Probabilistic scaling, propagation of randomness and invariant Gibbs measures

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Collaborators

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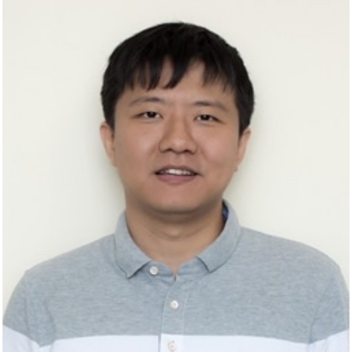

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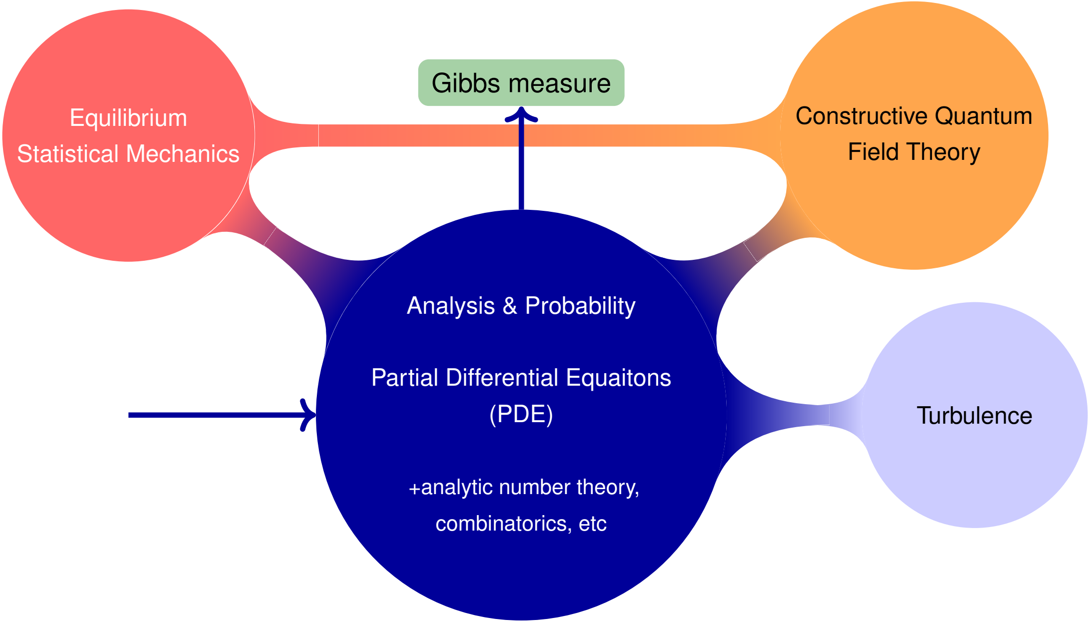
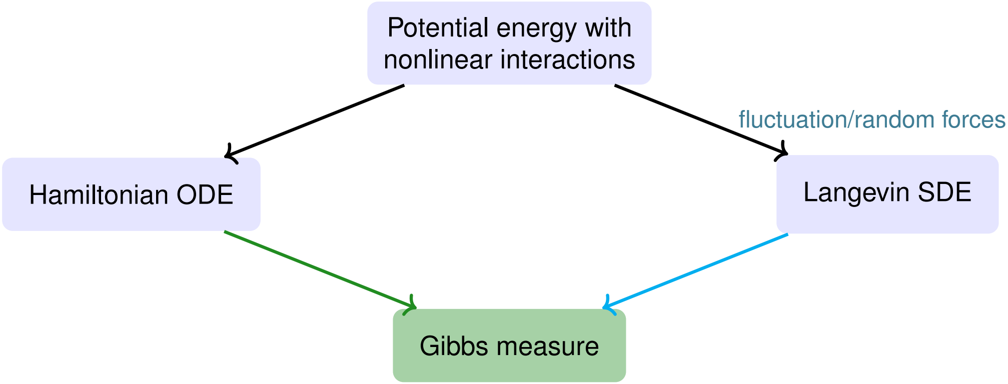

Gibbs measure
Equilibrium
Statistical Mechanics
Constructive Quantum
Field Theory
Analysis & Probability
Partial Differential Equaitons
(PDE)
+analytic number theory,
combinatorics, etc
Turbulence

Hamiltonian ODE and Langevin SDE

- Physical system



How does this work: Hamiltonian ODE -> Gibbs measure

- Potential energy:

$$V(\mathfrak{q}) = \frac{\mathfrak{q}^2}{2} + \frac{\mathfrak{q}^4}{4}, \quad \mathfrak{q} \in \mathbb{R}$$

- Hamiltonian:

$$H(\mathfrak{q},\mathfrak{p}) = \underbrace{\frac{\mathfrak{p}^2}{2}}_{\text{kinetic energy}} + V(\mathfrak{q}), \quad \mathfrak{p} \in \mathbb{R}$$

Hamiltonian ODE:

$$\frac{d}{dt}\begin{pmatrix} \mathfrak{q} \\ \mathfrak{p} \end{pmatrix} = \begin{pmatrix} \partial_{\mathfrak{p}} H \\ -\partial_{\mathfrak{q}} H \end{pmatrix} = \begin{pmatrix} \mathfrak{p} \\ -\mathfrak{q} - \mathfrak{q}^3 \end{pmatrix} \iff \frac{d^2}{dt^2}\mathfrak{q} = -\mathfrak{q} - \mathfrak{q}^3$$

Gibbs measure is defined as: for $\beta > 0$

$$\boxed{d\mu(\mathfrak{q},\mathfrak{p}) = Z_\beta^{-1} \exp\big(-\beta H(\mathfrak{q},\mathfrak{p})\big)\, d\mathfrak{q}\, d\mathfrak{p}}$$

