work. He also had a big influence in shaping my political views. Three of us, Ramanan, Narasimhan and myself would often go to a particular restaurant for snacks and coffee in the late afternoon and talk about diverse topics (even as mathematics took the lion's share), politics figuring inevitably. Narasimhan was of a leftist persuasion, Ramanan in those days did not lean as far left as Narasimhan. My own views did not go beyond an unqualified admiration for Jawaharlal Nehru, and these outings with them helped me towards a more sophisticated understanding of politics.

After my thesis my interests diverged from those of Narasimhan, but he was responsible for that as well. He had pointed out that my thesis problem had connections with some work of Andre Weil and asked me to explore that; which led me eventually to the theory of Discrete Subgroups of Lie Groups, a major preoccupation of my entire career. Our mathematical interactions continued, but were of a more general and less intense nature.

In the early sixties, Narasimhan and Ramanan organized a seminar on the Atiyah-Singer Index theorem in which I also gave some lectures. Narsimhan was at the same time working with Seshadri on the now famous Narsimhan-Seshadri theorem on stable vector bundles on a compact Riemann surface, directing my thesis and also working with Ramanan on a problem in Differential Geometry, an indication of his versatility that I mentioned earlier.

In 1964, he was responsible for getting me invited as a speaker in the prestigious International Colloquium in Differential Analysis held at TIFR. This was, of course big recognition for me. But his role went beyond that. Narasimhan and Chandrasekharan (the then Head of the School of Mathematics) made me rehearse my talk with them and that resulted in a well-received lecture – much to the surprise of colleagues who were aware of my poor track record as a speaker.

In 1966 – the year in which I received my PhD – I was appointed an Associate Professor at TIFR, and so became a colleague of Narasimhan's. From the beginning he treated me as an equal, but it was necessarily an asymmetric relationship: I continued to see him as a teacher. Our views on most issues were almost identical, but when occasionally we differed, I would defer to his views. He was a stickler for correct and dignified conduct on all occasions, and in Faculty meetings, in particular. He would wince when anyone failed on that front. He never raised his voice even while he was firm in giving his views on any subject. During the period he was Dean of the Mathematics Faculty he was meticulous in organizing and conducting Faculty meetings. He paid great attention to the wording of his letters to ensure that they said not just the right things, but in the right way. He treated the administrative staff courteously, extracting the best out of them.

Narasimhan, along with Seshadri, was one of the principal architects in building the School of Mathematics at TIFR into an international centre of excellence in mathematics, from the fledgling state it was in when they joined as students there in 1953. The two were about the same age and had been together at college. They continued to be together for some 30 years at TIFR, guiding the School of Mathematics there. They remained close friends even after they were physically far away from each other. It would appear that in death too they were close to each other – Seshadri died less than a year ago.

When in 1983 the DAE formed the National Board for Higher Mathematics (NBHM) as an agency for the promotion of higher mathematics in the country, Narasimhan was the natural choice to head it. NBHM under his chairmanship took many initiatives which went a long way in fulfilling its mandate. He took me in as a member of the Board, and later in 1986 made me Secretary. This gave me more opportunities to see his administrative skills from up close and that was of course a learning experience which was valuable to me when I succeeded him in 1984 to the Chairmanship of the Board.

In 1987, The Governing Council of TIFR passed up the opportunity of getting him to lead TIFR. It is a great pity, as I am sure that with his stature as a scientist and his proven skills as an administrator he would have taken TIFR to much greater heights than what was achieved in the subsequent ten years. But fortunately, his abilities did not go to waste even while TIFR denied itself his leadership. In 1992, he accepted an invitation from Abdus Salam to head the Mathematics Section at the International Centre for Theoretical Physics (ICTP) in Trieste. Under his leadership, mathematics at ICTP reached new heights. His advent ensured that initiatives for the promotion of mathematics in the developing world grew rapidly.

He played a big role in setting up programmes of cooperation in mathematics between India and other countries: France, Spain and Brazil. Some of this happened after he retired from ICTP and settled in Bengaluru. In Bengaluru, the Centre for Applicable Mathematics of TIFR, the Institute of Science and the International Centre for Theoretical Sciences benefitted immensely by his inspiring presence and advice on many matters.

Narasimhan received numerous awards and honours. I mention a few: Fellowship of the Royal Society, The World Academy of Sciences prize for Mathematics, the King Faisal Prize for Mathematics and the Padma Bhushan of the Government of India.

Despite his stature and achievements, he always remained accessible, especially so to anyone who wanted to discuss mathematics with him. He was always good company and had interesting things to say on a wide variety of subjects. As a liberal, he was in recent years much distressed at our country's slide away from the values he held dear. Mathematics was however his magnificent obsession, and perhaps shielded him from greater despondency. The way he kept abreast of the most recent developments in mathematics, was truly amazing.

I have dwelt at some length on my personal interactions with Narasimhan in the expectation that it will throw some light on some less known aspects of his personality. The internet has ample information on his public persona, in particular about his great achievements as a mathematician.

His passing away is for me a great personal loss – he was a close friend and mentor. □

M. S. Raghunathan was a student of M. S. Narasimhan and retired as Professor of Eminence from the School of Mathematics, Tata Institute of Fundamental Research. He is currently a Distinguished Visiting Professor, DAE-MU Centre for Excellence in Basic Sciences, Mumbai.

HIS PASSION FOR MATHEMATICS DEFINED HIM

SHOBHANA NARASIMHAN

I write this note in memory of my father, M.S. Narasimhan, from the rather unique perspective of someone who was his daughter, as well as scientific colleague, book buddy, guide to contemporary popular culture, and partner in many adventurous travel and culinary forays.

Those who saw in my father a rather patrician aloofness may be surprised to learn of his humble beginnings in a small village in Tamil Nadu, where there was not even a high school. He, therefore, had to drive a bullock cart every day to the school in the nearby town. At lunch time, he would emerge briefly from the classroom to eat his meal of *thayir sadam* (rice with yogurt) and feed the bullock hay – to find, occasionally, that the bullock would have come unmoored, and run all the way back home! An early childhood photo shows him with the traditional shaved head and *kudumi* (topknot) of the South Indian brahmin, and wearing diamond studs in his ears. However, after his father passed away when my father was twelve, years of financial hardship followed. Chellappa (his family's pet name for my father – it means 'cherished one') found his solace in books, reading everything that he could lay his hands on. A favourite family story is about the time he was sent to the town with money to buy some sugar, and discovered to his delight that a circulating library had just opened there. He spent all the money on a library membership instead, returning home much later with a bag of books (having forgotten all about the sugar). He also found joy in solving 'riders' or trigonometric puzzles, and declared that he would devote his life to mathematical research. His proud family responded by painting the walls of a room in their house black, so that he could scribble equations on the walls using chalk.

For college, he chose Loyola College in Madras over Vivekananda College; he was to tell me that this decision was determined largely by his discovery that students in Vivekananda College were expected to wake up before dawn for hours of morning prayers! This proved to be a life-changing decision, as at Loyola College, he encountered the French Jesuit priest Fr. Charles Racine, who mentored him, and, crucially, suggested that he join the newly-established TIFR in Bombay for a PhD. It was only earlier this year that I learnt that my father did not have enough money to pay the train fare from Madras to Bombay to attend the TIFR admissions interview; a teacher found out and lent him the money. He and his classmate CS Seshadri travelled to Bombay, and found free accommodation, for a few days, in a temple. Many people have remarked how dapper my father looked in his elegant suits and ties, however the first time he wore European trousers (as opposed to the Tamil *veshti*) was for his TIFR interview. He would later recall feeling intimidated during the interview, especially by KGR (KG Ramanathan) whose dark glasses made him appear particularly inscrutable. My father and Seshadri were both admitted to TIFR, their 'hostel' was the former servant quarters of the Old Yacht Club. My father was to wonder later whether the dismal living conditions there were responsible for him subsequently catching tuberculosis.

