The Complete 2026 IAS Lectures

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Abstract

This comprehensive 139-slide document merges the complete 5-part lecture series delivered at the Institute for Advanced Study (IAS) in March 2026. It presents a novel, geometric solution to large-N_c Planar QCD and the exact derivation of the QCD String. The lectures trace the evolution of the Makeenko-Migdal loop equations, demonstrating why coordinate-space singularities require a transition to Momentum Loop Space. I mathematically prove the exact breakdown of the Taylor-Magnus expansion for the Master Field at the 8th order, necessitating the introduction of 4D continuous geometry. The physical vacuum is identified as a rigid Hodge-dual minimal surface. By placing Majorana fermions (Elves) with chiral bag boundary conditions on this surface, planar Feynman graphs and asymptotically free QCD are dynamically induced, while non-planar intersections are annihilated via the Pauli principle. Finally, projecting the boundary into Minkowski twistor space allows an exact-WKB quantization. The resulting parametric Regge trajectories natively derive the non-relativistic "hook" for heavy quarks, matching the 36-state experimental PDG meson spectrum (including heavy B and D families) with a single universal string tension.(Includes historical context, explicit derivations, and detailed Q&A appendices addressing theoretical objections).

Series Overview:

1. **Loop Calculus & The MM Equation:** The generation of planar graphs and the singularity catastrophe.
2. **Momentum Loop Space:** Finite algebraic structure and off-shell recursion.
3. **The Hodge-Dual Minimal Surface:** Zero modes, additivity, and the area law.
4. **The Fermi String:** The “Elfin” determinant, phase space path integral, and Analytic Twistor String.
5. **WKB spectrum:** The WKB quantization of emergent rigid twistor string leads to nonlinear meson Regge trajectories matching those from PDG data with average accuracy 8%.
6. **Exact WKB:** We argue that this WKB spectrum is *exact* due to the large factor in front of Hessian for quantum fluctuations, resulting from a large number of winding around quark loop.

The 50-Year Challenge

- **The Premise:** In the limit $N_c \to \infty$, QCD should correspond to a string theory (planar diagrams dominate).
- **The Object:** The Wilson Loop functional:

$$W[C] = \frac{1}{N_c} \left\langle \text{tr } \hat{P} \exp \left(\oint_C dx_\mu A_\mu(x) \right) \right\rangle \tag{1}$$

- **The Goal:** Find a string effective action $S_{eff}[C]$ such that:

$$W[C] = \int \mathcal{D}X \exp\left(-S_{eff}[X]\right) \tag{2}$$

- **The Obstacle:** For decades, the string dual remained elusive due to the **ultraviolet singularities** of $W[C]$ in coordinate space:
  - Cusp singularities.
  - Perimeter divergences.
  - Ill-defined loop equation at self-intersection points.

Wilson loop and physical amplitudes

- **Path integral for $\bar{q}q$ currents**.
  The observable current-current correlator is related to Wilson loop as path integral over quark velocity $v = C'$

$$\langle \bar{q}q\bar{q}q \rangle \propto \int d^4k \int ds_1 ds_2 \int DvW[C] \exp\left(\mathrm{i}q \int_{s_1}^{s_2} dtv + \oint dt(\mathrm{i}kv - v^2/2) \right) \quad (3)$$

- **Perturbation theory:**
  With

$$W = 1 - \lambda \int \frac{d^4p}{p^2} \int ds_1 v(s_1) \int ds_2 v(s_2) \exp\left(\mathrm{i}p \int_{s_1}^{s_2} dtv(t) \right) + \dots$$

  we integrate (Gaussian) over velocity and we get standard Feynman graphs in momentum space.

- **Ordinary quark loop in momentum space**

$$\langle \bar{q}q\bar{q}q \rangle \propto \int d^4k/(k^2(k+q)^2 + \dots \quad (4)$$

Conclusions for $W[C]$

- **Brownian paths with infinite cusps:**
  The velocity distribution being Gaussian, these paths are Brownian, with infinitely many cusps.
- **Path integral leads to new UV divergences.**
  The Wilson loop $W[C]$ for piecewise smooth loop is not the relevant physical observable here; insisting on making it finite contour-by-contour is not the right renormalization problem for current correlators.
- **The Wilson loop, integrated over paths** given the set of external momenta injected into the loop, creates observable momentum amplitudes which are renormalizable (just anomalous dimensions of quark currents).
- The renormalization factors of the quark currents are path independent, so they have nothing to do with the cusp singularities of $W[C]$ at the wedge where these currents act.

Parallel Transport and Loop Calculus

- **Holonomy Identity**. The path-ordered exponential of the covariant derivative operator $D_\mu(x)$ along the closed loop $C(\theta)$ with $C(\theta_0) = x$ is equal to the Wilson loop (holonomy) along the path $C(\theta)$ multiplied by the identity operator $\mathbb{I}$ in the Hilbert space.

$$\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x)\right) = \mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) A_\mu(C(\theta))\right) \otimes \mathbb{I} \tag{5}$$

- **Product integral** The path-ordered exponential on the L.H.S. is formally defined as the limit of a product integral:

$$\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x)\right) = \lim_{N\to\infty} \prod_{k=N\to 1} \exp\left(\Delta\theta_k \dot{C}_\mu(\theta_k) D_\mu(x)\right) \tag{6}$$

where $\Delta\theta_k \to 0$ and the product is ordered from right to left. Let $dC_\mu(\theta) = d\theta \dot{C}_\mu(\theta)$.

- **The Infinitesimal Disentangling** We analyze a single infinitesimal factor $\exp\left(dC_\mu D_\mu(x)\right)$. Using the Lie-Trotter product formula, $e^{A+B} = e^A e^B + O([A,B])$, we can split the operator:

$$\exp\left(dC_\mu(\partial_\mu + A_\mu(x))\right) = \exp\left(dC_\mu \partial_\mu\right) \exp\left(dC_\mu A_\mu(x)\right) + O(d\theta^2) \tag{7}$$

  The $O(d\theta^2)$ terms vanish in the $N \to \infty$ limit. The $\exp\left(dC_\mu \partial_\mu\right)$ operator is an infinitesimal translation operator, T_{dC}.

- **The Translation Identity** We now use the fundamental operator identity that defines a translation:

$$T_{dC}\, f(x) = f(x + dC)\, T_{dC} \tag{8}$$

Parallel Transport and Loop Calculus

- **The Hopping Identity** Applying this to our $A_\mu(x)$ operator, we find the "hopping" identity:

$$\exp\left(dC_\mu \partial_\mu\right) \exp\left(dC_\mu A_\mu(x)\right) = \exp\left(dC_\mu A_\mu(x + dC)\right) \exp\left(dC_\mu \partial_\mu\right) \tag{9}$$

  This "disentangling" allows us to move the (Abelian) derivative operator to the right.

- **Iteration and Re-ordering** We now apply this identity iteratively to the full product. Let $C_k = C(\theta_k) = x + \int_0^{\theta_k} d\theta' \dot{C}(\theta')$. By periodicity, $C_N = x$.

$$\prod_{k=N\to 1} \exp\left(dC_k \cdot D(x)\right) \approx \prod_{k=N\to 1} \left(\exp\left(dC_k \cdot A(C_k)\right) \exp\left(dC_k \cdot \partial_x\right)\right) \tag{10}$$

$$= \left(e^{dC_N \cdot A(C_N)} e^{dC_N \cdot \partial_x}\right) \dots \left(e^{dC_1 \cdot A(C_1)} e^{dC_1 \cdot \partial_x}\right) \tag{11}$$

Parallel Transport and Loop Calculus

- **operators are non-Abelian and remain path-ordered** The crucial point is that all ∂_x operators are Abelian and commute with each other, while all $A(C_k)$ operators are non-Abelian and remain path-ordered.
- **Arguments of gauge field shifted along the path** Furthermore, the resulting argument x_k of each A in the product at k-th place will be

$$x_k = C_N + \sum_{n=N\to k} dC_n = C_k \tag{12}$$

- **two distinct, ordered parts** This process separates the product into two distinct, ordered parts:

$$= \left(\prod_{k=N\to 1} e^{dC_k \cdot A(C_k)} \right) \left(\prod_{k=N\to 1} e^{dC_k \cdot \partial_x} \right) \tag{13}$$

Parallel Transport and Loop Calculus

- **Local limit** $N \to \infty$ Taking the $N \to \infty$ limit, this becomes:

$$= \left(\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) A_\mu(C(\theta))\right)\right)\left(\exp\left(\int_0^{2\pi} dC_\mu \partial_\mu\right)\right) \tag{14}$$

- **The Closed Loop Condition** The first term is the Wilson loop $W[C]$. The second term is the identity operator because the loop C is closed:

$$\int_0^{2\pi} dC_\mu = \int_0^{2\pi} d\theta \dot{C}_\mu(\theta) = C_\mu(2\pi) - C_\mu(0) = 0 \tag{15}$$

- **Final form** The Hilbert space operator on the L.H.S. is equal to the Wilson loop multiplied by the identity operator $\mathbb{I}$.