In our example,

$$d\mu(\mathfrak{q},\mathfrak{p}) = Z^{-1} \exp\Big(-\mathfrak{q}^4/4\Big) \underbrace{\exp\Big(-\mathfrak{q}^2/2 - \mathfrak{p}^2/2\Big)\, d\mathfrak{q}\, d\mathfrak{p}}_{\text{Gaussian measure}:= d\rho}$$

$$d\mu \ll d\rho$$

can fail in infinite dimensional (PDE) setting

Invariance: Gibbs measure is invariant under the Hamiltonian flow Φ_t

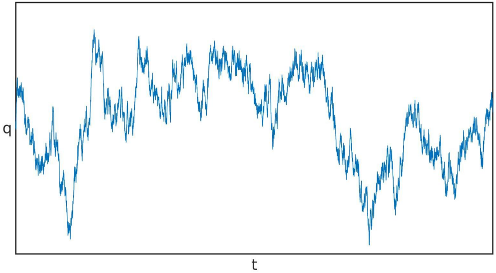
$$\mu\Big(\Phi_t((\mathfrak{q}_0,\mathfrak{p}_0) \in A)\Big) = \mu\Big((\mathfrak{q}_0,\mathfrak{p}_0) \in A\Big)$$

Langevin SDE -> Gibbs measure

- An overdamped Langevin SDE :

$$d\mathfrak{q} = -V'(\mathfrak{q})dt + \sqrt{2}\,dB,$$

where $B =$ Brownian motion.



- The measure

$$d\nu(\mathfrak{q}) = Z^{-1} \exp\Big(-V(\mathfrak{q})\Big)\, d\mathfrak{q}$$

is invariant under the Langevin dynamics.

- Note:

$$d\mu(\mathfrak{q}, \mathfrak{p}) = d\nu(\mathfrak{q}) \exp\Big(-\frac{\mathfrak{p}^2}{2}\Big)\, dp.$$

From ODE to PDE on Tori

Consider now $\phi : \mathbb{T}_x^d \to \mathbb{R}$ or $\mathbb{C}$ and energy

$$H(\phi) = \int_{\mathbb{T}_x^d} \left(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^{p+1}}{p+1} \right) dx.$$

- Formally, we can associate it to a Gibbs measure:

$$d\mu(\phi) \sim Z_\beta^{-1} \exp\left(-\beta H(\phi) \right) \prod d\phi(x)$$

as well as 3 different dynamical flows $\longleftrightarrow \Phi_d^{p+1}$ models

(a) A nonlinear stochastic heat equation. ($\leftrightarrow$ Langevin)

(b) A nonlinear wave equation. ($\leftrightarrow$ real-valued Hamiltonian)

(c) A nonlinear Schrödinger equation. ($\leftrightarrow$ complex-valued Hamiltonian)

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schrödinger
$d = 1$				
$d = 2$				
$d = 3$				
$d = 4$				
$d \geq 5$				

Timeline:

1960 2022

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schrödinger
$d=1$				
$d=2$	[Nel66]			
$d=3$	[GJ73]			
$d=4$				
$d\geq 5$	[Aiz81, Fro82]			

Timeline: 1982

1960 2022

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schrödinger
$d = 1$		[Iwa87]	[Fri85]	
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
$d = 4$				
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1987

Timeline:

1960 2022

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schrödinger	
$d=1$		[Iwa87]	[Fri85]	[Bou94]	
$d=2$	[Nel66]	[DPD03]	[Bou99]	$p=3$ [Bou96]	$p \geq 5$
$d=3$	[GJ73]				
$d=4$					
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Timeline: 2003

1960 2022

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

<table>
<tr><th>Dimension</th><th>Measure</th><th>Heat</th><th>Wave</th><th colspan="2">Schrödinger</th></tr>
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2013

Timeline:

1960 2022

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Dimension	Measure	Heat	Wave	Schrödinger	
$d=1$		[Iwa87]	[Fri85]	[Bou94]	
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$d=4$	[ADC19]				
$d \geq 5$	[Aiz81, Fro82]				

Timeline: 2019

1960 2022

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

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2019

Timeline:

1960 2022

The NLS on $\mathbb{T}^d$

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u(0), \qquad x \in \mathbb{T}^d \end{cases}$$

Here $p \geq 3$ is odd, eq. is defocusing, conserves mass and Hamiltonian.

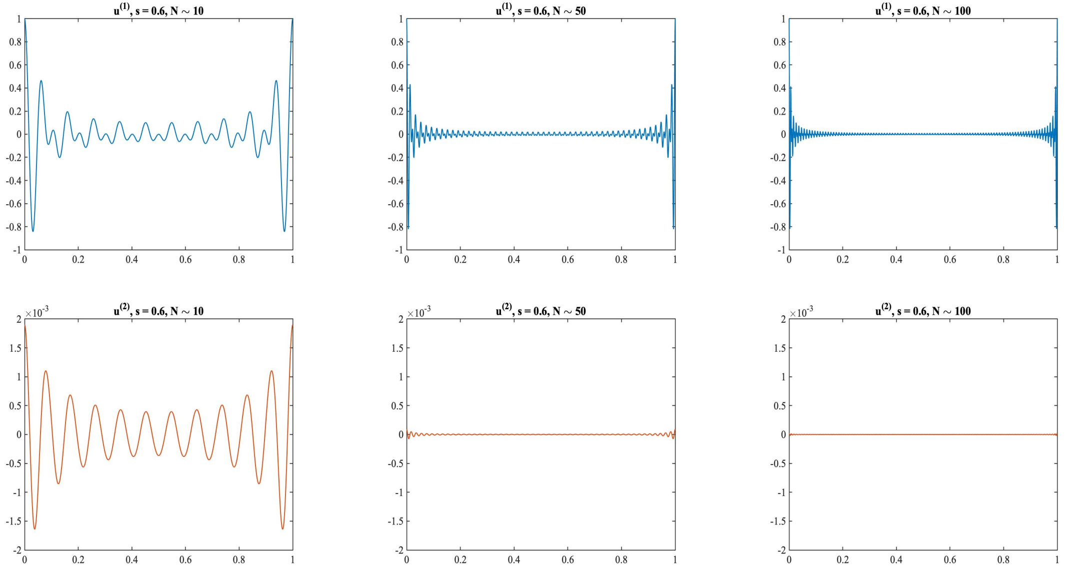
Deterministic scaling: $s_{cr} := \frac{d}{2} - \frac{2}{p-1}$ (heuristics from high-high-to-high)

Theorem: For $s_{cr} \geq 0$, NLS on $\mathbb{T}^d$ is locally well-posed for data in H^s when $s > s_{cr}$.
Ill-posedness may occur for $s < s_{cr}$.