Recently, after he passed away, a group of people were trying to find a single word to capture the essence of my father's personality. My friend Ralph Gebauer chose 'gentleman'. George Thompson chose 'remarkable'. After some thought, I chose 'mathematician'. For though there were many other aspects to his personality, his overwhelming love and passion for mathematics defined him, and mathematics brought him great joy. My abiding memory of my father is of him lying in bed, one leg crossed over the other, his nose buried in a yellow Springer Verlag volume, or scribbling equations on a notepad. One little-known idiosyncrasy of his was that he believed that mathematics could best be done with black Bic ballpoint pens. Though he was proud of his Mont Blanc fountain pen, he suffered from Bic anxiety: the fear that he would run out of Bic pens, as a result of which his mathematical creativity would dry up! I would therefore return from trips to Europe with dozens of these cheap ballpoint pens in my luggage. (I now find that his desk drawers are filled with packs of Bic pens, the sight of which seems particularly poignant and evocative to me, and brings tears to my eyes). When he was focusing on mathematics, the rest of the world receded: once, when he was about seven, my brother Mohan came to me with a funny look on his face, and said, *"I just went and told Appa that I thought I had broken my thumb, he was working and replied 'Don't worry, Mohan, I'll buy you another one!'"*

I was born prematurely and was very tiny, and my father used to call me epsilon (because, of course, $e \rightarrow 0$). One of my earliest memories of my father is related to mathematics. I must have been about four years old, I was in his office at TIFR, and, standing on a chair, was writing numbers on his blackboard. I wrote, "0, 1, 2, 3,..." and then turned and asked him, *"Appa, what number comes before zero?"* He got tremendously excited by this question, explained to me about negative numbers, and kept boasting about my precocity to everyone. *"I knew then,"* he was to tell my students, many years later, *"that she would one day become a mathematician ..."* There was then a small pause, before he continued, a little sadly, *"Or ... maybe ... a physicist."*

I did indeed go on to become a physicist, and my father tried very hard to understand what I was working on, reading through the introductory solid state physics books by Kittel (which he didn't like because of its lack of rigour) and Ashcroft and Mermin (which he approved of). Once, when he was in hospital with six broken ribs, he was in great pain, and to distract him, I asked for help with some mathematics I was stuck on. I tried to explain the problem I was attempting to solve, and failed completely! We seemed to speak totally different languages. I would say something like, *"The atoms sit on a triangular lattice,"* to which he would respond, *"That sentence makes no sense!"* Both of us found this very frustrating, but that night, as he lay in bed sleepless with pain, he suddenly grasped the question I was working on, and the next morning I had a neatly written solution waiting for me! However, he refused to be a co-author on the paper, explaining that he had a policy of only accepting authorship on a paper if he understood every word of it, and that he couldn't understand most of what I had written!

M.S. Narasimhan with his wife Shakuntala and daughter Shobhana. Credit: Shobhana Narasimhan

Indeed, his grasp of physics (as of many practical things) had astonishing lacunae. After he complained that he could not finish a bottle of wine in one sitting, I bought him a vacuum wine saver set. He assured me that he knew how to use it, and was very happy with it. It was only several years later that he happened to actually use it in front of me. He employed it correctly to pump out the air from the bottle, but then removed it, left the bottle open to the air, and then reinserted the cork. My jaw dropping, I howled: *"What are you DOING?"* Annoyed, he said, *"I pumped out the air!" "Yes,"* I replied, *"but then you left the bottle open! What then was the point in pumping out the air beforehand? Haven't you heard that 'nature abhors a vacuum'?!"* My father stared at me blankly, whereupon I was betrayed into exclaiming, *"Appa, you may be a great mathematician but you don't understand physics at all!"*. Quite upset with me, he retorted, *"Do you know that my work is said to have contributed greatly to the advancement of physics?!"* He was also quite baffled by computers. Once, I was trying to talk him through a series of computer operations via Skype. I said *"Now, press 'delete'!"*, whereupon he became very nervous, and kept exclaiming, *"Bayamaa irukku maa!"* (My dear, I feel scared!)

My father read widely and voraciously, in English, Tamil, French and occasionally Italian. He also enjoyed reading the Times Literary Supplement, the New York Review of Books, and the weekend arts and literary supplements in The Hindu, Il Sole 24 and the Financial Times. We shared a love for detective fiction, and would wait eagerly for the latest instalment in mystery series by our favourite authors. Surprisingly often, he and I (in different countries) bought the same book, by the same author, on the same day. After this happened several times, he began to faithfully send me an sms or email message as soon as he bought a book, to avoid such duplication in our libraries. We spent many happy hours browsing together, in bookstores in Mumbai, Bangalore, Paris, Cambridge MA, Cambridge UK, New York and Berlin. Inevitably, both of us would finally emerge from the stacks with a towering pile of books we wanted to buy, with his collection of books being more eclectic than mine.

In addition to mathematics and books, he enjoyed good food and wine, cricket and the company of friends. Well into his eighties, he was open to new experiences and new adventures. Once, he was visiting me in Cambridge UK, and as we were chatting, he mentioned that he had always been fascinated by the Rosetta Stone, both because of his interest in Ancient Egypt and because the mathematician Andre Weil invoked it when drawing analogies between number theory, function fields and Riemann surfaces. *"Shall we go to London to see the Rosetta stone?",* I suggested. His eyes lit up, *"Why not?"*, he replied. So, we immediately jumped on a train to London, and two hours later we were in the British Museum, paying homage to the Rosetta Stone! He also enjoyed greatly our trip to Cambodia to see Angkor Wat, another longtime dream of his, enjoying every aspect of our visit, from visiting the temples, to trying out Khmer cuisine, and chatting in French to two old gentlemen in a crêperie in Siem Reap.

My father was also interested in art, with his favourites being the French Impressionist painters. However, he also followed me uncomplainingly through several exhibitions of abstract art, graffiti art and conceptual art. He even suggested that we visit the Venice Biennale, brushing aside my reservations (I knew that the art there was unlikely to appeal to him). After visiting a few pavilions (featuring displays such as rooms of blank walls, to make a statement) he sighed, pronounced *'Interesting! So this is art, nowadays!'*, and then found a comfortable bench to sit on in the gardens, while I visited the rest of the exhibits. What I find truly remarkable, though, is that two years later, he insisted that we visit the Venice Biennale again!

My personality was shaped by him in important ways. I acquired from him a feeling of empathy for the disadvantaged and disenfranchised in our society, a non-jingoistic pride in being Indian, and a respect for rationalism and the power of science.

Above all, we were great friends, and I miss him terribly. □

Shobhana Narasimhan is Professor of Physics at Jawaharlal Nehru Centre for Advanced Scientific Research, Bengaluru.

M.S. Narasimhan greets S. Chandrasekhar during the latter's visit to TIFR in 1987. M.S. Raghunathan, B.V. Sreekantan, K.G. Ramanathan and K. Ramachandra are also present. Credit: TIFR Archives

M.S. Narasimhan with M.S. Raghunathan. Credit: Shobhana Narasimhan

www.icts.res.in

NARASIMHAN AND THE MATHEMATICS OF PHYSICS

T. R. RAMADAS

Narasimhan always insisted – here he differed from the view of people like von Neumann – that mathematics has its rich internal logic, and when capable people follow natural questions with good taste and a feel for structure, its landscape could be illuminated without recourse to physical intuition.

Among his many outstanding qualities was a remarkably disciplined curiosity. This was reflected in the breadth and depth of his work, which ranged over the analysis of differential and integral operators, representation theory, differential geometry, mathematical physics, and large parts of algebraic geometry. His scholarship in Mathematics was vast and penetrating. For example, although he never worked on number theory per se (and confessed to a lack of intuition for analytic number theory) he was fascinated by the interaction between geometry, representation theory, and arithmetic. After the appearance of Wiles' proof, he invested significant effort understanding its relationship with the Langlands program, and many of us benefitted by his aphoristic summaries.

Coming of age mathematically at TIFR, at a time when the influence of Dirac [via Bhabha] and Weyl [via K. Chandrasekharan] was certainly present, it was natural [I imagine] that Narasimhan engaged with the mathematics of physics. This had an impact on his mathematics, though analysis and geometry remained his major pre-occupations.

Let me list somewhat carefully the points of contact between Narasimhan and theoretical physics. These were not all equally consequential, of course. But their very number points to his ability for wide-ranging intellectual engagement, and of course, friendship!

1) Given a partial differential equation invariant under a group (of "symmetries"), one gets for free an action on its space of solutions. If the equation in question is linear, the space of solution is a vector space and one gets a representation. The discovery by Dirac of his equation, and consequences for physics are well-known. Harish Chandra's interest in representation theory was kindled by the ensuing interest in "invariant wave equations".

It was Narasimhan's fantastic idea, flawlessly executed by his student R. Parthasarathy, that a version of the Dirac equation formulated on an appropriate homogeneous space, would give a "realization" of certain discrete series representations, a cornerstone of Harish-Chandra's theory.

