Geometrization of the Local Langlands Correspondence

Special year learning seminar, Spring 2024

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Let E be a local field and let G/E be a reductive group. For simplicity, we assume G to be split. The local Langlands correspondence aims to parametrize the irreducible smooth representations of $G(E)$ in terms of the so-called Langlands parameters (L-parameters), that is, continuous homomorphisms

$$\varphi : W_E \to \widehat{G}(\mathbb{C}),$$

from the Weil group W_E of E to complex points of the Langlands dual group $\widehat{G}$. When E is nonarchimedean with a finite residue field $\mathbb{F}_q$ of characteristic p, which is the focus of this seminar, the Weil group is the dense subgroup of the absolute Galois group $\mathrm{Gal}(\overline{E}/E)$, given by the preimage of $\mathbb{Z} \subset \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, generated by the q-Frobenius.

Inspired by V. Drinfeld, L. Lafforgue and V. Lafforgue's work on global Langlands correspondence over function fields, the very rough idea of Fargues-Scholze's geometrization is to realize the Weil group of E as the étale fundamental group of a "curve", and consider moduli spaces of modifications of G-bundles on it, with various level structures. The étale cohomology of these spaces will provide smooth representations of $G(E)$ and will also be equipped with Weil group actions. It turns out that the collection of these data pins down the desired parametrization. To realize this idea for p-adic E's, they have worked in the framework of stacks on the v-site of perfectoid spaces in characteristic p, where the tilting equivalence helps one to pass freely between characteristic 0 and p.

More precisely, given an $\mathbb{F}_p$-perfectoid space S, one can define an E-adic space X_S, called the relative Fargues-Fontaine curve over S. Sending S to the groupoid of G-bundles on X_S defines the stack of G-bundles Bun_G. When S is a geometric point, the étale fundamental group of X_S agrees with the absolute Galois group $\mathrm{Gal}(\overline{E}/E)$. This case is extensively studied by Fargues-Fontaine [3] and has important applications to p-adic Hodge theory. To make the Weil group appear, one considers the "mirror curve" $\mathrm{Div}^1 = \mathrm{Spd}\breve{E}/\varphi^{\mathbb{Z}}$[1], whose S-points are in bijection with degree 1 closed Cartier divisors on X_S. One has $\pi_1(\mathrm{Div}^1) = W_E$[2].

Next, one defines the global (resp. local) Hecke stacks by parametrizing modifications of G-bundles on X_S (resp. on its completion at a Cartier divisor). Namely, for a finite set I, the global Hecke stack Hck_G^I sends S to the groupoid of pairs $(S \to (\mathrm{Div}^1)^I, \mathcal{E}_1 \dashrightarrow \mathcal{E}_2)$, for a selection of degree 1 closed Cartier divisors labelled by I, and a meromorphic isomorphism of G-bundles on X_S, undefined at the union D_S of these divisors. The local Hecke stack $\mathcal{H}\mathrm{ck}_G$ is defined in the same way, except that $\mathcal{E}_1$, $\mathcal{E}_2$ are G-bundles on the completion of X_S along D_S. There is a natural restriction map $\mathrm{Hck}_G^I \to \mathcal{H}\mathrm{ck}_G^I$.

[1]Here $\breve{E}$ is a complete maximal unramified extension of E, $\mathrm{Spd}\breve{E}$ is the sheaf that sends an $\mathbb{F}_p$-perfectoid space S to the set of untilts of S over $\mathrm{Spa}\breve{E}$; and φ is a geometric Frobenius that acts on the test objects.

[2]in the sense that $\mathrm{Loc}_\Lambda(\mathrm{Div}^1) \cong \mathrm{Rep}_\Lambda(W_E)$

The relation between these objects is described by the following diagram.

$$\begin{array}{ccccc} & & \mathrm{Hck}_G^I & \xrightarrow{\ q\ } & \mathcal{H}\mathrm{ck}_G^I \\ & \swarrow^{h_1} & & \searrow^{h_2} & \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{array}$$

Here the maps h_1, h_2 are obtained by projecting to $\mathcal{E}_1$, $\mathcal{E}_2$, and the structure map to $(\mathrm{Div}^1)^I$. For a suitable coefficient ring Λ, algebraic Λ-representations of copies of the Langlands dual group $\widehat{G}^I$ appear as flat (over Λ) perverse universal locally acyclic (over $(\mathrm{Div}^1)^I$) sheaves on $\mathcal{H}\mathrm{ck}_G^I$, via the geometric Satake equivalence. For an algebraic Λ-representation V of $\widehat{G}^I$, denote the corresponding Satake sheaf on $\mathcal{H}\mathrm{ck}_G^I$ by $\mathcal{S}_V$. We can define a Hecke operator on $D(\mathrm{Bun}_G, \Lambda)$ by the formula

$$T_V : A \mapsto Rh_{2,*}(h_1^* A \otimes_\Lambda^{\mathbb{L}} q^* \mathcal{S}_V).$$