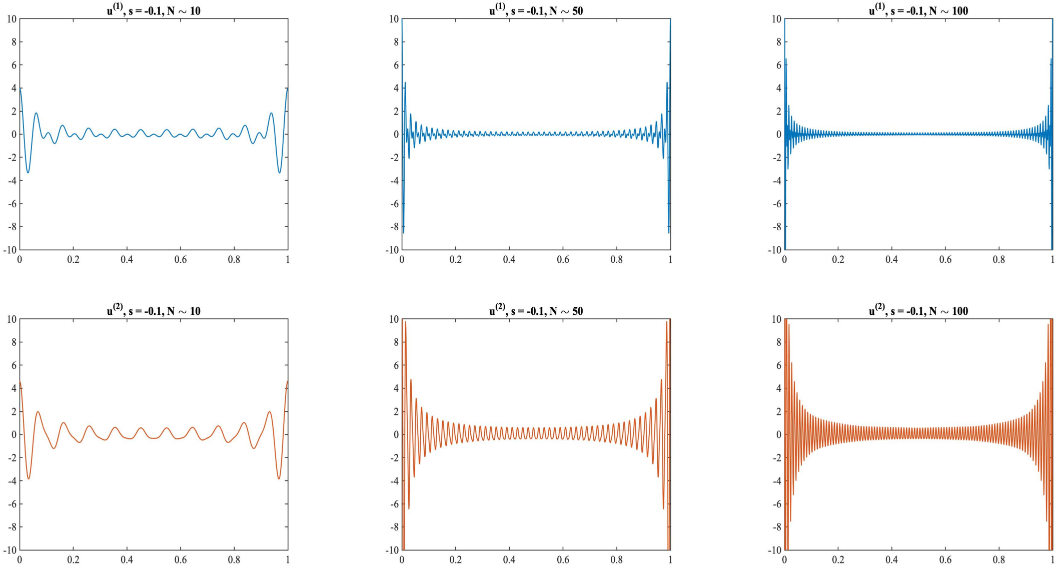
(Bourgain '93, Bourgain-Demeter '15; Christ-Colliander-Tao '03 . . .)

In BEC, binary collisions between the bosons, yields the cubic NLS on $\mathbb{T}^3$; while ternary collisions give a quintic NLS in $d = 1, 2, 3$ (L. Erdös–B. Schlein–H.T. Yau; T. Chen–N. Pavlović; Y. Hong–K. Taliaferro–Z. Xie; X. Chen-J. Holmer.)

The graphs of $u^{(1)}$ and $u^{(2)}$ when $t = 1$ of regularity $s = 0.6 > s_c = 0$



The graphs of $u^{(1)}$ and $u^{(2)}$ when $t = 1$ of regularity $s = -0.1 < s_c = 0$



Injecting Randomness

Want to understand and describe how randomness affects the behavior of solutions to PDE's. Such randomness may come into the problem in various ways but two common ones are:

- From the equation such as in stochastic problems with additive or multiplicative noise.
- From random initial data which obeys some canonical law of distribution (e.g.Gaussian law).

The NLS on $\mathbb{T}^d$: Random data theory

Canonical random data:

$$u^\omega(0) = f(\omega) = \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{\langle k\rangle^{\alpha}} e^{ik\cdot x}, \qquad \boxed{\alpha := s + \frac{d}{2}}$$

where $\{g_k\}$ are i.i.d . r.v. complex Gaussian $\mathbb{E}g_k = 0$, $\mathbb{E}|g_k|^2 = 1$, or uniformly distributed on the unit circle of $\mathbb{C}$.

The law of $f(\omega)$ is formally given by a Gaussian measure supported on $H^{s-}(\mathbb{T}^d)$, $\boxed{s = \alpha - \frac{d}{2}}$

$\boxed{\alpha = 1}$ special:

- Corresponds to statistical ensemble of Gibbs measures.
- In 2D and 3D such Gibbs measures are supported on distributions: $H^{0-}(\mathbb{T}^2)$ and $H^{-\frac{1}{2}-}(\mathbb{T}^3)$ respectively.

Random series on the torus $\mathbb{T}$

- 1920-1930: Paley, Zygmund, Rademacher, Kolmogorov, Khintchine ...

- Khintchine's inequality gives the square root cancellation[1], roughly stating that a sum of functions/complex scalars a_j with randomized signs/phases has magnitude comparable to its square function.

- For a sequence of i.i.d. random variables $\{X_j\}_j$ on a probability space $(\Omega, \mathcal{P})$, $\mathcal{P}(X_j = 1) = \frac{1}{2} = \mathcal{P}(X_j = -1)$ and any $1 \leq p < \infty$,

$$\boxed{\Big[\mathbb{E}\big(|\sum_j a_j X_j|^p\big)\Big]^{1/p} \sim \big(\sum_j |a_j|^2\big)^{1/2}}$$

[1] as in the CLT

Probabilistic scaling: a guiding principle

$$u^\omega(0) = f(\omega) = N^{-\alpha} \sum_{|k|\sim N} g_k(\omega) e^{ik\cdot x},$$

then f and the linear solution $u^{(1)} := S(t)f(\omega)$ have a.s. unit size in H^s.

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Let $u^{(2)}$ satisfy $iu_t^{(2)} + \Delta u^{(2)} = |u^{(1)}|^{p-1} \cdot u^{(1)}$. If NLS is a.s. locally well posed, the iteration $u^{(2)}$ should also be bounded in H^s for fixed time t. Fix $|t| \sim 1$ and $|k| \sim N$,

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$$\begin{aligned}\widehat{u^{(2)}}(t,k) &\sim N^{-p\alpha} \cdot \sum_{k_1,\dots,k_p \in \mathbb{Z}^d,\, |k_j|\sim N} \mathbf{1}_{A_{kk_1\cdots k_p}} g_{k_1}(\omega)\overline{g_{k_2}}(\omega)\cdots g_{k_p}(\omega) \\ &\lesssim N^{-p\alpha} \cdot \left(N^{d(p-2)+d-2+}\right)^{\mathbf{1/2}}\end{aligned}$$

so $u^{(2)}$ is bounded in H^s norm if and only if $s \geq -\frac{1}{p-1} := s_{pr}$.