2) Narasimhan had an abiding interest in the masterful synthesis (due to Hörmander and others) of PDE in geometric terms, and he was very aware of the uses of symplectic geometry, wave-front sets, the Hamilton-Jacobi equation, Maslov indices, etc. By the time, I met him in 1977-78, I think he had also understood the ideas of geometric quantisation, and was aware of their limited success.

3) Among Narasimhan's close friends was P.P. Divakaran, with whom he carried on a dialogue that spanned decades, a dialogue that covered mathematics, physics, politics, culture, and much else. Divakaran has described — in his tribute at the TIFR memorial meeting — how Narasimhan (and also S. Ramanan and M.S. Raghunathan) were his mathematical consultants. Many of their discussions turned on representation theory and the role of central extensions.

4) I joined TIFR as a graduate student in theoretical physics, with a good training from IIT Kanpur. Thanks to H.S. Mani and Tulsi Dass, I knew about gauge theories and the early work on their geometric aspects. Divakaran adopted me informally as a student, and through him I began hesitatingly to talk to Narasimhan. The Gribov ambiguity was in the air, and I explained to Narasimhan my rather primitive understanding of these matters. Narasimhan very quickly brought all the geometry into focus, and we proved (independently of I.M. Singer, whose paper appeared after we finished our manuscript) that gauge-fixing was not always possible. It was typical that he insisted on the "correct" analytical setting for infinite-dimensional geometry, and our work contained the earliest construction of the space of connections as an infinite-dimensional principal bundle modelled on a suitable Sobolev space. And much else.

5) During this period, Narasimhan [and I] interacted intensely with Pronob Mitter [then at the LPTHE, Jussieu] and his students O. Babelon and C. Vialet, trying to decipher the geometry and physics of gauge theories. But the path integral proved a bridge too far, though Narasimhan made repeated attempts to "crack the code."

6) Incidentally, my thesis also contains a very efficient précis of Dirac's theory of constrained systems, translated into modern geometry by Narasimhan from my accounts (learned from Dirac's little book and a wonderful set of notes by N. Mukunda) as a physics student. He was amused by how Dirac's intuition seemed guided by algebra, even though the situation is so intrinsically geometric.

7) In the late eighties and nineties the moduli spaces of semi-stable vector bundles on curves that Seshadri and he characterised with their epoch-making theorem, and then Ramanan and he studied in loving detail for over a decade, were identified as the spaces in terms of which the conformal blocks of the WZW models were defined. Narasimhan w approaches), though yet again the path-integral stood in the way of the mastery that he would have liked.

It is fair to say that these matters remained a major preoccupation from then on.

8) Together with J.-M. Drézet, he laid out the basics of the theta bundle on moduli spaces of vector bundles of arbitrary rank and degree. Soon after he and I undertook a proof of the Verlinde formula in purely algebro-geometric terms. This dealt only with rank two bundles and proofs of the main ingredients – constancy of the dimension of theta functions and "factorisation" — follow more naturally from the point of Kac-Moody groups. But our method of proof forced us to confront many issues for the first time. We had to give a careful construction of parabolic moduli spaces on singular curves, (which we did á la Simpson), definition of theta bundle theorem, a vanishing theorem (in a context where Kodaira vanishing could not be immediately applied), and finally a geometric proof of factorisation.

9) Narasimhan was quick to recognise the power of Kac-Moody groups in the context. He enlisted A. Ramanathan and S. Kumar in a series of works that carefully elucidated the relationship between the definitions of conformal blocks in algebro-geometric terms and in terms of loop groups. These papers, technically difficult and carefully written, remain standard references.

10) During his years in Trieste (at ICTP and SISSA), Narasimhan was as close to the physicists as to the mathematicians. George Thompson (and for a while, Mathias Blau) was his interlocutor in matters topological-field-theoretical, and Narasimhan and he continued their conversation on these matters even after Narasimhan's return to India.

11) Narasimhan was ever on the lookout for expositions of physics, particularly quantum mechanics and quantum field theory, that would make the fields accessible to him. He held Dirac's "Principles of Quantum Mechanics" in high esteem, and said on more than one occasion that notwithstanding the lack of analytical rigour, it had a rigorous and clear narrative. I think he was largely disappointed with other expositions, though.

12) Narasimhan served as mathematical consultant on topics as diverse as calculations of string amplitudes, Berry's phase, band theory of crystals, and general relativity. He was particularly happy when A. Raina used ideas from conformal field theory to give extremely elegant proofs of classical facts about Riemann surfaces, among them Fay's trisecant identity. Narasimhan was also proud of the occasions when he could provide a useful mathematical insight to Shobhana Narasimhan, (his daughter and well-known physicist) in her work on computational nanoscience. □

T. R. Ramadas was a student of M. S. Narasimhan. He is currently Professor of Mathematics at Chennai Mathematical Institute.

NARASIMHAN AND ICTS

SPENTA R. WADIA

I first got to know Narasimhan in 1984, when we began working on String theory. Our aim was to explore new models of String Theory and generalize the standard Nambu-Goto area action for string propagation in a manifold with the addition of a term that generalizes the motion of a particle in the field of a magnetic monopole. In this connection I had many very useful discussions with Narasimhan. It was a pleasure to discuss with him and he was very generous with his time. I was constantly wanting to connect with my mathematician colleagues at TIFR and once they asked me to give a mini course in Quantum Mechanics...I do not think I was very successful. Narasimhan later told me to my surprise at that time that "concepts of physics are hard to comprehend". That was a surprise to me as I thought that formal math was hard and textbooks demanded too much patience.

During the decade we overlapped at TIFR, he left an impression on me which I want to mention. I very vividly remember Narasimhan speaking continuously and intensely about mathematics to his younger students and colleagues as they walked down the corridors or on their way to the canteen or along the seaside. By example and perhaps even unconsciously he made the point that within the institute we should be intensely discussing subject and research rather than anything else!

During the years he was at ICTP in Trieste I always made it a point to meet him during my yearly visits and enjoyed his hospitality during many dinners with good Italian wine. The focus of our discussions was about how to do better in Indian science.

In August 2007 after the ICTS was formally approved I invited him to be on the International Advisory Board on which he served till March 2013. During the 5 years we spent at IISc in the 'one corridor institute' as ICTS was then called, I greatly benefitted from his advice. More than his words I felt supported by his presence. He was always there and was always very positive about ICTS and its trajectory. After we moved to the campus in Shivakote he visited us on several occasions.

His last visit to ICTS was in February of 2020 during the program on `Moduli of Bundles and Related Structures' and the inaugural `Madhava Lectures' by P. P. Divakaran. At that time, ICTS organized a discussion session on the early days of mathematics at TIFR in which Narasimhan, Seshadri, Raghunathan, Divakaran and Carlos Simpson participated along with some of us from ICTS. It was an amazing session of historical value that is reproduced in an issue of ICTS News. Here Narasimhan spoke at length about the early years that established the school of mathematics at TIFR and the mathematical excellence that was achieved there.

In his passing away we in India and the world at large have lost a great mathematician and an inspiring mentor and builder of mathematical institutions.

I have lost a friend and someone whom I looked up to. □

Spenta R. Wadia is Founding Director, Emeritus Distinguished Professor and Infosys Homi Bhabha Chair Professor at ICTS-TIFR, Bengaluru.

ANY INJUSTICE IN THE WORLD WOULD BOTHER NARASIMHAN

KUMAR S. NARAIN

I met Prof.M.S. Narasimhan in the early nineties just before he joined ICTP Math group after his retirement from TIFR. At that time the ICTP Math group was very small, he essentially developed the ICTP math group to what it is now.

From the very beginning Narasimhan was interacting a lot with HEP group and in particular with those of us who were working on string theory or mathematical physics. I remember, many times we would meet at the ICTP cafeteria, and he would be very keenly interested in what I was doing and what were the recent interesting results in string theory. Many times he wanted to know the details and would ask me to come to his office and explain it on the blackboard. And there were many times he would give some suggestions or some ideas that helped me to solve some of the problems those days.

I remember one occasion very vividly. During one of the conversations, it came up that there was some result that a mathematician and a theoretical physicist had obtained independently that contradicted each other. My own inclination was that the mathematician must be correct, but to my surprise, Narasimhan said the physicist must be correct. Seeing my stunned expression, he laughed and said that the physicists have a different intuition, many times they know already what the result should be, for good reason, even before embarking on trying to prove it. He of course said it half-jokingly, but it was also clear to me that he was making a serious statement.

