Project Report: Embedded Contact Homology And Billiards.

This project develops a formal framework to interpret billiard dynamics as piecewise-smooth Reeb flows on contact manifolds. The primary motivation is to apply powerful tools from contact topology, such as Embedded Contact Homology (ECH), for the study of billiards. Since these tools apply for smooth Reeb flows, such an approach requires constructing careful smooth approximations of the piecewise-smooth Reeb flows that arise from billiard systems. Given smooth approximations with certain useful features, one may apply contact topological tools to the approximations. Through a rigorous analysis of the limiting behavior, we can then translate the results into dynamical statements about the original billiard system.

The project is structured in three parts. The first establishes the theoretical foundations for piecewise-smooth Reeb vector fields. The second details applications of ECH to two-dimensional billiards. The third outlines future work aimed at applying ideas from symplectic homology to study higher-dimensional billiards.

Part 1: Piecewise smooth Reeb vector fields. At IAS, we worked on establishing convergence results, demonstrating that the dynamics of the smoothened flows approach the true billiard dynamics in the limit. This included a detailed analysis of the linearized flow. We also investigated the rotation numbers (or mean indices) of periodic orbits, which is essential to guarantee that orbits detected by Floer-theoretic methods correspond to genuine billiard trajectories rather than artifacts of the approximation, such as convergence to the boundary.

Furthermore, we developed necessary genericity statements. For instance, we worked on showing that a generic billiard table possesses a finite number of inflection points and non-degenerate orbits. While this is easy for two-dimensional flat tables, our aim is to generalize this to Finsler tables in any dimension. The majority of the work for this part is substantially complete, with some remaining technical details to be finalized.

Part 2: ECH and 2-dimensional tables We formulated an outline for a proof of the following dynamical results, using the theory developed in the first part, and known applications of ECH in the smooth setting. To state the results, fix a compact Riemann surface with boundary Σ with unit sphere bundle $S\Sigma$.

Theorem 1 (Chaidez-Tanny, in progress)**.** *The set of Finsler metrics on Σ whose closed billiard orbits are dense in $S\Sigma$ is C^∞-comeager in the space of all Finsler metrics.*

Theorem 2 (Chaidez-Tanny, in progress)**.** *The set of (possibly non-convex) billiard tables $T \subset \mathbb{R}^2$ whose closed billiard orbits intersect a dense set of points in the boundary ∂T is C^∞-comeager.*

Theorem 3 (Chaidez-Tanny, in progress)**.** *Any billiard table $T \subset \mathbb{R}^2$ has at least two billiard orbits.*

Theorem 1 is an example of a smooth closing lemma (cf. [2, 3] for our work on this subject). Theorem 2 may be viewed as a partial strengthening of a theorem of Arnaud [1]. Theorem 3 is a version of the low-dimensional multiplicity theorem for billiards.

Part 3: Future directions for higher dimensional tables We have identified several promising research directions for studying higher-dimensional tables using Symplectic Homology and Contact Homology. We anticipate that techniques developed for prominent problems such as the multiplicity conjecture or the Hofer-Zehnder conjecture, including local Floer homology and the study of symplectically degenerate maxima, can be successfully adapted to our framework. This part is substance for future work.

References

[1] M.-C. Arnaud. Cˆ 1-generic billiard tables have a dense set of periodic points. *Regular and Chaotic Dynamics*, 18(6):697–702, 2013.

[2] J. Chaidez, I. Datta, R. Prasad, and S. Tanny. Contact homology and the strong closing lemma for ellipsoids. *Journal of Modern Dynamics (to appear)*, 2022.

[3] J. Chaidez and S. Tanny. Elementary sft capacities and the strong closing property for near periodic flows. *arXiv preprint* `arXiv:2312.17211`, 2022.