It lands in $D(\mathrm{Bun}_G \times [*/W_E^I], \Lambda)$, which identifies with $D(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$ via v-hyperdescent using the formalism of condensed mathematics. Moreover, these Hecke operators commute and tensor product of $\widehat{G}^I$-representations corresponds to composition of Hecke operators. Varying V, this defines an exact monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G}^I) \to \mathrm{End}_\Lambda(D(\mathrm{Bun}_G, \Lambda)^\omega)^{BW_E^I}.$$

It turns out that the collection of these functors, natural in I, pins down the desired correspondence uniquely (up to semisimplification of the L-parameters) via "excursion": An excursion data is a tuple $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ consisting of a finite set I, $V \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, $\alpha : 1 \to V|_{\widehat{G}}$, $\beta : V|_{\widehat{G}} \to 1$ and $\gamma_i \in W_E$, $i \in I$. If $\Lambda = L$ is an algebraically closed field over $\mathbb{Z}_\ell[\sqrt{q}]$ ($\ell \neq p$) then for any Schur irreducible object $A \in D(\mathrm{Bun}_G, L)$, i.e. $\mathrm{End}(A) = L$, there is a unique semisimple L-parameter $\varphi_A : W_E \to \widehat{G}(L)$ such that for all excursion data as above the endomorphism

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

is given by the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i)_{i\in I})} V \xrightarrow{\beta} L.$$

The above finishes the construction of L-parameters. But the geometric setup also leads Fargues–Scholze to a categorical version of the local Langlands conjecture [4, p. X.3.5], which deserves further exploration.

In [4], Λ is allowed to be the ring of integers in a finite extension of $\mathbb{Q}_\ell[\sqrt{q}]$. For simplicity, we will focus on the case that Λ is a torsion $\mathbb{Z}_\ell[\sqrt{q}]$-algebra.

The main reference is [4]. Complimentary materials include [12], Scholze's lectures on Geometrization of the local Langlands correspondence, lecture notes and videos available here, 2022 IHES summer school lectures on "The Langlands program and the moduli of bundles on the curve" by Fargues–Scholze, videos available on Youtube, Fargues' lecture series in Beijing, Bonn and Columbia, lecture notes available here, Hansen's Beijing notes on the categorical local Langlands conjecture, available here.

Preliminary meeting: 1/25/2024

Talks

All unspecified references are [4].

Talk 1: The geometry of the classical Bun_G (Kęstutis Česnavičius), 2/1

The Harder–Narasimhan stratification, Beauville-Laszlo uniformizaton, relation to the affine Grassmannian. References: [5, Sec. 5.3-5.5], [7], [13].

Talk 2 : The Fargues-Fontaine curve (Guido Bosco), 2/8

Define the relative curve $\mathcal{Y}_S$ and sketch the proof of Proposition II.1.1, 1.2, 1.4. Prove the properties of $\mathcal{Y}_S$ when S is a geometric point, c.f. Theorem II.0.1. Define the (relative) Fargues-Fontaine curve X_S as in Definition II.1.15, Proposition II.1.16. State its diamond equation as in Proposition II.1.17. Define the "mirror curve" Div^1, see Definition II.1.19 and below.

Talk 3: Vector bundles on the Fargues-Fontaine curve I (Tasos Moulinos), 2/15

Discuss the map $\mathrm{Isoc}_k \to \mathrm{Bun}(X_S)$ and its properties and thus define the vector bundles $\mathcal{O}(\lambda)$, for $\lambda \in \mathbb{Q}$. Explain the relation to Lubin-Tate theory as in Proposition II.2.2 and establish the fundamental short exact sequence Proposition II.2.3. Sketch the proof of ampleness of $\mathcal{O}(1)$ (Theorem II.2.6). Construct the algebraic Fargues-Fontaine curve and prove the GAGA theorem (Proposition II.2.7). State Proposition II.2.9.

Talk 4: Banach-Colmez spaces (Bogdan Zavyalov), 2/22

Define Banach-Colmez spaces: present the original definition of Colmez [2, Section 5.2.2] and list the properties as in [2, Proposition 5.16]; state the characterization of Le Bras as v-sheaves and as the category Coh^- [9, Proposition 7.11, Theorem 7.1]. State the fully faithfullness result of Anschütz-Le Bras [1, Corollary 3.10] and list the basic nontrivial Ext-groups [1, Theorem 3.8]. Define the Banach-Colmez space associated to a morphism of vector bundles as in II.2 Page 59. Sketch the proof of Corollary II.2.4. Discuss the structure of absolute Banach-Colmez spaces as in Proposition II.2.5. State properties of (projectivized) relative Banach-Colmez spaces (Proposition II.2.16). Give examples II.3.12-13.