He was also very concerned about the social and political situation in the world. Any injustice anywhere in the world would bother him. We had many social occasions, having dinner at home or in a restaurant, where we would have long discussions and what I noticed was his seriousness and mathematical rigour even in these matters. He wanted precise references and he would go through them meticulously and we would discuss again at a later occasion.

We will all miss him. He was a great mathematician who helped the development of mathematics in India and all over the world, and he was a great human being. □

Kumar S. Narain is Professor Emeritus at ICTP, Trieste.

JOURNEYS: EPISODES IN THE MAKING OF A PAN-INDIAN CULTURAL IDENTITY

P. P. DIVAKARAN

Mathematics as Culture

The particular thread running through the episodes of my title is that of our mathematical culture. That will surely cause some eyebrows to rise. But before explaining why mathematics is an apt choice as a proxy for Indian intellectual concerns – or Indian culture more generally – let me note that no civilisation has had as long and continuous an engagement with mathematical thought as the Indian: from at least as early as about 1200 BC (early Vedic times) until about 1700 AD. And, if the first tentative outline we have begun to glimpse of the mathematical knowledge of the Indus Valley (Harappan) people can pass future critical review, we can push the beginning back to about 2500 BC. Over this long period, the loci of activity covered almost every region of cultural India: from its northwestern frontiers and the Indus river valley, over the northern plains, across the Vindhya to the Deccan plateau and, in a final burst of creativity, to farthest Kerala. What is unusual and remarkable is that this continuity did not emerge unintended out of the random events of history but was something quite consciously constructed, as a form of resistance to the vagaries of history itself. The works of the great masters of the past, the *pūrvācārya*, were taught to the initiates everywhere and at all times; the Indian scholar never had to rediscover his roots as Renaissance Europe had to rediscover Classical Greece before laying claim to its inheritance. Nothing of value was ever lost. The history of Indian thought is replete with references to the imperative of preserving and propagating validated knowledge, not just in sciences like mathematics and linguistics but in virtually every activity that called upon imagination and skill. What resulted is a distinctively Indian cultural DNA whose invariant backbone goes back to the original wellsprings of our cultural being; and that, despite minor mutations over our long history and over the vast distances of our geography, is what defines us, the quintessential Indian of today.

Mathematics has always played its due role in the growth of civilisations; one had, at the very least, to count, measure, weigh or otherwise quantify the increasingly complex objects civilisations produced and dealt in as they evolved. Figures 1 and 2 are some examples from our own early history.

(left) **Fig. 1** A set of weights from the Indus Valley civilisation and *(right)* **Fig. 2** Typical Indus Valley streetscape

The objects in Fig. 1 are a set of weights from the sophisticated urban phase (beginning ca 2600 BC) of the Indus Valley (or Harappan) civilisation, cut from semi-precious stone in the form of cubes, doubling in weight and volume as we go from the smallest to the heaviest. The remarkable fact about them (apart from their gem-like beauty) is that they are all cubes, to a fair degree of accuracy. To double the volume of a cube, one needs to multiply the side by the cube root of 2. Did the Indus artisans have an algorithm for it? What makes the question interesting is the obvious fact that if the aim was simply to double the weight without insisting on a cube at every stage of doubling, they could just have doubled one side while keeping the other two sides unchanged.

Fig. 2 shows a familiar Indus Valley streetscape, also from the same 'high' period. The geometric precision and regularity of the plan, based on straight lines intersecting perpendicularly (involving a certain nontrivial geometric property of circles), is immediately evident; so is the proliferation of bricks in the architecture. What the picture cannot show is that the grid pattern is aligned along the cardinal directions and that the dimensions of the standard bricks are, universally, binary in proportion, i.e., in the ratio 1 to 2 to 4 (and that they are kiln-fired bricks).

The use of elementary – from today's perspective – principles of mathematics in the creation of material objects ('technology') is a common characteristic of many ancient cultures. What is special to India is that it was not limited to technology alone; the intimate embedding of mathematical ideas in the general intellectual universe is a pervasive theme running through our entire cultural history, from the time of our earliest textual compositions onward, sometimes explicitly acknowledged, more often to be painstakingly extracted from analytical readings of (non-mathematical) texts. The miracle is that it is possible to do so – it is as though one were able to reconstruct the fundamental principles and results of early Greek geometry from a reading of Plato and Aristotle, not to mention Homer. The other miracle is that the connection survived the increasing complexity and specialisation of scholarly concerns, and lasted as long as traditional scholarship itself lasted; there never was a culture of two cultures in India.

There are very many examples, from disciplines ranging from prosody to philosophy, that can be cited in support of the case for the place mathematics had in our culture. The one I like best is an episode from *Lalitavistāra*, a mythic-historic account of the life of the Buddha, composed and recomposed over a span of about five centuries either side of the beginning of the common era. In preparation for his marriage, Prince Siddhartha is made to undergo an examination on his knowledge of the world in all its aspects, during which he dazzles the assembled court by his mastery of, among other skills, the principles of decimal enumeration. It is quite a long passage and running through it (as well as in some other Buddhist texts) are clear intimations of the future Buddha's recognition of the infinitude of numbers which, it is explicitly stated, is a necessary part of the knowledge that leads to enlightenment. Indeed, in the cosmic imagery of Mahayana Buddhism as it grew to maturity in Gandhara, the universe itself is infinitely replicated in time and space; the totality of all of them is presided over by Amitabha, no longer identified as just the historical Buddha but worshipped in his transfigured form, as the supreme and transcendental lord of everything that is or that can be.

The language of mathematics and the mathematics of language

Of all the ways in which a mathematical mode of thought insinuated itself into the general cultural matrix, none had as deep and extensive an impact as on language in its various manifestations. This too goes back to the earliest phase of our linguistic history. That began, as we all know, with the composition of the Vedas, the *Ṛgveda* first (ca 1300 -1200 BC, consolidated into its ten Books of poems a century or two later). It is less well known that the main body of the Vedas, the *saṃhitā* texts, especially the *Ṛgveda* itself, is a treasure trove of information about the interplay between the mathematical and the literary/linguistic universes. (I am not speaking here about the auxiliary Vedic texts known as the *Śulbasūtra*, on architectural ('technological') geometry, in which language does not play a role except in the making of an appropriate terminology). Particularly informative is the part played by the grammar of (Vedic) Sanskrit in the implementation of the rules that regulated the formation of names for numbers of which, surprisingly, there are a few thousand occurrences in the *Ṛgveda* alone; what are they doing in a text which is best read as a work of visionary poetry?

To give this somewhat unexpected phenomenon its cultural context, it is essential to remember that Vedic people had no writing; Vedic literature is oral literature, composed orally, memorised and recited. For numbers to have an unambiguous identity, they had to have unambiguous names, constructed by following the rules of Vedic grammar. What emerges when we analyse the names are rigorous *grammatical* evidence for a perfect understanding of a *mathematical* construct, the now-universal system of decimal counting in which all numbers are 'measured' by a unit, 10, rather as distances are measured in metres. The significance of this breakthrough is often undervalued because representing numbers bigger than 9 nominally is not a matter of writing down abstract symbols side by side as we have been taught; it is, instead, an exercise in grammar that starts with the names of the numbers 1 to 9 and of 10, 100, etc., and combines them in grammatically correct ways to arrive at the name of every conceivable number. The decipherment of a name so formed as a precisely identified number follows the converse grammatical process that is equally rule-bound though not always easy; famous people have made mistakes in the past.

The *Ṛgveda* is a veritable proving ground for, on the one hand, the mastery of the fundamental principles of decimal enumeration – true Vedic mathematics, greatly more sophisticated than the schoolboy tricks that pass for it these days – and, on the other, of the parallel evolution of the equally rigorous rules of the grammatical composition of sounds, syllables and words; and we must remember that we are speaking of a time 800-700 years before Panini tied up the loose ends and put Sanskrit grammar to sleep for all time to come. The grammar of Sanskrit and the grammar of numbers – the rules of enumeration – came into being at the same time and in mutual symbiosis, and were preserved for posterity in the same text. Who is to say which was science and which language?