Conclusions for $W[C]$

- **Base-point identity (unusual but correct):**
  The loop holonomy can be viewed as parallel transport of $D_\mu(x)$ *based at* $x = C(\theta_0)$. The gauge field is evaluated at the shifted points $C(\theta)$ after disentangling, even though the operator argument stays at x.
- **It is gauge invariant for the closed loop.** With the open loop, this parallel transport of covariant derivative operator is not gauge invariant: it transforms as bilocal operator

$$U(1,2) = T \exp\left(\int_1^2 d\theta C'_\alpha(\theta)\nabla_\alpha(x_1)\right); \tag{16}$$

$$U(1,2) \Rightarrow S(1)U(1,2)S^{-1}(2) \tag{17}$$

- **It becomes gauge invariant at closed loop**

$$\operatorname{tr} U(1,1) = W[C] \exp\left(\oint ds C'_\mu(s)\partial_\mu(x_1)\right) = W[C]\mathbb{I} \tag{18}$$

The Loop Space Derivative

To formulate dynamics, we treat the loop C as the fundamental degree of freedom.

- **The dot derivative :** D_μ.

$$\frac{\delta W}{\delta \dot{C}_\mu(\theta_0)} = \operatorname{tr} \mathbb{P} D_\mu(x) \exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x) \right) \tag{19}$$

- **The Area Derivative:** $[D_\mu, D_\nu]$ Defined geometrically as the anti-symmetric tensor part of the second variation:

$$\frac{\delta W}{\delta \sigma_{\mu\nu}(\theta)} = \frac{\delta^2 W}{\delta \dot{C}_\mu(\theta - 0) \delta \dot{C}_\nu(\theta + 0)} - (\mu \leftrightarrow \nu) \tag{20}$$

- **The Loop Diffusion Operator:** $[D_\mu, [D_\mu, D_\nu]]$

$$L = \oint d\theta \dot{C}_\nu(\theta) \left(\frac{\delta}{\delta \dot{C}_\mu(\theta + 0)} - \frac{\delta}{\delta \dot{C}_\mu(\theta - 0)} \right) \frac{\delta}{\delta \sigma_{\mu\nu}(\theta)} \tag{21}$$

Leibniz property

- $L(A[C]B[C]) = L(A)B + AL(B)$
- $Lf(A[C]) = L(A[C])f'(A[C])$
- Note: This allows for an ansatz $W[C] = W_{pert}[C]\, e^{-\kappa Area[C]}$.

The Makeenko–Migdal (MM) Equation

- The equation of motion for the Wilson loop ($N_c \to \infty$):

$$\boxed{L(W[C]) = \lambda \oint_C dx_\mu \oint_C dy_\mu \delta^{(4)}(x-y) W[C_{xy}] W[C_{yx}]} \tag{22}$$

- **Left Side:** The Loop Diffusion operator.
- **Right Side:** The interaction term (loop splitting).
- **Factorization:** $W_2(C_1, C_2) \to W(C_1) W(C_2) + O(1/N_c^2)$.

Graphic form of MM equation

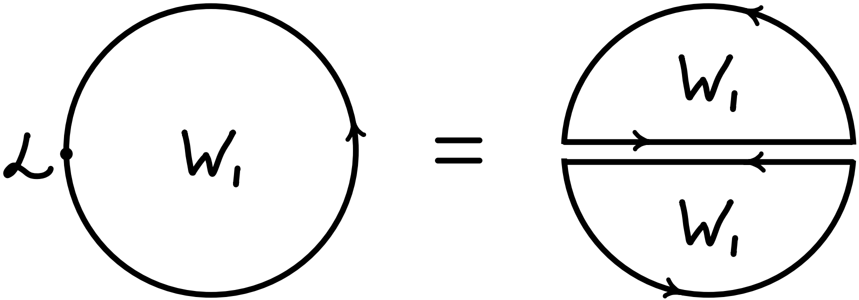


Figure: The MM equation with the loop diffusion operator and the integral term.

Generating Planar Graphs (The Bootstrap)

- Starting with $W = 1$, iterations generate the "Frame Diagrams."
- The inversion of ∂^2 generates the gluon propagator $1/(x-y)^2$.
- The commutators $[\partial, \partial]$ generate the correct tensor structures.
- **Result:** The equation sums all planar Feynman diagrams.

Discussion map: choose a fork

We can go in (at least) three directions. Please interrupt and pick:

Fork A: Planarity / combinatorics

- why RHS is a *product*
- cyclic order $\Rightarrow$ planar recursion
- how iterations reproduce planar graphs

Fork B: UV/contact + regularization

- $\delta(x-y)$ at self-intersection
- cusps / Brownian loops
- analytic point-splitting ("wires")

Fork C: Momentum loop space

- finite algebraic form
- off-shell recursion viewpoint
- continuum limit after solving

(We can take one fork deeply, then return and take another.)

Fork A: how the loop equation generates planar graphs

- Large-N input: *factorization* of multi-loop correlators turns the loop equation into a nonlinear recursion.
- The $\delta(x-y)$ forces two points on the *same oriented loop* to coincide in spacetime; this induces a canonical split $C = C_{xy} \cdot C_{yx}$ respecting cyclic order.
- The RHS becomes $W[C_{xy}]\, W[C_{yx}]$: "split one boundary into two boundaries" with preserved ordering.
- That preserved boundary order is the combinatorial hallmark of planarity (double-line viewpoint).
- Iterating from $W = 1$ builds planar "frame diagrams"; windows are then "glassed" by Wilson-loop factors.

Fork B: why coordinate space is singular, and what regularization does

- The loop equation's interaction term is a *contact* term: it lives at self-intersections ($x = y$).
- Dominant loops in the path integral are irregular (cusps / Brownian-like), so the contact term becomes ill-defined.
- Analytic regularization: soften the UV singularity without changing $d = 4$ (heuristically: $1/k^2 \to 1/(k^2)^{1+\varepsilon}$).
- Implemented as gauge-invariant point splitting: replace $\delta(x - y)$ by a smeared kernel and close split loops by short path factors ("wires").
- Purpose: make the loop equation well-defined *before* discussing continuum limits or nonperturbative solutions.

Fork C: momentum loop space (why amplitudes people should care)

- Coordinate-space pathologies are UV/contact issues (cusps, $\delta(x-y)$), not a failure of planarity.
- In momentum loop space the delta-contact catastrophe disappears and the loop equation becomes a finite local functional differential equation in $P(\theta)$.
- The recursion is naturally *off-shell*: splitting/merging operations act on loop-space kinematics, analogous in spirit to planar recursion for integrands.
- Practical message: solve dynamics first in the finite momentum-loop formulation, then take limits (e.g. remove smearing) after the solution is controlled.

Phys. Rep. '83 excerpt: what to look for (1–2 minutes)

- Next I'll show a few *verbatim pages* from the Phys. Rep. section *"Analytic regularization of the loop equation"* (no re-derivation on the slides).
- Purpose: make precise how the singular contact term in the MM equation is rendered well-defined (before taking any continuum limits).
- Watch for three ingredients:
  - **Analytic softening** of the UV singularity (a controlled ε-deformation).
  - **Gauge-invariant point splitting:** replace $\delta(x-y)$ by a smeared kernel.
  - **"Wires" / gluon path closure:** close the split loops with short path factors to preserve gauge invariance.
- After these pages, I will return to the compact "bootstrap" form (eqs. (18)–(23) in these slides) and explain how iteration reproduces planar graphs.

From Phys Rep '83 (1)

Analytic regularization of the loop equation

The most convenient regularization for perturbative gauge theory is the famous dimensional regularization, which was in fact implied above. However, it turns out to be ill-defined beyond perturbation theory. For example, it misses such an important phenomenon as the axial anomaly, which requires explicit point splitting.

The version of the point splitting procedure closest to dimensional regularization is analytic regularization. Now we are going to adapt analytic regularization for the loop equation. The idea is to replace the propagator $k^{-2} \to k^{-2-2\varepsilon}$ in a gauge invariant manner. The regularization will consist in analytic continuation in ε from the domain of convergence. The number of dimensions $d = 4$ will be unchanged, so the loop functionals will be well defined. Only the coefficient functions, i.e. the traces of gluon Green functions, will become nonsingular at coinciding points.

With the gluon propapator $k^{-2-2\varepsilon}$ in planar graphs the above line with a cross in (3.51) will correspond to $k^{-2\varepsilon}$ since the vertex (3.47) will remain $\sim k^2$ as before. At finite ε the points x and y would be split with the weight $\varepsilon|x-y|^{2\varepsilon-4}$ instead of the δ-function.

Naturally there should arise extra graphs to compensate the violation of gauge invariance at the point splitting. In the old formulation it corresponds to path factors $U(\Gamma_{xy})$ in the adjoint representation where Γ_{xy} is some path between x and y. In the loop equation the paths Γ_{xy}, Γ_{yx} should be added to close the loops

From Phys Rep '83 (2)

$$W(C_{xy}) \to W(C_{xy}\Gamma_{xy}^{-1}) \tag{3.54}$$

$$W(C_{yx}) \to W(C_{yx}\Gamma_{xy})\,. \tag{3.55}$$

From the point of view of gauge invariance these paths are arbitrary but space symmetry would be violated for arbitrary paths. There is a natural choice of these paths preserving the symmetry. Namely they may be chosen to coincide with the path of the gluon. The extra vertices arising from the factor $(U(\Gamma_{xy}))^{\mathrm{adj}}$ can now be interpreted as gluon interactions. Roughly speaking this is the same as replacing the δ-function by a covariant derivative in the adjoint representation $\nabla_\mu = \partial_\mu + [A_\mu,$ raised to the nonintegral power $(-\nabla_\mu^2)^{-\varepsilon}$.