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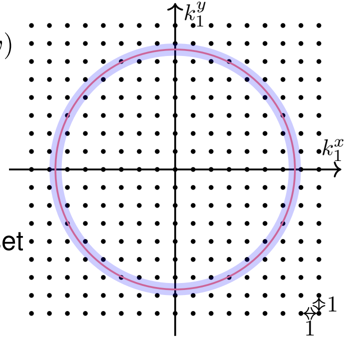

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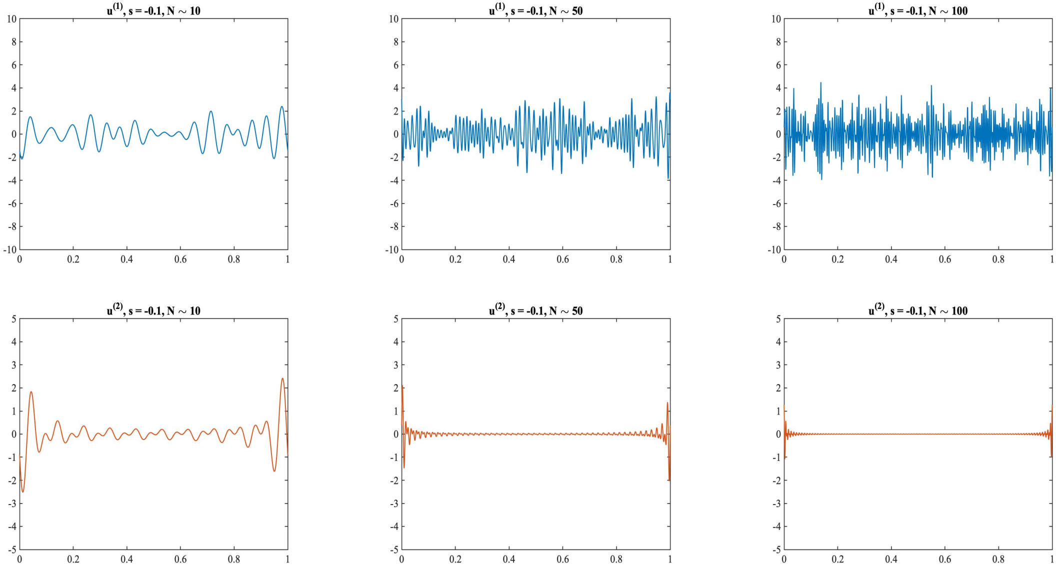
The $N^{d(p-2)+d-2+}$ is from counting the integer lattice point set $A_{kk_1\cdots k_p}$:

$$\left\{ \begin{matrix} k_1 - k_2 + k_3 - \cdots + k_p = k \\ |k_1|^2 - |k_2|^2 + |k_3|^2 - \cdots + |k_p|^2 - |k|^2 \approx \Omega \end{matrix} \right\}.$$

Random data: $u^{(1)}$ v.s. $u^{(2)}$ of regularity $s = -0.1 > s_{pr} = -\frac{1}{4}$



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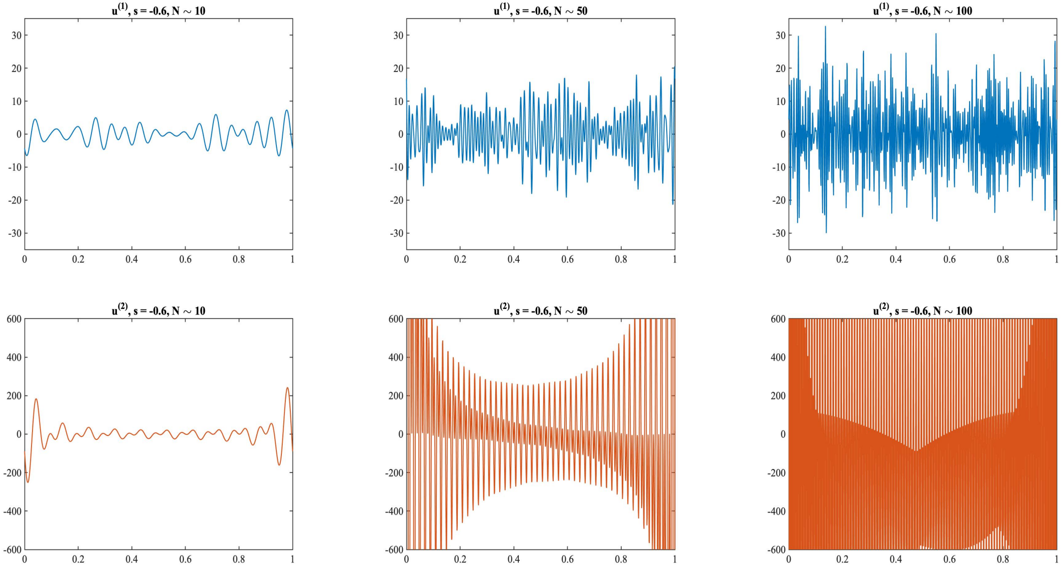

Parabolic and Hyperbolic Comparisons

- Stochastic heat equation:
$$(\partial_t - \Delta)u + \mathcal{N}_p(u) = \xi,$$
where ξ is some spacetime Gaussian noise on $\mathbb{R} \times \mathbb{T}^d$; $\quad s_{pr}^{H} = -\frac{2}{p-1}$

- Nonlinear wave equation:
$$(\partial_t^2 - \Delta)u + \mathcal{N}_p(u) = 0,$$
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- In most cases this is indeed true (e.g. NLS $p \geq 3$) but in some cases, especially in low dimensions and/or low degree nonlinearity, discrepancies may occur. These are not uncommon in other settings involving the notion of scaling including in (stochastic) parabolic and in deterministic settings.

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- Discrepancies mainly maybe caused by two mechanism: 1) *high-high-to-low* interactions in such cases ; 2) anomalies occurring in various counting estimates depending on the specific dispersive relation.

