INSTITUTE FOR ADVANCED STUDY - SUMMER COLLABORATORS RESEARCH PROJECT REPORT

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Our project, titled "*Diophantine geometry of integral and S-integral points on log-K3 surfaces*" was from June 15 – June 30, 2025 and involved the three named above, together with Peter Sarnak. The aim of this collaboration was the study of the topic in the title over number fields K. Much work has been done recently on aspects of these, in particular the Markoff-Hurwitz varieties and in general, SL(2)-relative character varieties of surfaces acted on by a mapping class group Γ. Prior contributions of the members of the group are as follows:

1. In [BGS16a, BGS16b] the reduction of the Markoff equation $\mathcal{M}(k)$: $x^2 + y^2 + z^2 - xyz = k$ modulo p is studied (strong approximation). It is shown that for almost all primes p, the induced graph is connected (this has culminated with the work of [Che24], where the main connectivity conjecture is proved, with finitely many possible exceptions; see also [Mar25]). In [BGS22], similar results for composite moduli shows that almost all of the Markoff numbers are composite. In [GMR19], asymptotic formula for integer points on Markoff-Hurwitz varieties $x_1^2 + ... + x_n^2 = x_1...x_n$ are obtained, giving an interpretation for the exponent of growth in terms of certain conformal measures on the projective space. An overview of these and related developments is in [Gam23].

2. In [GS22, GMS22], integral points on the Markoff surfaces are studied. In [GS22] it is shown that almost all such surfaces have a local-global "Hasse Principle", and that there are infinitely many Hasse failures, some coming from reciprocity obstructions (integral Brauer-Manin type), and other failures not of this type. In [GMS22], this theory of Markoff cubics is used to study and answer questions of commutators in group theory (diophantine word problems). It is shown that there is no profinite local-global principle for commutators in S-arithmetic groups (pertaining to a question of Shalev[Sha12]); the reciprocity obstructions from Markoff's equation are used to create such examples. Also in [GS22] a finite fundamental set of integral points for $\mathcal{M}(k, \mathbb{Z})$ (Γ-inequivalent integral points giving finitely many orbits) was constructed for almost all k. This was considerably extended by [Shi23] to general Markoff-Hurwitz varieties $x_1^2 + ... + x_n^2 - \alpha x_1...x_n = k$.

3. In [Wha20a] an effective analysis of integral points for algebraic curves on character varieties was established. In particular, it was shown that level sets of Markoff equations provide examples of algebraic varieties with Zariski dense sets of integral points, on which the solubility over the integers of any Diophantine equation is decidable. Other related work by Whang include (a) [Wha20b] proving a folklore conjecture that the character varieties are log-Calabi-Yau varieties, and connecting this to a reciprocity theorem for generating series for combinatorial counts of multicurves on surfaces, and (b) [Wha20c] obtaining a finite generation theorem for integral points on the character varieties.

We have/had several goals: (i) to study K-integral points on log-$\mathsf{K}3$ varieties as above. In particular, we wish to determine if the question of existence of such integral points is decidable; (ii) to study in some depth the arithmetic of character varieties that Whang has studied in the past few years. In particular to establish forms of strong approximation (generalizing [BGS16a, BGS16b, BGS22]) for character (and relative character) varieties associated with higher genus surfaces with punctures. Related to this is a novel type of Chebotarev density theorem; (iii) to find the analogue of the commutator problem for higher rank groups (Diophantine word problems) and it's relation to a suitable accessible variety from which one *lifts* integral points to the group, as was done in [GMS22] above and (iv) the study of the Markoff spectra of integral forms over number fields.

The most significant progress during our collaborative period at IAS was accomplished in the direction of (ii) above; we expect a manuscript in the near future tentatively titled "Strong approximation and Chebotarev density theorems on character varieties". We have also had extensive discussions on some of the other topics and have several avenues that we will explore in the coming months.

References

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