Let us now make these ideas more precise. First of all it is convenient to consider the integrated version of the loop equation (with $L = \int_C \mathrm{d}x_\nu\, L_\nu$)

$$L\, W(C) = \lambda \int_C \mathrm{d}x_\mu \int_{C_{xx}} \mathrm{d}y_\mu\, \delta^d(x-y)\, W(C_{xy})\, W(C_{yx})\,. \tag{3.56}$$

From Phys Rep '83 (3)

The original local version represented a vector equation for the scalar functional $W(C)$, so it was overcomplete. There were certain consistency relations following from the identities (2.88). The r.h. side satisfied the selfconsistency relations due to conservation of the vector current $j_\nu(C_{xx})$ in the sense of the loop calculus $\partial^\nu(x)\, j_\nu(C_{xx}) = 0$. Point splitting in the original equation should respect selfconsistency conditions.

There is no real problem, since there are no anomalies in the vector current but it leads to unnecessary complications. The scalar equation (3.56) is, in principle, equivalent to the overcomplete system, but it is more convenient for point splitting.

As a first step in the gauge invariant point splitting procedure let us introduce the heat propagation kernel

$$K_T(x-y) = \langle x|\exp(-T\hat{p}^2)|y\rangle \tag{3.57}$$

$$\hat{p}_\mu = -\mathrm{i}\,\partial/\partial x_\mu\,. \tag{3.58}$$

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

At vanishing proper time T, it reduces to a δ-function, so it may serve as a regularized definition of the δ-function. However, it would be more convenient to introduce a dimensionless cutoff ε by integrating the heat kernel over the proper time with the corresponding weight,

$$D_\varepsilon(x-y) = \varepsilon \int_0^\infty \mathrm{d}T\, T^{\varepsilon-1}\, K_T(x-y)\,. \tag{3.59}$$

This function is interpolated between the Feynman propagator ($\varepsilon = 1$) and the δ-function ($\varepsilon = 0$). At finite ε it corresponds to the propagator $\Gamma(\varepsilon+1)\,(p^2)^{-\varepsilon}$ with a branch point rather than a pole. The Feynman integrals with such propagators are defined as analytic continuation in ε from the domain of convergence.

Next recall the well known path integral representation of the heat kernel

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

From Phys Rep '83 (5)

$$K_T(x-y) = \int_x^y Dx(\cdot)\exp\left(-\int_0^T dt\,\tfrac{1}{4}\dot{x}^2\right). \qquad (3.60)$$

We shall use a special notation for such a path integral (including the proper time integration)

$$\int_0^\infty dT \int_x^y Dx(\cdot)\exp\left(-\int_0^T dt\,\frac{\dot{x}^2}{4}\right)\cdots \equiv \int D\Gamma_{xy}\cdots. \qquad (3.61)$$

With this notation

$$D_\varepsilon(x-y) = \int D\Gamma_{xy}\,\varepsilon T^{\varepsilon-1}. \qquad (3.62)$$

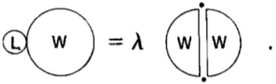
Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

From Phys Rep ’83 (6)

The reader is warned that sometimes other definitions of the path integral are used in the literature, in particular in the author’s papers. We use this one here, since it is most traditional and (therefore) the easiest to memorize. Now we are ready to write down the regularized loop equation

$$L\,W(C) = \varepsilon\lambda \int_C dx_\mu \int_C dy_\mu \int D\Gamma_{xy}\, T^{\varepsilon-1}\, W(C_{xy}\Gamma_{yx})\, W(C_{yx}\Gamma_{xy})\,. \tag{3.63}$$

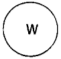
The iteration of this equation would produce regularized perturbation theory. By construction the first iteration yields the one gluon graph with propagator $\Gamma(\varepsilon+1)\,(k^2)^{-\varepsilon-1}$. In higher orders extra vertices will arise. The following graphical representation of (3.63) proves to be convenient:

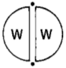


(3.64)

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

Here the functional integration over paths Γ_{xy} corresponding to the double line and the integration over the end points x, y is implied. Each window corresponds to the W-factor of the corresponding loop. We refer to such a graph as a "glassed" graph. Note that only planar graphs can be glassed. We have already dealt with glassed graphs in the random matrix model, in which they were only a technical device. Here the graphs have to be understood in the usual Feynman sense of summing the corresponding amplitudes over all histories. The iterations of this equation can be performed in an integral form



$= 1 + \lambda L^{-1}$


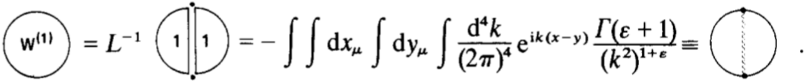
. (3.65)

In the first order on the right-hand side,

$$= \int_C \mathrm{d}x_\mu \int_C \mathrm{d}y_\mu \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik(x-y)}}{k^{2\varepsilon}} \Gamma(\varepsilon+1)\,. \quad (3.66)$$

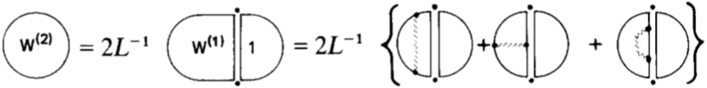
Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

The application of the inverse L operator (with the Euclidean boundary conditions) reduces to multiplication by $-k^{-2}$ in momentum space, i.e.



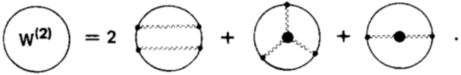
(3.67)

In the next order we find



(3.68)

where the wavy line denotes the regularized gluon propagator $\delta_{\mu\nu}\Gamma(\varepsilon+1)/k^{2+2\varepsilon}$. Application of L^{-1} is here a bit more tedious, but straightforward. The result has the structure



(3.69)

Bootstrap equation in detail

- $$\frac{\delta W[C]}{\delta \sigma_{\mu\nu}(x)} = \left(\partial_\nu^x \delta_{\mu\alpha} - \partial_\mu^x \delta_{\nu\alpha}\right) B_\alpha[C_{xx}] + B_{\mu\nu}[C_{xx}]; \tag{23}$$

- The linear terms related to the source term $J_\nu[C_{xx}]$:

$$B_\nu = \left(-\partial^2\delta + [\partial,\partial]\right)^{-1} \left(J_\lambda - \partial_\mu B_{\mu\lambda}\right); \tag{24}$$

$$J_\nu[C_{xx}] = \lambda \int dy_\nu \delta^4(x-y) W[C_{xy}] W[C_{yx}] \tag{25}$$

- Expansion of the inversion operator:

$$\left(-\nabla^2 + [\nabla,\nabla]\right)^{-1} = -\nabla^{-2}$$
$$- \nabla^{-2}[\nabla,\nabla]\nabla^{-2} - \nabla^{-2}[\nabla,\nabla]\nabla^{-2}[\nabla,\nabla]\nabla^{-2} + \dots \tag{26}$$

- We use the identity:

$$[\nabla_\alpha, \nabla_\beta] F[C_{xx}] = \frac{\delta F[C_{xx}]}{\delta \sigma_{\alpha_\beta}(y)}\Bigg|_{y=x+0}^{y=x-0} \tag{27}$$

- Representation as a sum over the Brownian path Γ:

$$-\nabla^{-2} J_\nu[C_{xx}] = \lambda \int d^4 z \int \mathcal{D}\Gamma_{xz} \oint_{\Gamma_{zx} \cdot C_{xy} \cdot C_{yx} \cdot \Gamma_{xz}} dy_\nu \delta^4(z-y) W[\Gamma_{zx} \cdot C_{xy}] W[C_{yx} \cdot \Gamma_{xz}] \tag{28}$$

Graphic Form of Bootstrap equation

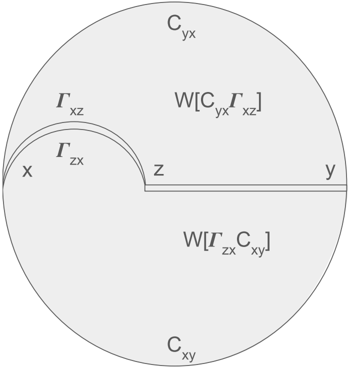


Figure: The path integral in (28) with Brownian paths.

Gluon graphs by iterations of Bootstrap equation

- In the old MM papers, the bootstrap equation was iterated in λ, starting with $W[0] = 1$.
- It faithfully reproduced conventional planar graphs, including ghost loops.
- One recovers the (planar) perturbative beta-function in this bootstrap iteration.

