2025 IAS SUMMER COLLABORATORS PROGRAM REPORT: ANALYSIS OF KINETIC EQUATIONS FOR WAVES

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1. Introduction

Systems with a large number of agents appear frequently in nature, and thus have many applications from gas dynamics and hydrodynamics (to predict navigation conditions, as well as weather patterns), to astrophysics (to model solar winds), as well as in plasma physics of fusion devices. Due to the large number of objects in these systems (e.g. 10^{23} particles in a dilute gas), it is practically impossible to understand the microscopic dynamics in a deterministic way. Instead, kinetic theory provides a statistical description of many-body systems by focusing on certain averaged quantities, such as the gas particle distribution or the wave energy distribution. This results in an effective kinetic equation which describes the out-of-equilibrium dynamics at the mesoscopic level.

Kinetic theory's foundations were set in the seminal works of Boltzmann [2] and Maxwell [7], who independently derived the Boltzmann equation, which describes the evolution of the probability density of a dilute gas in non-equilibrium. In the context of weakly non-linearly interacting waves (wave turbulence), the analogue of the Boltzmann equation is the wave kinetic equation (WKE), which describes the average energy spectrum of a system of many waves driven by a dispersive or wave equation at the microscopic level. The WKE was first introduced by Peierls [9] in his work on solid state crystals, and independently by Hasselmann [5, 6] who worked on gravity water waves. Later, Zakharov and L'vov [13] and Zakharov, L'vov and Falkovich [14] provided a broad framework applicable to various weakly interacting Hamiltonian systems. Of particular importance are the non-trivial stationary solutions called the Kolmogorov-Zakharov's spectra that were discovered by Zakharov [11] and resemble Kolmogorov's spectra of hydrodynamic turbulence.

In the context of the WKE, remarkable progress has been made in the rigorous derivation from the underlying dynamics, culminating in the recent works of Deng and Hani [3, 4] who derived the 4-wave kinetic equation from the cubic nonlinear Schrödinger's equation first up to the kinetic timescale, and then for as long as the kinetic equation remains well-posed. However, the analysis of the WKE from a PDE perspective is still largely underdeveloped, with the expected lifespan of solutions and the long time behavior still being unclear. Our focus during this summer collaboration is to make progress on understanding the well-posedness theory of wave kinetic equations.

2. Progress

We are motivated by the recent work of Ampatzoglou and Léger [1], where a sharp threshold of well/ill-posedness was obtained for the wave kinetic equation originating from quasilinear Schrödinger models. They show that the wave turbulence theory breaks down sooner than expected: the underlying system may be well-posed, while its kinetic counterpart is not. Our goal is to use the insights and tools developed in [1] to treat water wave equations describing wind-generated waves on the surface of the ocean. These waves can have a range of wavelengths - the small wavelength regime corresponds to capillary waves, while the larger wavelengths are associated to gravity waves. The dynamics of water waves can be affected by a number of phenomena, including the capillary/gravity wave interactions, energy input from wind, energy dissipation by viscosity and wave breaking.

During our stay at the Institute we focused on the weak wave turbulence of capillary waves, which was developed by Zakharov and Filonenko [12] and later reformulated by Pushkarev and Zakharov [10]. The equation in question reads

$$\partial_t f = Q[f], \tag{1}$$

where $f(t,k)$ is the wave density function at time t and wavelength $k \in \mathbb{R}^d$. Our focus is on the dimension $d = 3$. The collision operator $Q[f]$ describes resonant three-wave interactions and reads

$$Q[f](t,k) = \iint_{\mathbb{R}^{2d}} \Big(R_{k,k_1,k_2}[f] - R_{k_1,k,k_2} - R_{k_2,k,k_1} \Big) dk_1 dk_2, \tag{2}$$

where, with the abbreviated notation $f = f(t,k)$, $f_1 = f(t,k_1)$, $f_2 = f(t,k_2)$,

$$R_{k,k_1,k_2}[f] = 4\pi |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(\mathcal{E}_k - \mathcal{E}_{k_1} - \mathcal{E}_{k_2})\Big(f_1 f_2 - f f_1 - f f_2 \Big). \tag{3}$$