The millennium that followed saw several other manifestations of the close synergy between language and mathematics. There was, first, Panini's treatment of grammar (6th – 5th century BC) from a categorical point of view: rules were formulated not for individual linguistic units but for whole categories of them, a category (or a 'set', denoted in the modern manner by an otherwise meaningless syllable) consisting of all objects having a given linguistic function – noun stems or verb roots or various kinds of affixes for instance. This is an amazingly mathematical and amazingly modern thing to have done, a technique from what is today called abstract set theory in mathematics and logic but applied to the subtle complexities of a living language. That it was precisely that was recognised right away. Panini's great commentator Patanjali (3rd C. BC?) has a lovely little parable involving Indra and Brhaspati (the guru of the gods) whose moral is the indispensability of categorical, not enumerative, rules for the mastery of the infinitude of valid linguistic expressions. At about the same time, Pingala was constructing a theory for the classification of metres after first inventing methods that are the founding steps of the branch of mathematics now called combinatorics. It did not take long before Bharata in his *Nāṭyaśāstra*, till today the foundational text for Indian performing arts, extended these methods to melodic and rhythmic structures.

The primacy of the spoken word, made divine in the *Ṛgveda* as the goddess Vāc, runs through all of India's collective intellect and imagination, not least in mathematics, even after writing gained wide acceptance; mathematicians never got around to dealing in symbols and equations even in highly technical passages. Beyond that, the oral paradigm radically transforms the philosophy of how knowledge is acquired and passed on: the listener has to catch transient sequences of uttered sound on the fly, lodge them in his memory – that is the only existence they can have, as states of mind – and retrieve them as faithful reproductions of the original sound patterns. Such questions occupied linguists and other theorists at least from the time of Patanjali and found full expression in the work (named *Vākyapadīya*) of another great linguist (and philosopher of language and cognitive theorist), Bhartrhari (5th – 6th C. AD, possibly a contemporary of Aryabhata). For Bhartrhari, apprehension of *śabda* was the precondition for the description of the world, the basis of all ontology; to name was to know.

What has this to do with the content of mathematics itself? Take your minds back to (the memory of) what I said about numbers in the *Ṛgveda*: numbers are their names. And that brings in its wake all sorts of other issues, none more intractable than the notion of the infinite. Beginning as early as the *Taittirīya Saṃhitā* (ca. 1000 BC), one of the two recensions of the Yajurveda, we can glimpse the first uncertain recognition of the idea that numbers are without end. The paradox is that to bring them all into existence, they must be given names and that of course is a practical impossibility. A millennium after Bhartrhari and at the southern end of India, the mathematical text *Yuktibhāṣā* of Jyeshthadeva (which will have a role in one of my episodes below) has the startlingly unexpected statement:

. . . there is no end to the *names* of numbers; *therefore* there is no end to numbers themselves.

It is another story that, by the end of the book, Jyeshthadeva's mathematics obliges him to disregard his own dictum. The history of the mathematical infinity in India is a fascinating one; ironically, it never even got a proper name beyond adjectives meaning 'unending' or 'uncountable'. The zero, *śūnya*, in comparison is a dull thing, epistemically and historically.

The other striking fact is one I have already noted: the flow was not all one way, from linguistics to mathematics. Bhartrhari's model for the rule-bound process by which words (*pada*) cluster together to form sentences (*vākya*) is the process by which the numbers 1 to 9 (together with 0) coalesce, "like atoms", in their various positions to form larger numbers.

Journeys

Apart from its role in the shaping of a uniquely Indian philosophical substratum for our knowledge systems, the lack of writing had consequences at a more practical level. The rest of this talk is, in a sense, about how it contributed to the making of 'the pan-Indian cultural identity' of my title.

When the only repository of information is the individual mind, how else can knowledge travel but by the displacement of the body that envelops the mind? When people move, they bear their whole inner world – their knowledge, ideas right or wrong, practices mundane and sacral, skills, imagination, likes and dislikes – with them, individually and as a community. This realisation came surprisingly late to prehistorians but is currently the foundation of several models of cultural diffusion, for example the correlations between the spread of different cultural markers like a particular language or family of languages (the descendants of Old Indo-European for instance) and the cultivation of particular foodgrains (wheat and barley). What seems to be an Indian hallmark is the rich connotations the very idea of a journey evoked. Indians have always travelled: as pilgrims and proselytisers, invaders and those who sought to flee them, fugitives from calamities, those looking for patronage for their special skills etc., or simply in search of their imperishable selves in forest and mountain. There are many instances of such journeys in our narratives, often on an epic scale, some well-documented, others celebrated in myth and legend. Not all of them led to large cultural changes but there are those whose impact was truly transformational; very close to our own times, we know how the musical landscape of Maharashtra and Karnataka was redefined in the beginning of the 20th C. by just two or three gifted singers from north India

Based on the 5th Jehangir Sabavala memorial lecture, Chhatrapati Shivaji Maharaj Vastu Sangrahalaya (Mumbai), 4 November 2016. In this written version, I have tried to stay faithful to the open and informal style of the lecture. Readers who would like to get a more detailed idea of the historical and cultural background against which these journeys – geographical and intellectual – unfolded and of the essential evidence on which the emerging picture is based will find them in my book The Mathematics of India: Concepts, Methods, Connections (Hindustan Book Agency, New Delhi, 2018).

who found enlightened patronage in the courts of kings and chiefs far from their homes. The episodes of my title are three marathon migrations of whole populations, one from the remote past whose historicity is now being actively explored, the other two from historical times, where the facts on the ground only need a degree of informed interpretation. All three produced major cultural shifts and all three have a strong connection with our mathematical heritage; in fact, mathematical continuities are part of the basis of their reconstruction.

From the Indus to Damilica: the first great southern migration

When the highly evolved urban phase of the Indus civilisation collapsed around 1900–1800 BC, what happened to the people who built all those magnificent cities? No one believes any longer that they were massacred by invading Aryans. The archaeological evidence is that the emptying of the cities, probably as a result of climatic/ecological disruptions, was followed closely by the appearance of a small-town culture of petty settlements, devoid of any of the signatures of the high urban culture. A reasonable hypothesis is that a part of the population stayed around in the vicinity of their homeland, to be absorbed gradually in the culture of the Vedic Sanskrit speakers who succeeded them, while others, more robust in body and mind, moved away having lost the resources that had sustained them in their prosperity. Civilisations do not die totally and abruptly leaving only inanimate ruins behind; they either get overtaken by new arrivals or move off in search of greener pastures.

Unlike the Vedic Aryans, the Indus people had a written language though we only know what the characters look like, not how they were vocalised or what they meant. What was that language and is it possible to find fossilized bits of it in later Indian languages, preferably in those which are still alive? Not only will success establish a direct line connecting us to our remotest past, we will also have in hand the beginnings of a systematic approach to an eventual decipherment of the Indus script itself.

Almost all of the dozens of attempts that have been made to 'read' the Indus script have failed but one, more daring than the others at first sight, refuses to die. Its central thesis is that the Indus language was the original Dravidian tongue, the ancestor of Old Tamil. As far as I know, the hypothesis was first advanced by Father Heras of St. Xavier's College in Bombay – when its only support was the fact that a language currently spoken in a part of Balochistan, Brahui, has clear Dravidian roots – and taken up by several other scholars, not always with the necessary rigour. There is, however, one person who has brought exciting new credibility to the conjecture and that is Iravatham Mahadevan, in a series of identifications of a sample of individual Indus signs and even short phrases with expressions in Dravidian languages. The reasoning is intricate, too intricate for me to try to explain, but persuasively logical.

For a language, even one with a rudimentary form of writing like Harappan, to be transported over such huge distances, there must have been a substantial migration of its speakers and they must have left other, material, traces of their displacement. The last few decades have seen several discoveries in Tamil Nadu and Kerala of incisions on rock and pottery of groups of characters having an affinity with Indus signs. Some of the inscriptions appear to be written from right to left like the Indus writing (which is one of the few things we do know about it). But more convincing perhaps are certain numerical commonalities between Indus artefacts and south Indian trade and commercial practices dating back to the earliest historical times. For instance, traditional weights in south India followed the same doubling as in the Indus culture and were also the same in absolute terms: the unit weight is identical, 0.87 grams, 8 times (remember the 8!) the weight of a seed called *kunri-maṇi* or *kunni-kkuru* in Tamil-Malayalam (they were still in use by goldsmiths when I was a boy) and *gunja* in the North. The binary progression is seen in linear dimensions as well: the common Indus brick, of which there are millions scattered over the city ruins, are standardised to this proportion to an amazing degree, with the longest side about 28cm (allowing for erosion, etc.). Some months back I was at Pattanam, the recently excavated site near Kochi of a port trading with Mediterranean and west Asian emporia (possibly the fabled Muziris). Among the amphorae with ancient residues of wine and olive oil were also found baked bricks (of indigenous fabrication) of the same dimensions. And, embossed on a potsherd or two, a little swastika, whose first appearance is also in the Indus Valley.