Summary of the Bootstrap Approach 1

- The planar MM equation (22) transforms into a path integral equation.
- Frame diagrams look like planar trees with windows glassed with W functionals.
- Sums over Brownian paths produce gluon propagators.

Summary of the Bootstrap Approach 2

- The frame diagram is manifestly gauge invariant.
- Iterative solutions recover uniquely Faddeev-Popov perturbation theory.
- Various parts of the loop integral correspond to graphs including ghost loops.

The Coordinate Space Catastrophe

1. **The Delta Function:** $\delta^{(4)}(x-y)$ forces interaction at a single point.
2. **The Cusp Singularity:**
   - Physical quark paths are nowhere differentiable.
   - Perturbative QCD dictates a divergence:

$$\log W \sim -\Gamma_{cusp}(\gamma)\log(\Lambda_{UV} L) \tag{29}$$

3. **Natural question:** do cusp/perimeter divergences of $W[C]$ imply the loop equation itself must be renormalized?

Self - intersection and four wedges

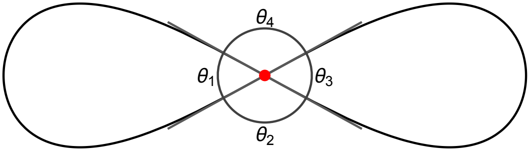


Figure: Four wedges at self intersection

Why multiplicative cusp renormalization is not closed at intersections (1)

- Consider a self-intersecting contour $C = C_{xy} \cdot C_{yx}$ with intersection at $x = y$.
- Geometrically, there are **four** local wedge angles $\theta_1, \ldots, \theta_4$ at the crossing.
- Cusp renormalization is angle-dependent: $W[C] \to \prod Z(\theta_i)\, W_{\text{fin}}$.
- **The Mismatch:**
  - **LHS:** In the MM equation the LHS is $L(W[C])$:

$$\text{LHS} \sim Z(\theta_1)Z(\theta_2)Z(\theta_3)Z(\theta_4)L(W_{\text{fin}}) + W_{\text{fin}}L(Z(\theta_1)Z(\theta_2)Z(\theta_3)Z(\theta_4))$$

  - **RHS:** The equation factorizes: $W[C_{xy}]W[C_{yx}]$. Each W sees only its own single cusp:

$$\text{RHS} \sim \underbrace{Z(\theta_1)}_{\text{from } W(C_{xy})} \cdot \underbrace{Z(\theta_3)}_{\text{from } W(C_{yx})} \lambda \int W_{\text{fin}}(C_{xy})W_{\text{fin}}(C_{yx})\delta(x-y)$$

- The missing factors $Z(\theta_2)Z(\theta_4)$cannot be absorbed into the bare coupling λ, because λ is universal and cannot depend on local angles θ_2, θ_4. These factors must cancel themselves on the left side.

How the loop diffusion operator cancels divergent diagrams

- The operator L cancels divergent factors on the LHS, related to the θ_2, θ_4 wedges.
- This operator, when applied to the planar graphs in $W[C]$, generates extra terms, exactly canceling all the graphs with gluon lines connecting C_{xy} with C_{yx}.
- The remaining gluon line connects points x, y on the loop, and this line, after the application of ∂_x^2 in L, becomes $\delta(x-y)$ matching the RHS. The other lines crossing this one, all get cancelled.
- The renormalization factors $Z(\theta_2), Z(\theta_4)$ depend on $\theta_{2,4} = \arccos\left((v(x) \cdot v(y))/|v(x)||v(y)|\right)$. So, one can apply the functional derivative $\frac{\delta}{\delta v(x)}, \frac{\delta}{\delta v(y)}$. Results are singular, so one has to keep regularization finite.
- So, the renormalization of $W[C]$ by cusp factors does not lead to a finite renormalized loop equation: operator L does not have a local limit.
- **Conclusion:** A single closed coordinate-space equation for $W[C]$ is *not stable* under renormalization. One has to keep regularization finite in that equation, and only go to local limit later, in observables, such as amplitudes for quark bilinears.

RG: local operators vs. Wilson loops on Brownian paths

- **QCD renormalization (RG) is formulated for local gauge-invariant operators:**

$$\mathcal{O}_i^{\text{bare}} = Z_{ij}(\mu)\,\mathcal{O}_j(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{O}_i(\mu) = \gamma_{ij}(g(\mu))\,\mathcal{O}_j(\mu).$$

- **Wilson loops are nonlocal line functionals.** Their UV structure depends on local contour geometry (perimeter/cusps/intersections) rather than on a local operator basis .
- **In physical correlators the loop variable is integrated over a Brownian ensemble:** typical quark paths are nowhere differentiable $\Rightarrow$ infinite cusp density . Therefore a "cusp counterterm per cusp (or per wedge)" cannot define a sensible RG notion for $W[C]$ on the paths that actually dominate the observable path integral .
- **Conclusion / strategy:** do not demand a renormalized coordinate-space $W[C]$ contour-by-contour; instead regularize geometrically and pass to **momentum loop space** where the loop equation is finite (no contact $\delta(x-y)$ singularity) and the local limit is taken *after* solving .

Summary of Lecture I

1. **The MM Loop Equation** provides the defining planar loop dynamics (Schwinger–Dyson form) for Wilson loops in large-N_c QCD.
2. **The observable amplitudes** are given by singular path integrals over Brownian loops involving $W[C]$ and Dirac amplitudes for the free Dirac particle propagation around the loop.
3. **Coordinate Space:** Fixed-contour $W[C]$ has cusp/perimeter UV singularities; in the physical Brownian-loop path integral these are not the renormalization problem that controls observable current correlators.
4. **The Way Forward:** Regularization by geometry and momentum space, where there are no singularities in the loop equation.

Next Lecture:

Momentum Loop Dynamics and the Finite Algebraic Structure of QCD.

Lecture II: Momentum Loop Space and the Failure of Taylor–Magnus

Discussion map: choose a fork

We can go in (at least) three directions today. Please interrupt and pick:

Fork A: Momentum Space

- The physical quark loop measure
- Fourier transform of $W[C]$
- Eliminating $\delta^{(4)}(x-y)$

Fork B: Algebraic MLE

- Magnus forms and Shuffle Ideal
- Algebraic recurrence relations
- Exact solutions up to $\mathcal{O}(P'^7)$

Fork C: The Catastrophe

- The breakdown at $\mathcal{W}^{(8)}$
- Vector vs. Scalar mismatch
- Why naive string theory fails

(Let's walk through the finite algebraic structure, and you'll see why it forces us into a non-perturbative regime.)

Regularization, Renormalizability, and Lattice QCD

- **The Standard Approach:** Lattice QCD provides a rigorous microscopic definition.
- **The Cost:** It explicitly breaks the continuous symmetries of 4D Euclidean space ($O(4)$ rotations) and subtle topological structures like **Hodge duality**.
- **Exact geometry requires the continuum:** Exact instanton solutions or Hodge-dual minimal surfaces simply do not exist on a discrete grid; they only emerge in the continuum limit.
- **Our Strategy:** Keep the theory regularized at the level of the Wilson loop $W[C]$ by a finite fermion mass m (which acts as a geometric UV cutoff).

Shifting the Paradigm: From Coordinates to Momentum

- **Is $W[C]$ physical?** The coordinate-space Wilson loop $W[C]$ is an intermediate, off-shell quantity.
- **Physical observables** involve path integrals over quark trajectories governed by a Brownian measure (an infinite density of cusps!).
- Attempting to renormalize $W[C]$ contour-by-contour for a smooth curve is physically artificial.
- **The Solution:** Transition to **Momentum Loop Space**.

$$W[P] = \int \mathcal{D}C\, W[C] \exp\left(\mathrm{i} \oint dx\, P_\mu(x)\dot{C}_\mu(x)\right); \tag{30}$$

$$\mathcal{D}C = \delta^4\left(\int_0^{2\pi} \dot{C} d\theta\right) \prod_{\theta=0}^{2\pi} d^4\dot{C}(\theta) \tag{31}$$

- By integrating over coordinate loops, we integrate out the geometric singularities (cusps, contact terms) natively!

Quark Loop Amplitudes in Phase Space

- In momentum loop space, physical observables are clear and finite.
- The amplitude for the propagation of a free Dirac particle around the phase space loop:

$$\mathcal{K}\left[P\right] = \operatorname{tr} \hat{P} \exp\left(-\int_0^\infty dt \left(\mathrm{i}\gamma_\mu P_\mu(t) + m_q\right)\right) \tag{32}$$

- The full scattering amplitude with injected external momenta q_k:

$$A[q_1, \ldots, q_n] \propto \int \prod dt_k \int \mathcal{D}P\, W[P+Q]\, \mathcal{K}\left[P\right] \tag{33}$$

where $Q(t) = \sum_k q_k \Theta(t - t_k)$ and $\sum_k q_k = 0$

Feynman Wheel

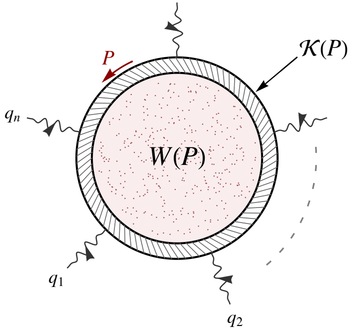


Figure: The momentum loop amplitude, $A(q_1, \dots q_n)$ with the inside part of the wheel corresponding to Momentum loop $W[P]$, and the outer rim corresponding to Dirac path amplitude $\mathcal{K}[P]$. The paths $P(t)$ are random, interacting with inner geometry of the string surface represented by amplitude $W[P]$.