Study of propagation of randomness

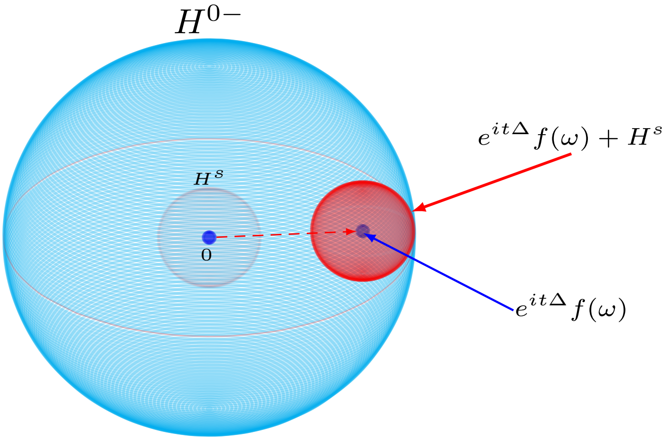
Start with random initial data distributed according to some canonical law (e.g. Gaussian, independent Fourier coefficients) then how does this random structure get transported when one moves along the flow of a nonlinear dispersive equation?

Some natural questions:

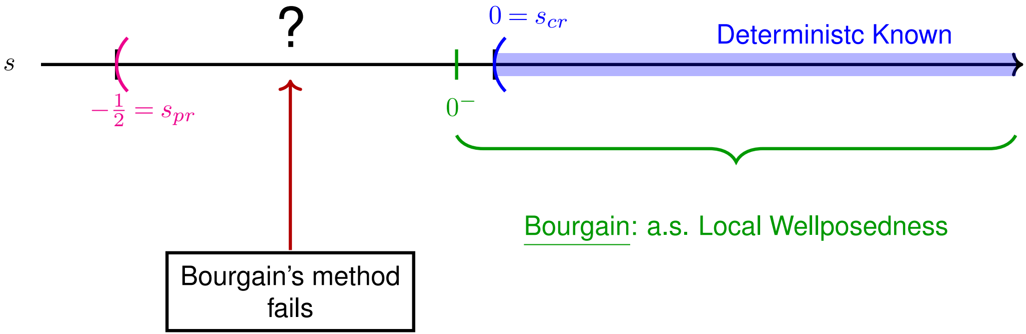
- 1. What is the optimal regime where the solution exists and is unique almost surely, at least locally in time?
- 2. Can one describe the solution in terms of the random structure of the initial data -for short times?
- 3. If there are (formally invariant) Gibbs measures, can we justify their invariance?

Bourgain's Seminal Work ('96)

- He considered the invariance of the Gibbs measure for the cubic NLS equation on $\mathbb{T}^2$: $s_{pr} = -\frac{1}{2} < 0- < 0 = s_{cr} < s$.



- multilinear large deviation estimates
- integer lattice counting estimates $\leftrightarrow$ analytic number theory
- TT^* arguments $\leftrightarrow$ random matrix estimates



- Why: If we start with random data f a bit rougher than Gibbs and proceed with linear-nonlinear ansatz:

$$u = e^{it\Delta} f + v.$$

Then remainder v is not as regular as before (stays below L^2).

- So one cannot close the estimates of the fixed point argument as in Bourgain by relying solely on the (poor) regularity of v.

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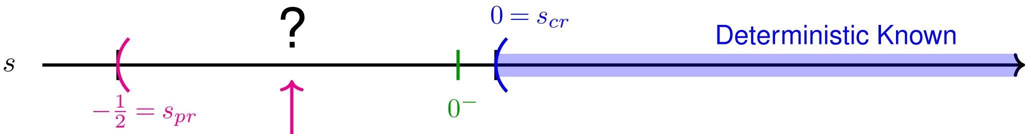
- We need to understand the intrinsic random structure of v. First question is where does this poorer regularity of v comes from?

- Just as it was the case in the study of singular stochastic heat equation (by Hairer's regularity structures or Gubinelli et al.'s paracontrolled calculus) the culprits are wave interactions such as (simplest form):

$$(u^{\omega}_{\text{lin}})_{\text{high}}\, v_{\text{low}}\, v_{\text{low}} \qquad \longleftarrow \text{need to remove (but, all of them!)}$$

How did we resolve this problem?



Bourgain: A.S. Local Wellposedness

Random Tensor Theory (Y. Deng, A.N, H. Yue, 2019–2020)

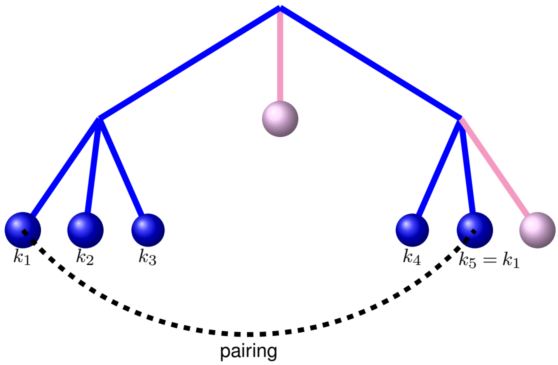
- Unveil the random structure of the remainder v above.
- Detailed information on propagation of randomness.

Works for any $d \geq 2$ and any number of wave interactions $p+1$ (unified theory).

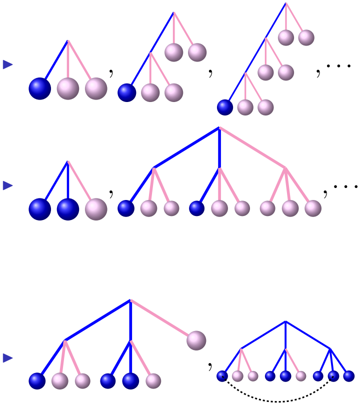
Components of the solutions: all kinds of iteration trees

To find the solution expand nonlinearity using the equation itself; keep expanding until we hit a low frequency input dictated by how close we are to s_{pr} $\rightarrow$ iteration procedure represented by Feynman-type diagrams or tree expansions such as:

- an example iteration tree:



- more trees:

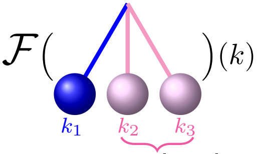


- To find the solution on the full probabilistic subcritical regime $s > s_{pr}$ we need arbitrary long (finite) high-order expansions. This gives rise to what we call random tensors where t is viewed as a parameter, which are highly nonlinear objects arising from the high-order iteration trees.