Talk 5: Vector bundles on the Fargues-Fontaine curve II (Kalyani Kansal), 2/26 (Monday)

Recall the Harder-Narasimhan theory on the Fargues-Fontaine curve and prove the classification of vector bundles (Section II.2.4: Proposition II.2.10-Lemma II.2.15). Prove Theorem II.2.19, Corollary II.2.20. Discuss Proposition II.3.1 and its variants. State the vanishing result in Proposition II.3.4, the relation to Div^d in Proposition II.3.6 and the structure of general punctured Banach-Colmez space in Proposition II.3.7. Include as many proofs as time permits.

Talk 6: The topological space $|\mathrm{Bun}_G|$ (Zeyu Liu), 2/29

Give the definitions of G-bundles from Definition III.1.1 (see also [12, Appendix to Lecture 19]). Define Bun_G. Define the Kottwitz set $B(G)$, its Newton and Kottwitz points. Thus define the topological space $|B(G)|$ and basic elements in $B(G)$, c.f. [10], [8]. Give the example of GL_n. Define G-isocrystals and recall the map $\mathrm{Isoc}_G \to \mathrm{Bun}_G$. Prove Theorem III.2.2, 2.3, 2.7. State the theorem of Viehmann/Gleason-Ivanov [14, Theorem 1.1], [6, Theorem 10.8]. Combine Theorem III.2.3 and 2.7 to prove Theorem III.2.4.

Talk 7: Mixed characteristic affine Grassmannian (Sally Gilles), 3/21

Define $\mathbf{B}_{\mathrm{dR}}^+$-affine Grassmannian and its Schubert cells as in [12, Lecture 19]; recall their basic properties. Define Beilinson-Drinfeld type affine Grassmannian over $\mathbb{Z}_p$ and explain briefly its relation to the Witt vector affine Grassmannian as in [12, Section 20.3]. Construct the Beauville-Laszlo morphism, prove Proposition III.3.1. State Lemma III.3.5. See also [11, Lecture 12]. Define moduli spaces of shtukas as in [12, Lecture 23] and explain how they arise as fibers of the Beauville-Laszlo map [11, Lecture 13].

Talk 8: The (non-)semistable locus of Bun_G (Juan Esteban Rodriguez Camargo), 4/1

Explain pure inner twist as in Section III.4.1, prove Proposition III.4.2 and thus deduce Corollary III.4.3. Give the example of GL_n (III.4.4). Prove Theorem III.4.5. Describe the structure of $\widetilde{G}_b$ as in Proposition III.5.1 and sketch the proof. Prove Proposition III.5.3.

Talk 9: Universal locally acyclic sheaves (Longke Tang), 4/4

Review the notions of Artin v-stacks and cohomologically smooth maps between them (Definition IV.1.1, 1.11). Apply this to Bun_G: prove Theorem IV.1.19; state Proposition IV.1.22, Corollary IV.1.23. Introduce universal local acyclicity as in Definition IV.2.1 (c.f. 2.22, 2.31) and review relevant properties. Discuss Proposition IV.2.15, IV.2.19. Explain the relation to dualizability as in Theorem IV.2.23. State Corollary IV.2.25, Proposition IV.2.26, and the characterization of ℓ-cohomological smoothness in Proposition IV.2.33. See also [11, Lecture 18].

Talk 10: Étale sheaves on Bun_G I (Alexander Petrov), 4/11

State V.0.1(i). Discuss $D_{\text{ét}}$ for classifying stacks of locally pro-p groups as in Theorem V.1.1, Corollary V.1.4. Prove Proposition V.2.1, V.2.2. Construct the local charts $\mathcal{M}_b$ as in Section V.3 and give Example V.3.1. Explain their properties as in Proposition V.3.5, 3.6. Show that they give cohomologically smooth charts of Bun_G as in Theorem V.3.7 (assuming Theorem IV.4.2).

Talk 11: Étale sheaves on Bun_G II (Sean Howe), 4/18

Use the charts $\widetilde{\mathcal{M}}_b$'s to construct compact generators in $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ (see proof of Theorem V.4.1) and prove Theorem V.4.1. Explain Remark V.4.5 and the example below. Discuss Verdier duality and ULA sheaves on Bun_G: state Theorems V.6.1, V.6.2, and sketch the proof of Theorem V.7.1. See also [11, Lecture 20].

Talk 12: The Hecke action (Dmitry Kubrak), 4/25

Define the local Hecke stacks (Definition VI.1.6), the Satake category (Definition VI.7.8), and state the Geometric Satake Theorem IV.0.2. (It would be nice if key ingredients of the proof could be mentioned). Define the Hecke operators as in Section IX.2 (ignore the formalism of $\mathcal{D}_{\text{lis}}$ and focus on torsion coprime-to-p coefficients), state Theorem IX.0.1. Comment on the relation to cohomology of local Shimura varieties: state Theorem IX.3.1 and sketch its proof. Define the geometric/spectral Bernstein center and explain briefly the idea of constructing L-parameters from Corollary IX.0.3.

References

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