Factorization of the Measure in Momentum Space

- How does the $\delta^{(4)}(x-y)$ contact term from the MM equation disappear?
- Consider the linear functional measure in loop space. For a self-intersecting loop $C(\theta_1) = C(\theta_2)$, the measure natively factorizes!

$$\delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \mathcal{D}C = \left(\delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \prod_{\theta'=\theta_1}^{\theta_2} d^4\dot{C}(\theta')\right)\left(\delta^4\left(\int_{\theta_2}^{\theta_1+2\pi} \dot{C} d\theta\right) \prod_{\theta'=\theta_2}^{\theta_1+2\pi} d^4\dot{C}(\theta')\right)$$

$$\implies \delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \mathcal{D}C = \mathcal{D}C_{12}\mathcal{D}C_{21} \tag{34}$$

- The singular contact constraint $\delta^{(4)}$ is strictly absorbed by the measure of the sub-loops.

The Momentum Loop Equation (MLE)

- Under this Fourier transform, the coordinate Makeenko-Migdal equation:

$$\mathcal{L}_\nu(W[C]) = \lambda \int_0^{2\pi} d\theta\, C'_\nu(\theta)\delta^4(C(0) - C(\theta))W[C_{0,\theta}]W[C_{\theta,2\pi}] \tag{35}$$

  becomes the **Momentum Loop Equation**:

$$\hat{L}_\nu[P]W[P] = \int_0^{2\pi} d\theta\, \frac{\delta}{\delta P_\nu(\theta)} \left(W[P_{0,\theta}]W[P_{\theta,2\pi}]\right) \tag{36}$$

- **Left Side (Kinematics):** Algebraic multiplication by a kinematic tensor $\hat{L}_\nu[P]$.
- **Right Side (Dynamics):** Functional derivative acting on the convolution of two sub-loops.
- **Crucial feature:** *No delta functions. No cusp logarithmic divergences.* The equation is purely algebraic-differential and completely finite.

The Kinematic Tensor and the Shuffle Ideal

- The loop diffusion operator $\mathcal{L}_\nu = \partial_\mu \frac{\delta}{\delta \sigma_{\mu\nu}(0)}$ corresponds to a specific trilinear product:

$$\hat{L}_\nu[P] = T^{\alpha\beta\gamma}{}_\nu \Omega^{(3)}_{\alpha\beta\gamma}[P], \tag{37}$$

$$\Omega^{(3)}_{\alpha\beta\gamma}[P] = \int_0^{2\pi} d\tau_1 P'_\alpha(\tau_1) \int_0^{\tau_1} d\tau_2 P'_\beta(\tau_2) \int_0^{\tau_2} d\tau_3 P'_\gamma(\tau_3) \tag{38}$$

  Here $\Omega^{(3)}$ is the third-order Magnus form (ordered triple integral of P').
- The tensor $T^{\alpha\beta\gamma}{}_\nu$ implements the exact Dynkin projector onto the nested double commutator $T \cdot (D \otimes D \otimes D) \sim [\hat{D}, [\hat{D}, \hat{D}]]$:

$$T^{\alpha\beta\gamma}{}_\nu = \delta_{\alpha\beta}\delta_{\gamma\nu} + \delta_{\gamma\beta}\delta_{\alpha\nu} - 2\delta_{\alpha\gamma}\delta_{\beta\nu} \tag{39}$$

- By Ree's Theorem for Free Lie Algebras, this tensor identically annihilates the **symmetric shuffle ideal** (symmetrized path integral) associated with the path-ordered integral, leaving just $\Omega^{(3)}$.

Algebraic Recurrence for the MLE

- We construct the functional Taylor series by expanding $W[P]$ in parametric-invariant **Magnus forms** $\Omega^{(n)}$:

$$W[P] = \operatorname{tr}\left\{\hat{T}\exp\left(\left(\int d\theta \hat{X}_\mu P'_\mu(\theta)\right)\right)\right\} = \sum_n \mathcal{W}^{(n)}_{\alpha_1\dots\alpha_n}\Omega^{(n)}_{\alpha_1\dots\alpha_n}[P] \tag{40}$$

- The constant operator $\hat{X}_\mu$ has the meaning of the position of quark. The MLE provides recurrent equations between multiple traces $\operatorname{tr}\hat{X}^{\otimes n}$.
- When the functional derivative acts, it brings down an exact commutator insertion:

$$\frac{\delta}{\delta P_\mu(a)}\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\} = P'_\nu(a)\operatorname{tr}\left\{[\hat{X}_\mu,\hat{X}_\nu]\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\}; \tag{41}$$

$$\frac{\delta}{\delta P_\mu(b)}\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\} = -P'_\nu(b)\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)[\hat{X}_\mu,\hat{X}_\nu]\right\} \tag{42}$$

- We dynamically equate LHS kinematic shuffle products against RHS discrete commutators. The MLE collapses into a finite, recursive combinatorial algebra!

The Magnus forms MLE

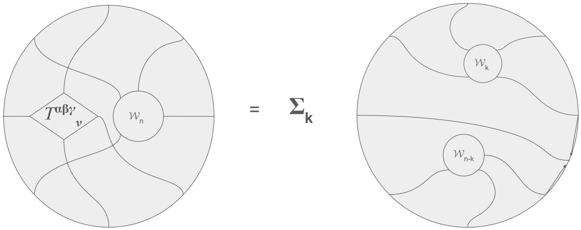


Figure: The recurrent equation for the coefficient tensor parameters $\mathcal{W}^n$ in the Magnus expansion. The external circle represents the unit circle where the angular parameter belong. The inner round blobs with wavy legs landing on a unit circle represent the $\mathcal{W}^n$ tensors, the rhombus with four wavy legs represents the 4-tensor T . Two arrows on the right side correspond to functional derivatives $\frac{\delta}{\delta P_\nu}$ bringing commutators by momentum area derivatives.

Topological Closure and Exact Solutions

- Because the momentum loop is physically closed, $\Omega^{(1)} = \oint P' d\theta = 0$.
- The functional derivative on the RHS splits the loop into two *open* subloops. We analytically enforce their closure by introducing a boundary gap:

$$P'_{closed}(\theta') = P'(\theta') + \Delta P \delta(\theta' - \theta_0) \tag{43}$$

- This subtracts the product $\Omega^{(1)}\Omega^{(n-1)}$ (interleaved operators) from the n-th order form.
- **Order** $\mathcal{O}(P'^3) \implies \mathcal{W}^{(4)}$**:** Solved exactly.
- **Order** $\mathcal{O}(P'^5) \implies \mathcal{W}^{(6)}$**:** Solved exactly. The geometric stress cascading from the lower-order loop derivatives is exactly absorbed by the independent Kronecker pairings.

Exact Algebraic Solutions up to $\mathcal{O}(P'^5)$

- We solve the algebraic MLE by expanding $W[P]$ in the full $(n-1)!!$ general non-cyclic Wick-contraction basis.
- To display the tensors compactly, we use the shorthand $\delta_{ij} \equiv \delta_{\mu_i \mu_j}$.

Order $\mathcal{W}^{(4)}$ (from $\mathcal{O}(P'^3)$):

$$\mathcal{W}^{(4)}_{1234} = \frac{1}{6}\delta_{13}\delta_{24} + c_{4,1}\left(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23}\right) \tag{44}$$

Order $\mathcal{W}^{(6)}$ (from $\mathcal{O}(P'^5)$):

$$\begin{aligned}\mathcal{W}^{(6)}_{123456} = {} & c_{6,1}\left(\delta_{12}\delta_{34}\delta_{56} + \delta_{16}\delta_{23}\delta_{45}\right) \\ & + c_{6,2}\big(\delta_{13}\delta_{24}\delta_{56} + \delta_{12}\delta_{35}\delta_{46} + \delta_{15}\delta_{23}\delta_{46} \\ & \qquad + \delta_{13}\delta_{26}\delta_{45} + \delta_{16}\delta_{24}\delta_{35} + \delta_{15}\delta_{26}\delta_{34}\big) \\ & + c_{6,3}\left(\delta_{14}\delta_{23}\delta_{56} + \delta_{12}\delta_{36}\delta_{45} + \delta_{16}\delta_{25}\delta_{34}\right) \\ & + \left(2c_{6,2} - \frac{1}{2}c_{6,1} - \frac{1}{2}c_{6,3}\right)\left(\delta_{13}\delta_{25}\delta_{46} + \delta_{14}\delta_{26}\delta_{35} + \delta_{15}\delta_{24}\delta_{36}\right)\end{aligned}$$

The Breakdown: Mathematical Contradiction at $\mathcal{W}^{(8)}$

- The geometric stress rigidly locks the coefficients, perfectly absorbing the constraints.
- **At $\mathcal{W}^{(8)}$, this linear absorption violently fails!**
- **The Source Term:** The RHS generates a massive, inhomogeneous geometric stress sourced by the non-linear cross-terms of the massive 4-point functions: $\mathcal{W}^{(4)} \times \mathcal{W}^{(4)}$.
- **The Trap:** The loop derivative acts effectively as a commutator: $\mathcal{D}(\mathcal{W}^{(8)}) \approx \mathcal{W}^{(8)}_{\nu\alpha_1\dots} - \mathcal{W}^{(8)}_{\alpha_1\nu\dots}$. This strictly annihilates any symmetric index structures in the Ansatz.
- **The Contradiction:** Even using the most general $(n-1)!! = 105$ non-cyclic Kronecker pairings, the surviving matrix rank of the variables is crushed. It is *algebraically impossible* to absorb the rigid, symmetric cross-term stress generated by $\mathcal{W}^{(4)} \times \mathcal{W}^{(4)}$.