$\mathcal{E}_k$ denotes the dispersion relation of the waves, which in the case of the surface with infinitely deep liquid, is of the form

$$\mathcal{E}_k = \sqrt{\sigma |k|^3}, \tag{4}$$

where σ is the surface tension coefficient. The presence of the delta functions in the collision operator is there to capture resonant conditions. Finally, the collision kernel V_{k,k_1,k_2} is of the form

$$V_{k,k_1,k_2} = \frac{1}{8\pi\sqrt{2\sigma}} \sqrt{\mathcal{E}_k \mathcal{E}_{k_1} \mathcal{E}_{k_2}} \left(\frac{L_{k_1,k_2}}{|k|\sqrt{|k_1||k_2|}} - \frac{L_{k,-k_1}}{|k_2|\sqrt{|k||k_1|}} - \frac{L_{k,-k_2}}{|k_1|\sqrt{|k||k_2|}} \right), \tag{5}$$

where $L_{k_1,k_2} = k_1 \cdot k_2 + |k_1||k_2|$.

Problem (1) is quasilinear, thus falls into the framework considered in [1]. It is also widely used in practice, and therefore we expect the results to be of interest to other scientific communities.

The overall strategy of the method used in [1] is to first study the gain-only equation – a modification of the wave kinetic equation that keeps only positive terms of the collision operator – and then focus on the full equation with both gain (positive) and loss (negative) terms. The ill-posedness of the gain-only equation in L^∞ polynomially weighted spaces is achieved by choosing initial data of the type $\langle k \rangle^{-M}$, where $\langle k \rangle = \sqrt{1 + |k^2|}$, for large enough M. A choice of initial data that will cause an instantaneous blow up for the full wave kinetic equation is a more subtle problem due to cancellations present between the gain and loss terms. However, adding oscillations to initial data overcomes this difficulty.

In our effort to implement the tools and strategy of [1] to the capillary water wave equation, during our stay at the Institute we first considered a toy model (1) in which the dispersion relation is quadratic instead of having power $\frac{3}{2}$. This starting point is convenient because for such dispersion relation resonant manifolds are easier to parametrize and can lead to explicit calculations. Namely, one parametrization of the resonant manifold determined by $k = k_1 + k_2$ and $|k|^2 = |k_1|^2 + |k_2|^2$ is of the form

$$k_1 = \frac{k}{2} + \frac{|k|}{2}\sigma, \qquad k_2 = \frac{k}{2} - \frac{|k|}{2}\sigma,$$

where σ is a parameter on the unit sphere $\sigma \in \mathbb{S}^{d-1}$. With the help of the above parametrization, by setting initial data to be $\langle k \rangle^{-M}$, for large enough M, our preliminary calculations show that the $\langle k \rangle^M$-weighted L^∞ norm of the solution is infinite, which implies that the gain only equation of the toy model blows up.

Encouraged by the positive outcome of the toy model, we next considered the capillary water wave equation (1) with the dispersion relation (4). Parametrization in such regime is more subtle problem and results in more complex expressions, as has been shown in [8]. Namely, one can parametrize resonant manifold S_k determined by $k = k_1 + k_2$ and $|k|^{\frac{3}{2}} = |k_1|^{\frac{3}{2}} + |k_2|^{\frac{3}{2}}$ as follows

$$w \in S_k \qquad \Leftrightarrow \qquad w = \alpha k + s(\alpha)\hat{\ell}, \quad \alpha \in [0,1], \ |\hat{\ell}| = 1, \ k \cdot \hat{\ell} = 0,$$

for a smooth function $s(\alpha)$. Calculations with this parameterization are more challenging and often require estimates in order to obtain explicit calculations. However, by choosing initial data to be $\langle k\rangle^{-M}$ again, we were able to do preliminary calculations that show that the gain-only equation of the capillary water wave equation (1) also blows up.

3. Future work

In addition to writing up the details of the proof that the gain-only capillary water wave equation blows up, we next plan to consider the full capillary water wave equation (1) and look for special initial data that will offset cancellations happening between gain and loss to still have a blow up. Inspired by [1], we will try to add carefully chosen oscillations. However, unlike [1] where the equations had a well-known isotropic formulation, the more complex structure of the equation (1) forces us to work in the anisotrpic setting. We expect this to be significantly more challenging than [1], but we are optimistic. We will also consider the wave kinetic equation stemming from 2-dimensional irrotational gravity water waves.

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