- The random tensors allow us to get a handle of the exploding complexity that arises from the high-order iteration trees.

- The random tensors carry the (random) information of the low frequency components of the solution and are independent from the Gaussians $g_{k_j}^{\pm}(\omega), j = 1, \ldots, q$. .

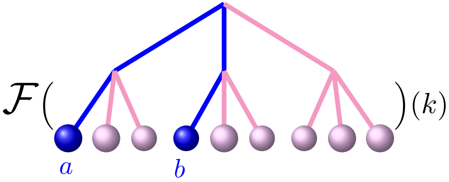
Random tensors $\longleftarrow$ High-order iteration trees



sol. at low freq.

$$= \sum_{k_1} h_{kk_1}(t)\, g_{k_1}(\omega)$$

$(1,1)$-tensor mapping $k_1 \to k$



$$= \sum_{a,b} h_{kab}(t)\, g_a(\omega) g_b(\omega)$$

$(2,1)$-tensor mapping $a, b \to k$

$$e^{it\Delta} f_{\text{high}}(\omega) \cdot \underbrace{u_{\text{low}} \cdot u_{\text{low}}}$$

Fourier picture $\longrightarrow$ $h_{kk_1}(t)$

- Random tensors are probabilistically independent from the Gaussians (which live on high frequencies).

Solution Ansatz:

- We make the Ansatz for the Fourier coefficient: $u_k(t) := \widehat{u}(k,t)$ of the solution:

$$u_k^\omega(t) = \frac{g_k(\omega)}{|k|^\alpha} + \sum_{k_1} h_{kk_1}(t)\, g_{k_1}(\omega) + \sum_{k_1 k_2} h_{kk_1k_2}(t)\, g_{k_1}(\omega)\overline{g_{k_2}}(\omega) +$$
$$\cdots\cdots + \sum_{k_1 k_2 \cdots k_q} h_{kk_1k_2\cdots k_q}(t)\, g_{k_1}(\omega)\overline{g_{k_2}}(\omega)\cdots g_{k_q}(\omega) + (\text{Remainder})_k$$

- The convergence of expansion is completely determined by the properties of these tensors.
- Heuristically the difficulty of covering the full probabilistically subcritical range $s > s_{pr}$ (for fixed p) can be measured by the order of the (finite) expansion needed, which tends to infinity as $s \to s_{pr}$.
- Each iteration of the equation gains regularity $\sim (p-1)(s - s_{pr})$ (as in probabilistic scaling heuristcs argument).

Random tensors framework

- We develop an algebraic theory: structure of the tensors and how they are built from smaller tensors using certain operations such as tensor products, contractions, etc. giving rise to two algebraic operations: merging and trimming.

- We also develop the analytic theory, which behaves well with our algebraic theory and entails choosing suitable norms for the tensors $h = h_{kk_1\ldots k_q}$, for which we prove several estimates that provide suitable bounds for the tensor terms and remainder.

Random tensors framework

- We develop an algebraic theory: structure of the tensors and how they are built from smaller tensors using certain operations such as tensor products, contractions, etc. giving rise to two algebraic operations: merging and trimming.
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- Proof proceeds then by induction relying on the above $+$
  - LDE
  - integer lattice counting estimates,
  - high-order TT^*/random matrix estimates
  - subtle selection algorithm needed to exploit the flexibility we build into the estimates proved in our analytic theory.

RTT Analytic Theory

Given a set of input variables A, $h_{k_A} : (\mathbb{Z}^d)^A \to \mathbb{C}$ is a function.

For (B, C) a partition of A, we define the operator norm $\|h\|^2_{k_B \to k_C}$ of h viewed as an operator mapping functions of k_B to functions of k_C.

For example for a tensor $h = h_{kxyz}$ we define

$$\|h\|^2_{kx \to yz} := \sup \left\{ \sum_{y,z} \left| \sum_{k,x} h_{kxyz} \cdot z_{kx} \right|^2 : \sum_{k,x} |z_{kx}|^2 = 1 \right\}$$

In some instances, we just use the ℓ^2 norm of h in all its variables (Hilbert-Schmidt norm), for example for $h = h_{ab}$, we have

$$\|h\|^2_{ab} = \sum_{a,b} |h_{ab}|^2.$$

A crucial component of the RTT is the choice of norms for the tensors $h = h_{kk_1 \dots k_q}$ that behave well with the algebraic process of merging and trimming.

Analytic theory: merging estimates

Suppose we consider the merged tensor

$$\mathfrak{h}_{bczw} = \sum_{a,e,f} (h^1)_{abc}(h^2)_{aef}(h^3)_{efzw}$$

Then we have the following multilinear estimates

$$\|\mathfrak{h}\|_{bz\to cw} \leq \|h^1\|_{ab\to c}\|h^2\|_{ef\to a}\|h^3\|_{z\to wef}, \text{ or } \leq \|h^3\|_{efz\to w}\|h^2\|_{a\to ef}\|h^1\|_{b\to ac}, \ \ldots$$

- The formula of $\mathfrak{h}$ does not depend on the order of h^j, but the right hand sides of the inequalities do. So we get a set of inequalities by reordering the tensors, from which we may choose at our disposal.
- Here is where a delicate selection algorithm comes in to optimize the choice one makes in the multilinear merging estimates.

Analytic theory: random matrix estimates (moment method)

- Similarly for trimmed tensors:

$$h'_{kxz} = \sum_{yw} h_{kxyzw} \cdot g_y(\omega)\overline{g_w(\omega)},$$

where the random tensor $h = h_{kxyzw}$ is independent with g_y and g_w, we have with high probability that

$$\|h'\|_{kx\to z} \lesssim N^{\varepsilon} \max\big(\|h\|_{kxyw\to z}, \|h\|_{kxy\to zw}, \|h\|_{kxw\to zy}, \|h\|_{kx\to zyw}\big)$$

where N is the max size of $kxyzw$, and $\varepsilon > 0$ is arbitrarily small.