The Root Cause: Vector vs. Scalar Mismatch

Why does the general expansion fail?

The fundamental root of this non-analyticity is that the loop equation is inherently a **vector** equation:

$$\hat{L}_\nu W = \int \frac{\delta}{\delta P_\nu}(W \times W) \tag{46}$$

But it governs a **scalar** functional $W[P]$.

- The number of independent tensor structures available for a scalar functional grows strictly slower ($\sim 4\times$ slower in $d = 4$) than those required for a free-index vector equation.
- At some critical order ($\mathcal{W}^{(8)}$), there will inevitably be more vector equations than available scalar parameters. The system becomes fatally overdetermined.

The Consequence for Naive String Theories

- The Loop Equation descends directly from the vector Yang–Mills equations of motion, dictating that a vector equation must be satisfied at every fixed point on the contour.
- Resolving this overdetermined system demands a deep mathematical “conspiracy” among the internal degrees of freedom within $W[P]$.
- **A Profound Corollary:** This exact vector-versus-scalar mismatch is fatal to all naive bosonic string theories of QCD!
- The Nambu-Goto string Laplace operator is a *scalar*. A scalar area-derivative constraint fundamentally cannot capture the full vector loop equation without generating an anomaly.

Physical Meaning: The Non-Analytic Vacuum

- The geometric contradiction at the 8th order rigorously proves that the exact solution $W[P]$ is **not an analytic functional** of the bounding loop $P(\theta)$.
- It cannot be indefinitely expanded in a continuous Taylor–Magnus series.
- **The Elfin Solution:** Our Elfin theory natively provides the required conspiracy! By placing Majorana fermions on a rigid Hodge-dual minimal surface, the internal degrees of freedom identically satisfy the vector equations across all components.
- **Ultra-local Integrals:** The integration over the momentum loops is not dominated by smooth, small P variations. It factorizes point-by-point. The true continuum solution is inherently non-perturbative.

Summary of Lecture II

1. **Momentum Space Cures the UV:** Transitioning to $W[P]$ integrates out cusp and contact singularities, yielding a finite algebraic MLE.
2. **Magnus Expansion is Finite but Limited:** The system is analytically solvable up to 6th order, proving the equations are singularity-free in the UV.
3. **The Vector/Scalar Mismatch:** The scalar nature of $W[P]$ cannot satisfy the vector nature of the Yang–Mills loop equations perturbatively at the 8th order.
4. **Naive Strings Fail:** Bosonic string Laplacians are scalars; they lack the geometric degrees of freedom to satisfy the vector MLE.

Next Lecture:

How the Elfin/Twistor theory provides the missing "conspiracy" via Hodge Duality and Majorana fermions on a rigid minimal surface.

Addendum: Questions & Answers

From the extended discussion during Lecture II

Q&A 1: Does the Trace Ansatz Restrict Generality?

Question raised during the seminar:

Is the ansatz $W[P] = \operatorname{tr} \hat{T} \exp\left(() \,\mathrm{i} \int \hat{X}_\mu dP_\mu\right)$ *the most general one? Doesn't a trace of finite matrices impose polynomial trace identities (like Cayley-Hamilton) that artificially restrict the Magnus expansion tensors* $\mathcal{W}^{(n)}$*?*

- **Answer:** If $\hat{X}_\mu$ were finite $N \times N$ matrices, yes, the finite trace identities would fatally restrict the expansion.
- However, in the $N_c \to \infty$ limit, the Master Field operates in an **infinite-dimensional** Hilbert space.
- As shown rigorously in *(Migdal, 1994, "Second Quantization of the Wilson Loop")*, the true non-perturbative Fock space of the string endpoint is the space of **"words"** generated by $d = 4$ Cuntz algebra operators: $a_\mu a_\nu^\dagger = \delta_{\mu\nu}$.
- In the infinite-dimensional free Cuntz algebra, **there are zero trace identities.**

Q&A 2: The Single Universal Operator

- **Follow-up:** *But to match independent $\mathcal{W}^{(n)}$ tensors at every order n, wouldn't you need an n-dependent operator or n-dependent Hilbert spaces?*
- **Answer:** No. A single, universal operator $\hat{X}_\mu$ is sufficient.
- The exact construction *(Migdal, 1994)* represents the position operator as an infinite cascading sum of Cuntz creation operators:

$$\hat{X}_\mu = a_\mu + \sum_{k=1}^{\infty} Q_{\mu,\mu_1\dots\mu_k} a^\dagger_{\mu_1} \dots a^\dagger_{\mu_k} \tag{47}$$

- The single operator $\hat{X}_\mu$ contains an infinite tower of independent parameters (the planar connected moments Q).
- Therefore, the trace of this path-ordered exponential is **completely unrestricted** and exhaustively spans the functional space of any closed loop.

Q&A 3: Can the MLE be Absorbed into a Closed Algebra?

Question:

The Right-Hand Side splits the trace. This is equivalent to a non-linear operator variation $\delta \hat{X}_\mu = \epsilon_\nu([\hat{X}_\mu, \hat{X}_\nu]|0\rangle\langle 0| + |0\rangle\langle 0|[\hat{X}_\mu, \hat{X}_\nu])$. *Could the entire MLE just be the invariance of the Cuntz algebra under some non-linear symmetry generated by* $[\hat{X}_\mu, \hat{\Gamma}_\lambda]$*?*

- **Answer:** This beautiful idea works for the **scalar** (projected) loop equations *(Migdal, 1996)*. The RHS loop-splitting is indeed a perfect non-linear automorphism of the 1D Cuntz algebra.
- **But it fails for the exact Vector MLE.** The exact Left-Hand Side is a macroscopic geometric stress driven by a 3rd-order Magnus form (a shuffle product):

$$\hat{L}_\nu W[P] = T^{\alpha\beta\gamma}{}_\nu \Omega^{(3)}_{\alpha\beta\gamma}[P] \times W[P] \tag{48}$$

- A purely internal operator variation acts as a degree-1 derivation. It fundamentally **cannot** reproduce a degree-3 geometric shuffle product for an arbitrary path $P(\theta)$.

Q&A 4: The Necessity of 4D Geometry

- **The Physical Meaning:** Planar QCD **cannot** be reduced to a purely 1-Dimensional Topological Field Theory of words and letters.
- While the Master Field theorem guarantees $W[P]$ can be formally represented by $\operatorname{tr} \hat{T} \exp(() i \int \hat{X} dP)$, the 1D internal space of the Cuntz operators fundamentally lacks the geometric degrees of freedom to represent the 2D spatial stress of the true Yang-Mills vector equation.
- The Vector MLE is strictly incompatible with a closed 1D algebra for $\hat{X}$. This is the exact, rigorous reason why the Taylor-Magnus expansion violently fails at $\mathcal{W}^{(8)}$.
- **The Way Out:** There must be specific, 4D continuous geometry involved to absorb this stress. As we will see in Lecture III, this geometry is the **Hodge-dual minimal surface**.

Lecture III: The Hodge-Dual Surface and the Physical Vacuum

Discussion map: choose a fork

In Lecture II, the 1D algebraic Master Field violently broke down at $\mathcal{W}^{(8)}$. This catastrophe mathematically mandates the introduction of 4D continuous geometry. Today we construct it:

Fork A: Loop Zero Modes

- Area derivative without divergences
- Exact factorization in MM equation
- The physical multi-instanton vacuum

Fork B: Topological Stability

- Self-intersecting loops
- Plateau vs. Goldschmidt
- The soap film phase transition

Fork C: Hodge-Dual Geometry

- The Holomorphic map to $\mathbb{C}^4$
- Virasoro constraint & uniformization
- Symmetrization by Parity

(Let's build the exact rigid 4D geometric background of the QCD string, step by step.)

The Big Picture: The Continuum Solution

- **The Goal:** A continuum framework for confining planar QCD ($N_c \to \infty$).
- **The Method:** Quantizing the Fermi-string (1981) degrees of freedom on the **rigid Hodge-dual minimal surface**.
- **Why a Rigid Surface?** The fermions are *not* placed on a random fluctuating surface. The bulk geometry is rigid and fixed holographically by the boundary loop.
- **The Payoff:** This completely avoids the Liouville instability of early random-surface models! There is no summation over fluctuating metrics.