From the ubiquity of the binary ratio, it is a reasonable guess that the Indus people counted using a standard (a 'base') that is a power of 2: 2, 4, 8, 16 and so on, rather than the Vedic 10 (though there are also some weights in decimal order). The consensus view is that it was 8: 2 and 4 are inconveniently small and 16 is too big for cognitive comfort. Fact: in Dravidian languages, the verb 'count' and the noun 'eight' have the same etymology; the verb root is *eṇ* and the name for 8 is its noun form *eṭṭu*. Another fact: the name for 9 means '1 less than 10', as though it was an afterthought (and, intriguingly, in Sanskrit *nava* is both 9 and 'new', already in the *Ṛgveda*).

There are, however, questions about chronology which are as yet unanswered. The archaeological evidence from Tamil Nadu-Kerala (Damilica to the Mediterranean people), as of now, is not older than about the 3rd C. BC and that includes the site of Keezhadi near Madurai, very recently unearthed and very Harappan in its use of fired bricks in the making of perpendicularly intersecting, cardinally aligned walls. Perhaps future work will help fill in the time gap. (Note added: The continuing excavations at Keezhadi have now unearthed artefacts datable to ca 600 BC). Or perhaps, more interestingly, the travellers took their time over the long journey, groups of them putting down roots at places along the way. Let us only note that there are first millennium BC archaeological sites with a strong Indus Valley signature on the natural routes from Sind-Gujarat to Damilica, for example Daimabad, east of Pune.

I have spent some time over the technicalities of the Indus-Dravidian connection because there are competing narratives about what the Indus civilisation evolved into, though they are, in terms of evidence, nowhere near as persuasive. And because, if and when fully established, its implications will be so shatteringly disorienting: Madrasis as the descendents of Panjabis and Sindhis and Gujaratis? Each single fact that I have laid out may be less than absolutely convincing; together, they merge into a cohesive picture of what can happen to cultural identities when people in the mass journey into the unknown, to uncertain destinations along uncharted roads. Mahadevan has several examples of Indus Dravidian equivalences but the icing on the cake has to be his extension of linguistic concordances to include Vedic Sanskrit as well. The reasoning is again subtle but what he ends up with is, tentatively for the present, tripartite equivalences of a few Indus signs with words in both Old Tamil and Sanskrit, the result, according to him, of direct borrowings from Indus to Vedic. It has long been known that the *Ṛgveda* already had a fair sprinkling of loan words from Old Tamil. The mechanics of how that happened – across thousands of kilometers in the conventional view – has been a mystery for as long. If the stick-at-homes among the suddenly impoverished Indus-Dravidian speakers were next-door neighbours of the Vedic speakers, there is no more mystery; we would know where the Vedic Panjabis learned their Madrasi. And, if the journey took the natural route down western India, we will have the answer to another longstanding puzzle: why Marathi is so abundantly rich in words of Dravidian origin which, according to several scholars, are not relatively recent borrowings from, say, Kannada but of prehistoric, Old Tamil, provenance.

Aryabhata and the great eastern migration

Everybody knows that Aryabhata was the greatest mathematician and astronomer that India produced; he was in fact one of the greatest mathematicians of all time, anywhere. We know from his own words that his magnum opus, the *Āryabhaṭīya*, the only work of his to have survived, was completed in 499 AD when he was 23 years old, in a place called Kusumapura close to Pataliputra, modern Patna, in Magadha. We know very little else. The image below (courtesy: Arvind Paranjape) is of a sculptural portrait of him as imagined at the Inter-University Centre for Astronomy and Astrophysics in Pune and it does him justice, conveying not only his youthfulness and vigour, but also the supreme self-assurance that runs through the 121 two-line verses of his book. Later astronomers tell us that he was of the region of Ashmaka, but not where this Ashmaka was. The problem is that texts, from a very early time (Panini onwards), have reference to two Ashmakas, one more or less in the neighbourhood of Gandhara in the north-west, the other between the rivers Narmada and Godavari in northwestern Maharashtra.

The quest for Aryabhata's roots quickly turns into a

Fig. 3 A modern sculptor's rendering of Aryabhata (in the IUCAA campus, Pune)

fascinating exercise in the reconstruction of the cultural history of his time. The background is this. In the first half of the 5th C. AD, the Gupta empire was in its pomp, extending right across the north Indian plains and into Afghanistan. The northwestern part of the empire in particular had become home to a remarkably cosmopolitan intellectual and artistic life, nurtured equally by Mahayana Buddhism and Hellenic Alexandria, well before the Guptas rose to power. Mahayana sculpture and architecture in the so-called Indo-Greek style reached great heights and Taxila cemented its place as one of the cultural crossroads of the antique world – it was thus that Ptolemaic astronomy reached India, thereby making it possible for Aryabhata to construct his synthesis of Greek astronomy and Indian geometry. But the first half of the 5th C. was also the time when the seeds of the eventual disappearance of the Gupta dynasty were being sown. The Hunas of central Asia, encamped in northern Afghanistan, were making increasingly frequent raids into Gandhara and beyond and, by the middle of the century, had made serious inroads into the western part of the Gupta empire. The Hunas were implacably hostile to Buddhism and devastated Gandhara – and, of course, they burned the libraries as others had earlier burned the library of Alexandria and still others were, in due course, to burn Nalanda. Taxila was reduced to ashes and rubble, never to come to life again; the ruins we see there today are the ruins left behind by the Hunas.

What happened to the monks and the professors and the students? In the absence of documented information, we are forced, as so often in India, to extract what we can from the indirect records carved in indestructible stone, the architecture and sculpture of the period. Of that there is a profusion, and of an extraordinarily high quality, from Mathura in the north, to Sarnath and farther east, to Ajanta and Kanheri and several other places south of the Vindhya. The iconographic and stylistic unity we see in Gandharan and in late Gupta and early post-Gupta art from virtually every region on the periphery of the empire is, given the distances involved, nothing short of startling. I restrict myself here to one theme out of many: the benevolent standing Buddha in *abhaya* or *varada mudra*.

Fig. 4 is a particularly lovely example from late (4th-5th C.) Gandhara. Figs. 5 – 7, equally if not more graceful, are less than a century later and are, from left to right, from Mathura, Ajanta and Sarnath.

Even to the untutored eye, even without any analytic comparisons, their thematic and aesthetic congruence is obvious.

There are very many other representations in these (and other) Mahayana sites which have close affiliations to Gandhara: especially popular in Ajanta and Kanheri is the Buddha in his transcendental Amitabha form, the supreme lord of the cosmos in its infinite multiplicity, accompanied by Bodhisattvas, also deified. Let us recall again that the region of Gandhara was where the final apotheosis of the Buddha occurred.

The only sensible explanation I know of for this sudden upsurge of explicitly Gandharan Mahayana piety where none existed before – some of the sites, (Kanheri, Ajanta, etc.) had a Hinayana presence earlier but had fallen into disuse – is that the Huna depradations forced a mass exodus of monks and scholars out of their northwestern homeland to the safe edges of the Gupta dominions. The timing is right: inscriptions as well as stylistic considerations establish quite satisfactorily that the rebirth of Buddhism at Ajanta is to be dated to the second half of the 5th century. Likewise at Kanheri. The scale and opulence of the monuments (as well as some inscriptions) leave no doubt that they had royal or quasi-royal patronage. If the Gupta kings, in particular Skandagupta who led the resistance to the Huna advance, gave sanctuary to the homeless monks, they were only being true to the royal ideal of the *dharmarāja*, the guardian of faiths, in the plural. Gupta kings were Vaishnavas in the beginning but, before the 5th century was out, some of them had names like Buddhagupta and Tathagatagupta.

Graphic confirmation of the hardships of the journey comes from one particular sculptural theme – very popular in western India – that had no prototype in Gandhara. Figure 8 is a superbly executed example.