  - Proof goes back to Bourgain's '96 paper and relies on **high order** TT^* argument and multilinear estimates above.

Invariance of 2D Gibbs NLS for all $p \geq 5$

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$					
$d = 2$			[Bou99]	$p = 3$ [Bou96]	$p \geq 5$ Open
$d = 3$			Open	Open	
$d = 4$					
$d \geq 5$					

Timeline: 2019

1960 2022

Invariance of 2D Gibbs NLS for all $p \geq 5$

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$					
$d = 2$			[Bou99]	$p = 3$ [Bou96]	$p \geq 5$ **[DNY19]**
$d = 3$			Open	Open	
$d = 4$					
$d \geq 5$					

Timeline: 1960 — 2019 — 2022

Hyperbolic Φ^4_3 problem: Invariance of 3D Gibbs cubic NLW

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$					
$d = 2$			[Bou99]	$p = 3$ [Bou96]	$p \geq 5$ **[DNY19]**
$d = 3$			**[BDNY22]**	Open	
$d = 4$					
$d \geq 5$					

Timeline: 2022

1960 2022

Hyperbolic Φ_3^4 problem: Invariance of 3D Gibbs cubic NLW

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$					
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$d = 3$			**[BDNY22]**	Open	
$d = 4$					
$d \geq 5$					

Invariance of Gibbs measure under 3D cubic NLW dynamics. Challenges:

- It is harder because now $d\mu \perp d\rho \quad \rightarrow$ probabilistic dependent Fourier modes.
- Spatial regularity $-\frac{1}{2}-$. On the hand, the problem is probabilistically subcritical $s_{pr}^{W} = -\frac{3}{4} < -\frac{1}{2}-$

Heart of the matter: establish local in time existence and uniqueness of solutions on the statistical ensemble:

- Find a suitable representation of the statistical ensemble which is achieved by relying on the parabolic Φ^4_3 (3D cubic stochastic heat equation + invariance of the measure under its flow): $\phi_{wave}(0)$ and $\phi_{heat}(\tau)$ have the same law $\to$ good representation!
- A para-controlled Ansatz (thanks to the smoothing effect of NLW) as opposed to the more delicate random tensor Ansatz (Y.Deng–A.N.–H.Yue '20).
- The analytical framework of the RTT.
- An analysis of heat-wave stochastic objects. The 'caloric data' comes from the cubic stochastic heat equation: some interactions of the explicit stochastic objects contain both heat and wave propagators $\to$ synergy between parabolic and hyperbolic theories.
- The discovery/existence of a hidden cancellation between sextic stochastic objects
- A new bilinear random tensor estimate.
- The combinatorial molecule estimates (as in Deng-Hani's full derivation of the WKE for cubic NLS).

Open Challenges

- Gibbs measure for 3D cubic NLS.
  - As before $d\mu \perp d\rho$, measure lives in $H^{-\frac{1}{2}-}$.
  - But now $s_{pr} = -\frac{1}{2} \to$ probabilistically critical problem: no gain of regularity with each iteration of HHH.
- Out-of-equilibrium long time dynamics
  - Is there any part of the random description of the solution that propagates for longer times? Is it possible to extend for longer times the random structure lying in the high frequencies components?

Many thanks for your attention!!

Appendix

A case study: 2D Quintic NLS Gibbs: what's happening?

- Let us consider $p = 5$, and recall $s_{cr} = \frac{1}{2}$ and following Bourgain write $u = u_{\text{lin}} + v$.

  Then $u_{\text{lin}} \in H^{0-}$ but v can only be put in $H^{\frac{1}{2}-}$ which is still (det.) supercritical: one cannot close the estimate by itself.

- This poor regularity comes <u>only</u> from high-low frequency interactions so we may try to identify a term X from v that is paracontrolled by u_{lin},

$$X = \mathcal{I}\pi_{>}(u_{\text{lin}}, :|u|^{p-1}:) \qquad \mathcal{I} = (i\partial_t - \Delta)^{-1}$$

  and hope that X behaves like u_{lin} and that $Y := v - X$ is smoother.

- But for this, need some control on the lower frequency parts of $:|u|^{p-1}:$ which itself contain $:|X|^{p-1}:$ whose regularity is $H^{\frac{1}{2}-}$, still supercritical $\to$ <u>no way</u> of controlling it assuming only this. Instead one needs to zoom in/unveil and invoke the structure of X .

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- But for this, need some control on the lower frequency parts of $:|u|^{p-1}:$ which itself contain $:|X|^{p-1}:$ whose regularity is $H^{\frac{1}{2}-}$, still supercritical $\to$ no way of controlling it assuming only this. Instead one needs to zoom in/unveil and invoke the structure of X .

  - For heat -and to some extent wave- X has higher regularity due to smoothing so the low frequency part above is a nice function and can be place directly into a good function space.

Method of random averaging operators

- To tackle this conundrum and prove the invariance of the Gibbs measure for 2D NLS (any $p \geq 5$) we introduce the method of random averaging operators (RAO) which also lays the foundation for the more general random tensors theory (RTT) mentioned above.

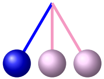
- Instead of trying to place the high-low interactions above in a low-regularity space (unsuccessful), RAO views these interactions as an operator applied to the high frequency linear evolution.

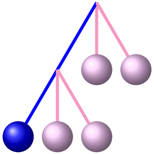
- RAO has a much simpler form than random tensors and less notation-heavy, and suffices in many cases where one is not too close to probabilistic criticality.

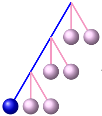
- The RAO come in the RTT as they are the simplest and basic cases of random tensors $\rightarrow$ the$(1,1)$-tensors.