The Singular Solution of the MLE and the QCD Vacuum

- In Lecture II, we found that the Momentum Loop Equation (MLE) does not allow a regular analytic solution expandable as a Taylor–Magnus series in P'.
- The Taylor–Magnus expansion emerged from iterations of the MLE starting with the **trivial zero mode**:

$$\mathcal{L}_\nu(W)[P] = \mathcal{O}(P^3) \quad \Longrightarrow \quad W[P] = \textit{ const } + \mathcal{O}(P^2) \tag{49}$$

- The mathematical alternative is a **singular solution** governed by the nontrivial zero-mode:

$$\mathcal{L}_\nu(W)[P] = 0 \quad \Longleftrightarrow \quad \mathcal{L}_\nu(W)[C] = 0 \tag{50}$$

- Physically, a nontrivial zero mode corresponds to a non-perturbative QCD vacuum—i.e., a background gauge field (Master Field) solving the classical Yang–Mills equations:

$$[D_\mu, F_{\mu\nu}] = 0 \tag{51}$$

- This vacuum state is notoriously difficult to build in the pure gauge-field representation, and equally hard in the momentum loop equation (MLE). However, as we shall see today, there is **an exact analytic solution** in coordinate loop space!

The Area Derivative and Loop Equation

- Historically, the area derivative was plagued by kinematic divergences. We have found a regular definition based on the discontinuity of the second variation of the area:

$$\frac{\delta W}{\delta \sigma_{\mu\nu}(\theta)} = \frac{\delta^2 W}{\delta C'_\mu(\theta - 0)\delta C'_\nu(\theta + 0)} - (\mu \leftrightarrow \nu) \tag{52}$$

- The planar loop equation for Yang–Mills gradient flow reduces to the loop space diffusion equation:

$$\mathcal{L}_\nu(W) = \left(\frac{\delta}{\delta C'_\mu(\theta + 0)} - \frac{\delta}{\delta C'_\mu(\theta - 0)} \right) \frac{\delta W}{\delta \sigma_{\mu\nu}(\theta)} = 0 \tag{53}$$

- **The Discovery:** This operator admits an exact solution (a zero mode) $W = \exp\left(-\kappa S[C]\right)$ provided $S[C]$ satisfies $\mathcal{L}_\nu(S) = 0$.

Self-Duality in Loop Space

- **The Golden Key:** The loop space diffusion equation is identically satisfied for any functional $S[C]$ with a **Self-Dual (SD) or Anti-Self-Dual (ASD) area derivative**:

$$\star \frac{\delta S_{\pm}}{\delta \sigma_{\mu\nu}(\theta)} = \pm \frac{\delta S_{\pm}}{\delta \sigma_{\mu\nu}(\theta)} \tag{54}$$

- Why? Due to the kinematic **Bianchi identity** (a Jacobi identity for a triple commutator), the divergence of an (anti)self-dual area derivative identically vanishes:

$$\epsilon_{\alpha\mu\nu\lambda} \left(\frac{\delta}{\delta C'_{\alpha}(\theta+0)} - \frac{\delta}{\delta C'_{\alpha}(\theta-0)} \right) \frac{\delta S_{\pm}}{\delta \sigma_{\mu\nu}(\theta)} \equiv 0 \tag{55}$$

- This is the exact same mathematical reason why the self-dual gauge field solves the Yang–Mills equations of motion: the Jacobi identity for the double commutator

$$\epsilon_{\alpha\beta\gamma\delta} D_{\alpha} D_{\beta} D_{\delta} \equiv 0 \tag{56}$$

Compatibility with the Makeenko–Migdal Equation

- Due to the **Leibniz property** of the loop differential operator, we can dress the fluctuative part of the Wilson loop with this zero mode:

$$W[C] = \exp\left(-\kappa S_{\pm}[C]\right) W_{\mathsf{fluct}}[C] \tag{57}$$

- In general, the zero mode factor corresponds to a superposition (with unknown weights w_i) of the (anti)self-dual gauge fields $A^{(i)}$ with fixed Hodge chirality χ

$$\exp\left(-\kappa S_{\chi}[C]\right) = \sum_{\star F^{(i)} = \chi F^{(i)}} w_i \operatorname{tr} \hat{T} \exp\left(\oint_C A^{(i)}_{\mu}(x) \mathrm{d}x_{\mu}\right) \tag{58}$$

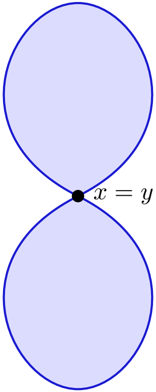
- The full Makeenko–Migdal equation has a singular integral term at self-intersections $x = y$:

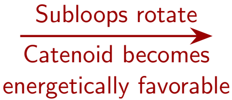
$$\mathcal{L}_{\nu}(W[C]) = \lambda \int_C dy_{\nu} \delta^4(x - y) W[C_{xy}] W[C_{yx}] \tag{59}$$

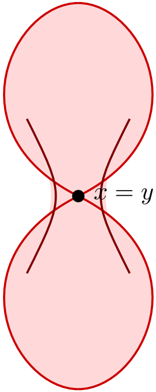
Topological Stability: The Area Functional at Intersections

- **Crucial Additivity Requirement:** This dressing **only** works if the surface area is **strictly additive** at self-intersections ($S[C] = S[C_{xy}] + S[C_{yx}]$). Only then does the confining factor exactly factor out of the RHS integral and cancel!
- Consider a self-intersecting loop $C = C_{xy} \circ C_{yx}$ (e.g., a figure-8).
- There are two competing topological classes of minimal surfaces bounding this loop:
  1. **Two Disks (The Goldschmidt Solution):** The surface pinches off completely at the intersection. The area is strictly additive: $S[C] = S[C_{xy}] + S[C_{yx}]$.
  2. **A Connected Surface (Plateau Minimum):** A catenoid-like cylindrical surface bridging the gap between the two sub-loops.
- **The Phase Transition:** As the sub-loops rotate around the intersection point to get closer, the connected cylindrical shape may become energetically favorable (the absolute global minimum).
- Does the QCD vacuum undergo this phase transition?

Visualizing the Topological Phase Transition



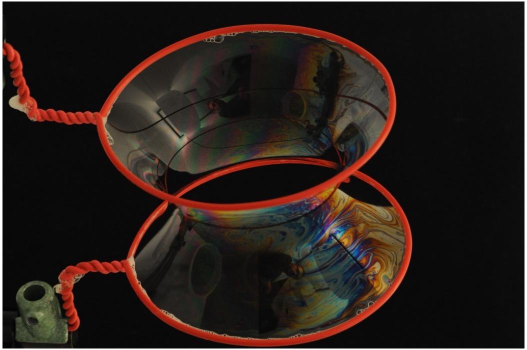




Goldschmidt Solution
Two Disks (Local Min)
$S_{add} = S[C_{xy}] + S[C_{yx}]$

Plateau Solution
Connected Cylinder (Global Min)
$S_{conn} < S[C_{xy}] + S[C_{yx}]$

Classical Physics vs. Planar QCD: The Catenoid



A classical soap film finds the absolute global minimum (the Plateau solution).

- **Classical Physics (Soap Films):** Nature minimizes physical energy. When two loops are close, the film dynamically transitions into this connected **catenoid**.
- **The QCD String:** This beautiful classical solution is **strictly forbidden** as the vacuum geometry of Planar QCD!
- **Why?** A connected cylinder destroys exact planar factorization. The RHS of the MM loop equation requires the amplitude to factorize into two disjoint pieces: $W[C] \to W[C_{xy}]W[C_{yx}]$.
- Therefore, the QCD vacuum geometry must remain trapped in the additive **Goldschmidt branch** (two separate disks), even when the loops are arbitrarily close.

Mathematical Stability of the Goldschmidt Branch

The Loop Equation dictates the Topology

We are not solving the absolute Plateau problem; we are solving the QCD equations of motion. We **must** select the additive **Goldschmidt branch** (disjoint disks) over any complex topology.

- **But is it mathematically stable?** Yes.
  - **Douglas & Radó:** A disconnected solution always exists for separated/touching boundaries.
  - **Gulliver's Barrier:** A topological barrier ensures the additive Goldschmidt branch remains a stable *local minimum* in functional space.
  - **White's Bridge Principle:** The limit of an infinitesimal neck smoothly converges to the touching disks in the varifold sense, maintaining a continuous, divergence-free loop calculus and avoiding "catenoid collapse".

The Physical Vacuum: Multi-Instanton Resummation

What is the physical origin of this rigid background?

The MM equations are exact equations of motion. They possess an inherent ambiguity corresponding to the choice of the **physical vacuum**. Multiplying by a zero mode factor is equivalent to specifying this vacuum.

- In standard gauge theory, the non-perturbative QCD vacuum is a complex mixed state of multi-instantons.
- A single instanton explicitly breaks macroscopic translation invariance. Restoring Poincaré symmetry requires a highly complex integration over the entire multi-instanton moduli space.
- **Loop Space Magic:** Just as instantons have $F_{\mu\nu} = \pm \star F_{\mu\nu}$, our geometric zero mode has $\frac{\delta S}{\delta \sigma_{\mu\nu}} = \pm \star \frac{\delta S}{\delta \sigma_{\mu\nu}}$.
- Because our minimal area functional $S_\chi[C]$ depends only on the relative geometry of the loop and is invariant under rigid translations, it **is the exact, translation-invariant holographic representation** of the implicitly resummed multi-instanton vacuum!