It is from Mumbai's own backyard (cave 90 in Kanheri, on top of the hill). What it portrays is the Bodhisattva Avalokiteśvara, granting protection (the secondary panels on either flank) to the traveller beset by the dangers of the trackless forest. One cannot possibly doubt that it is a shrine of thanksgiving to the infinitely




(left to right) **Fig. 4** A standing Buddha from late (4th-5th C) Gandhara, **Fig. 5** Standing Buddha from Mathura, **Fig. 6** Standing Buddha from Ajanta and **Fig. 7** Standing Buddha from Sarnath

www.icts.res.in

Fig.8 Bodhisattva from Cave 90 in Kanheri

compassionate Bodhisattva for having led the wanderer safely across perilous lands to a new haven. The sedentary monks of Gandhara would have had no use for this particular avatar; he is the travellers' Bodhisattva.

It is in these chaotic times that Aryabhata the Ashmakiya announces himself in Kusumapura: a man of the northwest, who or whose gurus probably learned their Alexandrian astronomy in the colleges of Gandhara, and who found himself, by the time he was 23, at the other end of the rapidly declining Gupta empire. What is more natural then than that he or his immediate ancestors formed part of the great migration eastward? Aside from his name, *bhaṭa*, not the Brahmin title *bhaṭṭa*, there are hints in his work that he might himself have had Mahayana leanings. Two examples: his idea of time as without beginning or end, accommodating infinitely repeating equal cycles, *yuga*, of planetary motion – quite distinct from the four conventional *yugas* of Hindu mythology – is also part of Mahayana cosmogony; and his description of the bejewelled garden on the axis of the world that is the Meru mountain, the axis mundi on whose heights the gods dwelt, is reminiscent of the Buddhist 'Land of Bliss' *Sukhāvatī*. There is no sign at all that he was a Puranic Hindu; no benediction is sought from an identifiable god or goddess as was the common custom of Hindu authors.

Towards the last quarter of the 5th century monastic settlements and universities were coming up all over the eastern half of north India. Huen Tsang counts in his famous travelogue dozens of sangharamas each with hundreds of scholar-monks; monasteries from those days are still being unearthed today. The great university in Nalanda was founded by a Gupta king, possibly Skandagupta, less than two decades after the destruction of Taxila, and its first rector was Vasubandhu, a celebrated spiritual philosopher from Gandhara. In the midst of all the turmoil, the great migration thus also initiated a renewal: north India's intellectual and aesthetic life shifted its centre of gravity decisively across its breadth, then to thrive afresh over six or seven centuries until that too came to an end at the hands of Muhammad Ghori.

The Sanskritisation of Kerala: Madhava

One must not imagine that transformational journeys were always driven by calamities and invasions. People of talent had always gone where they found a market. Especially willing travellers were the temple builders, guilds of architects and sculptors and other artisans who criss-crossed the land looking for patronage and finding it, creating in the process the quintessential Indian sacral landscape that surrounds and subsumes us: think of the soaring spires of Khajuraho and Bhubaneswar and the gopurams of Tamil Nadu. There may be local variations in the detail but the underlying unity of vision cannot be missed, and that comes primarily from the theoretical and symbolic conception, unchanged across India at least from the 5th C. AD onwards, of the temple as the location where the worldly and the cosmic are in connection, a man-made terrestrial replication of the *axis mundi*. The canonical manuals of sacred architecture, going all the way back to the *Śulbasūtra*, are no more than the encoding of this vision. Figure 9 is a relatively early example of architectural universality, not just at the conceptual level, but extending right down to the smallest of details.

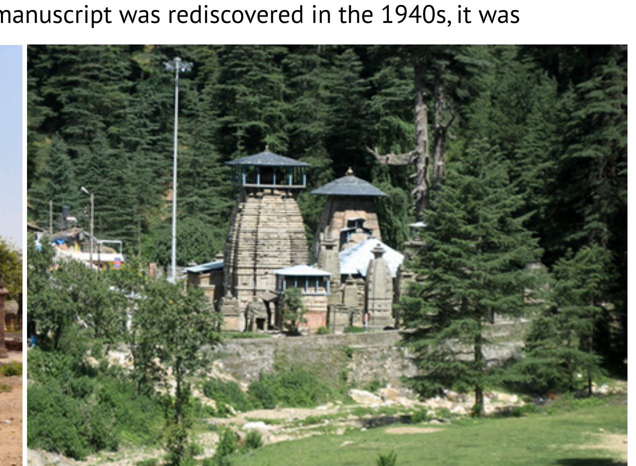
In Fig. 9 (photo courtesy: Baerbel and Guenter von Gehlen) are two temples in a complex of nine, mostly of the 7th C., in a place called Alampur on the Tungabhadra river, just across from Kurnool. In Fig. 10 is part of another complex, built perhaps a century later, but not in arid Telengana as the vegetation attests. It is in fact in Jageswar in Uttarakhand, northeast of Almora, 1500 kilometers north of the Tungabhadra. Who will be willing to swear that it was not designed and built by the descendents of the same guild?

The fact is that, between the two halves of India, there has always been a traffic of people and ideas, at least from the time of Ashoka, more north–south than the other way, mostly but not always pacific. There are any number of tidemarks of this civilisational ebb and flow, covering a whole spectrum of fields and activities. My third episode kicks off around the 5th-6th centuries AD, from when we get the first recorded indications of the increasingly powerful hold of northern Brahmins on the kings and chieftains along the West Coast – and, to a smaller extent, elsewhere in south India – and of how they brought with them, along with their special gifts as middlemen between this world and the other, their learning in the sciences and the arts as well as the language, Sanskrit, that was its vehicle. Between the 8th and the 13th centuries, the slow flow turned into a flood. A large number of the migrants ended up in Kerala which was never to be the same again.

There is no good understanding of what attracted this alien tide, wave upon wave, over seven or eight centuries; perhaps it was nothing other than the promise of verdant lands ruled by gullible chieftains. Several places turn up in the records as possible staging areas in their journeys down the coast but, according to a late document submitted by the Brahmins of Kerala to the British to buttress their claim to their adopted land, their original home was very far away indeed, Ahichhatra on the Ganga, near Kannauj. By the 13th C. their north Indian past was no more than a fading memory as they became socially and culturally integrated, in fact dominant, in their new home. Massive changes were brought about. The local language was transformed almost beyond recognition by wholesale borrowings from Sanskrit – thus was modern Malayalam born – as were literary forms and the performing arts. Kerala as a whole assumed a fresh cultural identity, a fruitful graft of its Dravidian roots and the discipline and rigour brought in by Sanskrit.

One may once again ask: what has all this to do with mathematics?

Keeping aside Shankaracharya (8th C.), of an early migrant family and a true wanderer himself, the first book written in Sanskrit in Kerala is, astonishingly, an astronomical treatise named, after its author, *Śaṇkaranārāyaṇīya*. It is a dated (869 AD) commentary, at second hand, on the *Āryabhaṭīya* but its historical significance goes well beyond astronomy. Parts of the book are in the form of a scientific dialogue between the author and his patron, the Chera emperor himself (a king who was a learned astronomer!), in the capital city of Mahodayapuram, the same as the Muziris that traded with Rome in pre-Sanskritisation antiquity. When the manuscript was rediscovered in the 1940s, it was



(left) **Fig. 9** Two temples in Alampur and *(right)* **Fig. 10** Temple in Jageswar

immediately realised that the many incidental details in it were of immense value in firming up the history of the so-called second Chera empire, until then very poorly known.

Within the mathematical context too, and in its own time, the book turned out to be path-breaking: it marked the first implantation of Aryabhatan mathematics in Kerala, in fact in any part of India south of the Vindhya. (The wanderings of Aryabhata's book will make for an excellent case study in itself). The seed sprouted, largely out of sight, over five centuries before it burst forth in a riot of brilliant new mathematics. That happened around 1400 AD, in a cluster of villages in the lower basin of the river Nila in central Kerala, in the heart of the area that had become and remained the nursery of the new hybrid culture, home not only to astronomer-mathematicians – we should no longer be surprised to learn that some of them distinguished themselves as philosophers, philologists and grammarians, etc. – but also great writers and poets, over several centuries.