Simple example $(p = 3) : (1, 1)$ tensor terms

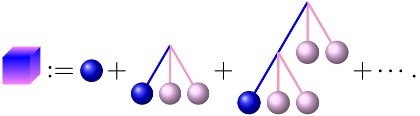
Let $\mathcal{I} =$ Duhamel operator. Denote  $:= e^{it\Delta} f_N(\omega)$ and  $:= u_{N^\delta}$, and define

 $:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}),$

 $:= \mathcal{I}\mathcal{C}\big(\mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}), u_{N^\delta}, u_{N^\delta}\big),$

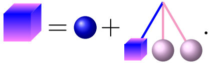
and so on  $, \cdots$.

The sum of these trees forms on an infinite series of trees:

$\Psi_{N,N^\delta} =$ 

which is equivalent to the para-linearized equation:

$$\begin{cases} (i\partial_t + \Delta)\Psi_{N,N^\delta} = \mathcal{C}(\Psi_{N,N^\delta}, u_{N^\delta}, u_{N^\delta}); \\ \Psi_{N,N^\delta}(0) = f_N(\omega). \end{cases} \quad \Longleftrightarrow$$

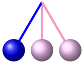
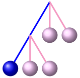


By solving this equation, we have that the k-th Fourier mode of Ψ_{N,N^δ} is in the following form:

$\mathcal{F}($  $)(k) = \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha}$

where h_{kk_1} is the $(1,1)$ random tensor (matrix); indep. of $g_{k_1}(\omega)$.

The solution Ansatz in RAO

$u = \sum_N$  $+ \text{remainder} = \quad u_{\text{lin}} + \Big(\sum_N$  $+$  $+ \cdots\Big) + \text{remainder}$

$$= \underbrace{u_{\text{lin}}}_{\in H^{0-}} + \underbrace{\mathcal{P}(u_{\text{lin}})}_{\in H^{\frac{1}{2}-}} + \underbrace{\text{remainder}}_{\in H^{1-}}$$

New paradigm: We view the key high-low interactions where the high frequencies come from u_{lin} as a random operator $\mathcal{P}$ applied to u_{lin}. We expand the solution u in Fourier space, where $u_k(t) := \widehat{u}(t,k)$, as

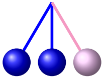
$$\boxed{u_k(t) = \frac{g_k(\omega)}{\langle k\rangle^\alpha} + \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1\rangle^\alpha} + (\text{remainder})_k} \tag{RAO}$$

where h_{kk_1} is the $(1,1)$ random tensor (matrix) independent from g_{k_1} and containing all the randomness information of the low frequency components of the solution u and prove suitable operator norm estimates for h_{kk_1}.

We also globalized the local-in-time random averaging operator structures: Bourgain's globalization + the stability of the random structures (RAO Ansatz) under small H^{1-} perturbations of the data.

Simple example (p=3) of a (2,1) tensor term

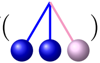
- The $(2,1)$ tensors have 2 terminal leaves which are Gaussian term $e^{it\Delta} f_N(\omega)$; the other terminal leaves are low frequency components u_{N^δ}.
- Denote



$$:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_{N_1}(\omega), e^{it\Delta} f_{N_2}(\omega), u_{N^\delta})$$

where $\mathcal{I}$ is the Duhamel operator, $N = \max(N_1, N_2)$ and $N_1, N_2 > N^\delta$, $\;0 < \delta < 1$ fixed. Then (modulo details about the temporal frequency):

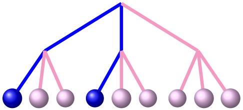
The k-th Fourier mode is

$\mathcal{F}($  $)(k)$

$$\sim \sum_{k_1,k_2} \underbrace{\left(\sum_{|k_3| \le N^\delta} \mathbf{1}_{\left\{ \substack{k = k_1 - k_2 + k_3 \\ |k|^2 = |k_1|^2 - |k_2|^2 + |k_3|^2} \right\}} \widehat{u}(k_3) \right)}_{h_{kk_1k_2}} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2 \rangle^\alpha}$$

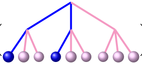
where $|k_1| \sim N_1$, $|k_2| \sim N_2$ and $|k_3| \le N^\delta$. Note that here $h_{kk_1k_2}$ is a (2,1) random tensor -say- maps $k_1, k_2 \to k$. Another Example:

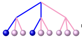
$$\mathcal{IC}\big(\mathcal{IC}(e^{it\Delta}f_{N_a}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(e^{it\Delta}f_{N_b}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(u_{N^\delta}, u_{N^\delta}, u_{N^\delta})\big)$$

$:=$ 

where $N = \max(N_a, N_b)$ and $N_a, N_b > N^\delta$.

$$\boxed{\mathcal{F}\Big(\;\Big)(k) = \sum_{\substack{|a|\sim N_a \\ |b|\sim N_b}} h_{kab} \cdot \frac{g_a(\omega)}{\langle a\rangle^\alpha} \frac{\overline{g_b(\omega)}}{\langle b\rangle^\alpha}}$$



Note that h_{kab} is a $(2,1)$ random tensor which maps $a, b \to k$ associated to the term .

$$h_{kab} = h_{kk_{11}k_{21}} = \Big(\sum_{\substack{k_{12},k_{13},k_{22},k_{23}, \\ k_{31},k_{32},k_{33}}} \mathbf{1}_{(\star)_{kk_{11}k_{21}}} \prod_{\substack{\mathfrak{l}\in\{12,13,22, \\ 23,31,32,33\}}} \widehat{u}(k_{\mathfrak{l}}) \Big)$$

$$\begin{aligned}(\star)_{kk_{11}k_{21}} := \{(k_{12}, k_{13}, k_{22}, k_{23}, k_{31}, k_{32}, k_{33}) : |k_{\mathfrak{l}}| \le N^\delta, \mathfrak{l} \in \{12, 13, 22, 23, 31, 32, 33\}\\ k = (k_{11} - k_{12} + k_{13}) - (k_{21} - k_{22} + k_{23}) + (k_{31} - k_{32} + k_{33})\\ |k|^2 = (|k_{11}|^2 - |k_{12}|^2 + |k_{13}|^2) - (|k_{21}|^2 - |k_{22}|^2 + |k_{23}|^2) + (|k_{31}|^2 - |k_{32}|^2 + |k_{33}|^2)\}\end{aligned}$$