The mathematical hero of that story and the founder of what we can call the Nila school was another Brahmin migrant, as great a mathematician as Aryabhata but nowhere near as famous: Madhava, unknown except to a handful of local scholars even as late as 30 years ago. Madhava (ca. 1360-1430; not the same as the religious philosopher Madhvacharya), very simply, invented calculus. Now, the world has always believed that calculus was invented by Newton and Leibniz in 17th C. Europe. It will take me into tedious technicalities to explain why that is wrong history, but that is the case. The reason the misunderstanding took so long to correct – and still persists in some quarters – throws a revealing sidelight on reverse acculturation. After a period of self-imposed insularity, the Sanskrit speaking Brahmins of Kerala gradually mastered the local language which they had themselves helped shape; they finally became Malayalis. And it is in a Malayalam text with the Sanskrit/Malayalam title *Yuktibhāṣā* (which I mentioned earlier), written in about 1520 by Jyeshthadeva, a Brahmin who was totally comfortable in his new skin, that we find the best account of Madhava's calculus. Its first translation into another language (English) is only 10 years old. The world has taken a long time to wake up to the richness of its contents and to Madhava's own place in the pantheon of the truly great.

Appropriately, Madhava himself was a recent arrival in Kerala; his full name, Madhavan Empran, tells us that he was a Tulu Brahmin by origin. The Tulu country was a settlement area as well as a staging post on the long trek from Ahichhatra to the farthest southwestern corner of India. It was also a part of the Vijayanagar kingdom which oversaw a rejuvenation of traditional knowledge systems after they were driven out of north India, for the last time, by Muhammad Ghori. Once again, the cultural centre of gravity moved, this time from the north to the south.

In the end, what the journeys I have spoken about (and others like them) bring home to us most vividly is how little India's fragmented polity through most of its history – which king ruled over which part of it when – mattered to the diffusion of creative ideas that slowly but ever so surely built up the mosaic that is our civilisational inheritance. Very few of them were journeys of armed conquest. Kerala, where Indian mathematics went, first to flourish and then to die, but which at the same time renewed itself in so many ways in response to these stimuli coming from so far away, was never part of any kingdom based outside it. It was something of a shock when I first realised this but no longer; I like to think that it best illustrates what it means, and has always meant, to be Indian. □

P. P. Divakaran is a former Professor of Physics at TIFR, Mumbai.

PROGRAMS

Elliptic Curves and the Special Values of L-Functions
2–7 August 2021 ✦ *Organisers* – Ashay Burungale, Haruzo Hida, Somnath Jha, Ye Tian

Quantum Fields, Geometry and Representation Theory 2021
5–23 July 2021 ✦ *Organisers* – Aswin Balasubramanian, Indranil Biswas, Jacques Distler, Chris Elliott, Pranav Pandit

ICTS Summer School on Gravitational-Wave Astronomy
5–16 July 2021 ✦ *Organisers* – Parameswaran Ajith, K. G. Arun, Bala R. Iyer, Prayush Kumar

Bangalore School on Statistical Physics - XII
28 June–9 July 2021 ✦ *Organisers* – Abhishek Dhar, Sanjib Sabhapandit.

OUTREACH

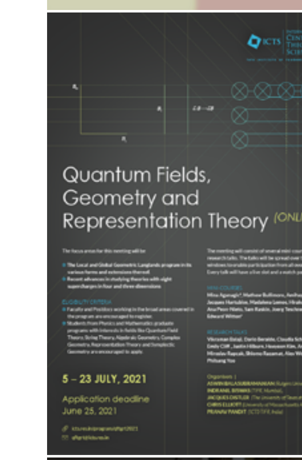
Kaapi With Kuriosity has been temporarily renamed Kuriosity During Quarantine. All talks are held online.

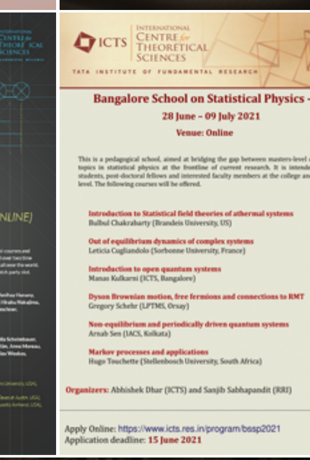
KURIOSITY DURING QUARANTINE

Metallurgical Heritage of India
25 July 2021 ✦ *Speaker:* Sharada Srinivasan *(National Institute of Advanced Studies, Bengaluru)*

PUBLIC LECTURE

Technology & Cosmic Frontiers
19 August 2021 ✦ *Speaker* – Kip S. Thorne *(Richard P. Feynman Professor of Theoretical Physics (Emeritus), Caltech; Nobel Laureate in Physics, 2017)* and Rana Adhikari *(Professor of Physics, Caltech)*





ICTS INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

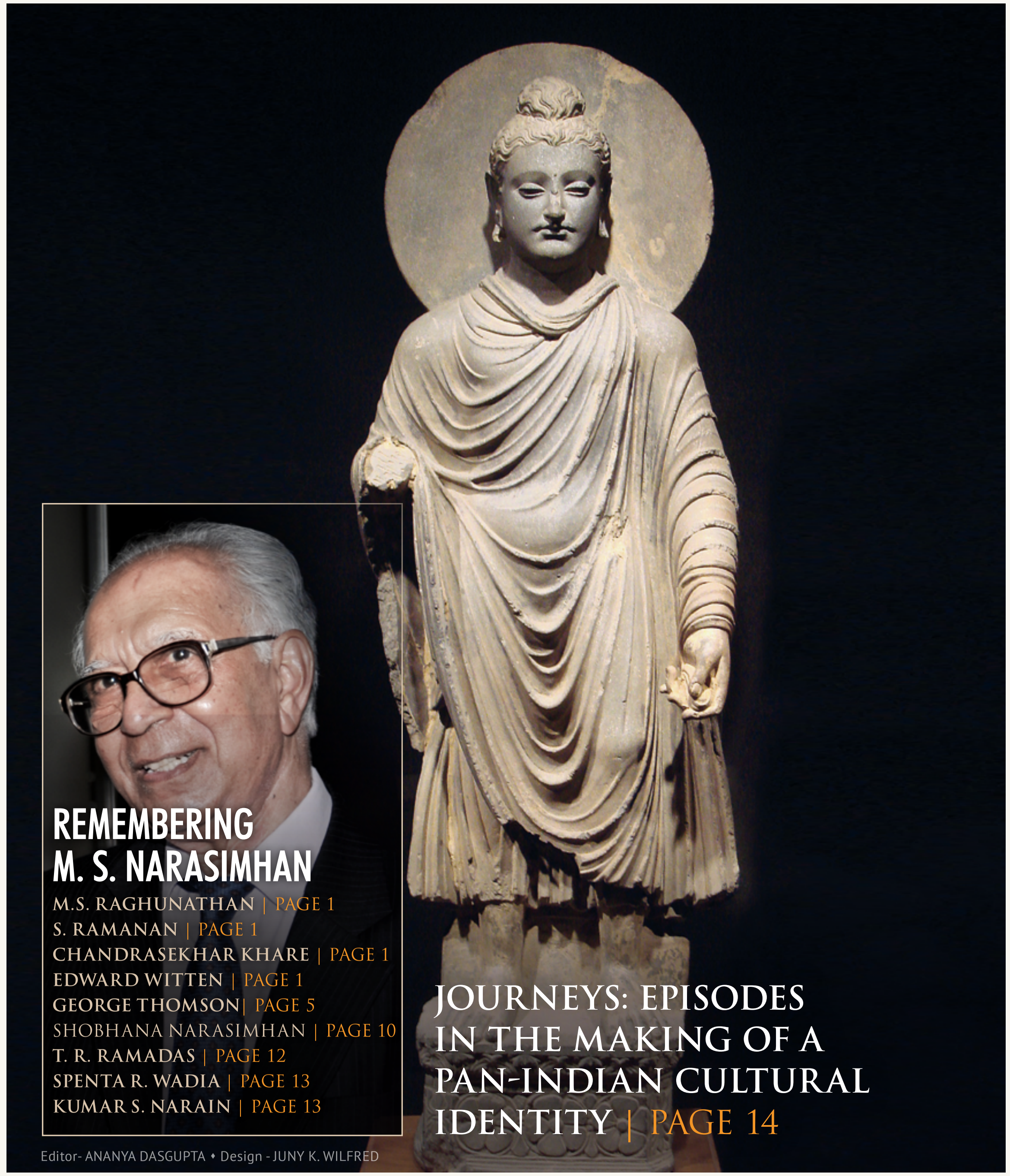
ICTS Address
Shivakote, Bengaluru, India 560089
Website www.icts.res.in

ICTS NEWS

INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES

VOLUME VII
ISSUE 2
2021

TATA INSTITUTE OF FUNDAMENTAL RESEARCH



REMEMBERING M. S. NARASIMHAN

Editor- ANANYA DASGUPTA • Design - JUNY K. WILFRED