Constructing the Hodge-Dual Minimal Surface

- **We define a surface** mapping the unit disk $\mathcal{D}$ to an extended space $\mathbb{R}^3 \otimes \mathbb{R}^4$ via coordinates $X^i_\mu(\xi), i = 1, 2, 3$.
- **The area is defined** as $S_\chi[C] = \min_X \int_{\mathcal{D}} d^2\xi \sqrt{\Sigma^2_{\mu\nu}/2},\ \Sigma_{\mu\nu} = e_{ab}\partial_a X^i_\mu \partial_b X^i_\nu$.
- **Hodge duality constraint**: $\star\Sigma_{\mu\nu} = \chi\Sigma_{\mu\nu}$
- The Hodge chirality $\chi = \pm 1$ enters exactly through 't Hooft matrices $\eta^{\chi,i}_{\mu\nu}$ in the **boundary conditions**:

$$X^i_\mu(\partial\mathcal{D}) = \eta^{\chi,i}_{\mu\nu} C_\nu, \qquad \eta^{\chi,i}_{\mu\nu} = \delta_{i\mu}\delta_{\nu 4} - \delta_{i\nu}\delta_{\mu 4} + \chi\epsilon_{i\mu\nu 4} \tag{60}$$

- **The Holomorphic Ansatz:** We resolve the duality constraint exactly by mapping to $\mathbb{C}^4$:

$$X^i_\mu = \eta^{\chi,i}_{\mu\nu}(f_\nu(z) + \bar{f}_\nu(\bar{z})), \quad z = \xi_1 + \mathrm{i}\xi_2 \tag{61}$$

Boundary Conditions and Area Duality

- The boundary conditions for the Hodge-dual surface reduce to a conventional Dirichlet problem in $\mathbb{C}^4$:

$$2\,\mathrm{Re}\, f_\mu(e^{\mathrm{i}\theta}) = C_\mu(\theta) \tag{62}$$

- Our surface area reduces to the induced metric in this space:

$$\Sigma_{\mu\nu} = 2(F_{\mu\nu} + \chi \star F_{\mu\nu}), \qquad F_{\mu\nu} = \mathrm{i}(f'_\mu \bar{f}'_\nu - \bar{f}'_\mu f'_\nu) \tag{63}$$

- The area element is automatically Hodge-dual for *any* arbitrary holomorphic function $f_\mu(z)$:

$$\star\Sigma_{\mu\nu} = \chi \Sigma_{\mu\nu} \tag{64}$$

- This guarantees the self-duality of the area derivative, making the Hodge-dual minimal area an exact zero mode of the loop diffusion operator.

The Virasoro Constraint & Uniformization

- The geometric area functional $S = \int \sqrt{-\det g}\, d^2 z$ possesses full reparametrization invariance $z \to w(z, \bar{z})$.
- By the **Uniformization Theorem**, we can always choose isothermal (conformal) coordinates where the induced metric is diagonal: $g_{zz} = 0$.
- For our specific holomorphic ansatz, this is precisely the **Virasoro null constraint**:

$$(f'_\mu(z))^2 = 0 \tag{65}$$

- **The Magic of the Conformal Gauge:** The highly non-linear geometric Nambu-Goto area density strictly collapses to an exact **quadratic Lagrangian**:

$$\mathcal{L} = \sqrt{(g_{z\bar{z}})^2 - |g_{zz}|^2} \xrightarrow{(f')^2=0} g_{z\bar{z}} \propto |f'_\mu|^2 \tag{66}$$

Exact Zero Modes and the Hilbert Transform

- The linear boundary problem $2\,\mathrm{Re}\, f_\mu = C_\mu$ is explicitly solved by the **Hilbert Transform** ($\mathcal{H}$):

$$f_\mu = \frac{1}{2}(1 + \mathrm{i}\mathcal{H})C_\mu; \qquad \mathcal{H}[u](\theta) = \frac{1}{2\pi} P.V. \int_0^{2\pi} d\theta' \cot\left(\frac{\theta - \theta'}{2}\right) u(\theta') \tag{67}$$

- In the holomorphic gauge, the minimal value of BOTH S_+ and S_- reduces identically to the standard, chirality-independent Dirichlet functional (the Goldschmidt area): $S_+[C] = S_-[C] \propto \int_D |f'(z)|^2 d^2 z$.
- Because their area derivatives are strictly self-dual and anti-self-dual, the kinematic **Bianchi identity** applies to each term individually.

$$\mathcal{L}_\nu(S_+) = 0 \qquad \text{and} \qquad \mathcal{L}_\nu(S_-) = 0 \tag{68}$$

- Thus, each chiral surface S_+ and S_- solves the exact Makeenko-Migdal equation **completely on its own**!

Spinor Factorization and Gauss Maps

- The twistor factorization of the null tangent vector $f'_{a\dot{b}} = \lambda_a \mu_{\dot{b}}$ leads to an exact factorization of the induced metric on the worldsheet:

$$e(z, \bar{z}) = 2\sqrt{2} f'_\mu \bar{f}'_\mu = \sqrt{2}(\bar{\lambda}\lambda)(\bar{\mu}\mu)$$

- In this spinor representation, the area derivative simplifies beautifully:

$$\frac{\delta S_\pm}{\delta \sigma_{\alpha\beta}} = -n_i^\pm \eta^{i\pm}_{\alpha\beta}$$

- Here, the maps $n_i^\pm$ are identified with the **Left and Right Gauss maps** of the minimal surface:

$$n_i^+ = \frac{\bar{\lambda}\sigma_i\lambda}{\bar{\lambda}\lambda}, \quad n_i^- = \frac{\bar{\mu}\sigma_i\mu}{\bar{\mu}\mu}$$

- For the Hodge-dual minimal surface S_χ, the area element $\Sigma_{\mu\nu}$ transforms under $SO(4)$ strictly as a single Gauss map, corresponding to its chirality.

The True Difference: Active vs. Frozen Twistors

- **The crucial point:** The Hodge-dual solutions **ARE** different from the ordinary minimal area! The difference manifests directly in the holomorphic twistor parametrization $f'_\alpha(z) = \lambda(z)\sigma_\alpha\mu(z)$.

- **Self-Dual (SD) Solution:** The left $SU(2)$ is active. $\lambda(z)$ is given by a set of poles, while $\mu(z)$ is **constant**, leading to an active left Gauss map n_i^+.
- **Anti-Self-Dual (ASD) Solution:** The right $SU(2)$ is active. $\mu(z)$ has poles, while $\lambda(z)$ is **constant**, leading to an active right Gauss map n_i^-.

- The matrix $\lambda \otimes \mu$ transforms exclusively by either left or right $SU(2)$ multiplications. This permanent freezing of one chiral sector makes these solutions fundamentally distinct from the usual, generic minimal surface.
- The symmetric combination (required by parity) will later be used as the classical geometric background determining the mass spectrum of QCD, leading to observable predictions supporting the Hodge-dual surface!

Parity Restoration & Why Not Nambu-Goto?

- **Restoring Parity:** If S_+ and S_- independently solve the loop equations, why symmetrize?
- A single Hodge-dual surface explicitly breaks macroscopic Parity ($\chi = \pm 1$). To preserve the exact parity invariance of the physical QCD vacuum, we take the symmetric combination:

$$S[C] = \frac{1}{2}(S_+[C] + S_-[C]) \tag{69}$$

- **Why not just use Nambu-Goto?** The conventional Nambu-Goto area lacks this chiral decomposition, so the Bianchi identity does not apply.
- The NG area produces $\frac{|\delta x|}{(\delta y)^2 + (\delta x)^2}$ contact singularities in the loop diffusion operator $\mathcal{L}$. The limit depends on the relation between the normal δy and the tangent to the curve δx distances from the point on a boundary of the disk, where the $\mathcal{L}$ operator is applied.
- Our chiral solution uniquely bypasses this singularity and resolves this ambiguity, similar to the invariant point splitting resolving anomalies in QFT.

String Scale and the Operator Product Expansion

- The string tension parameter κ in $W = W_{pert} \exp\left(-\kappa S[C]\right)$ is not fixed by the MM equation; it is a non-perturbative property of the QCD vacuum.
- To find its physical origin, we match the geometric vacuum to the Operator Product Expansion (OPE) at short distances.
- Evaluate the 2nd area derivative for a small circular loop at opposite points ($|x_{12}| = 2r$). The perturbative OPE gives:

$$\frac{\delta^2 W_{pert}}{\delta\sigma_{\mu\nu}(1)\delta\sigma_{\lambda\rho}(2)} \propto \langle \operatorname{tr} F_{\mu\nu}(x_1)U_{1,2}F_{\lambda\rho}(x_2)U_{2,1}\rangle \tag{70}$$

- The leading short-distance conformal term scales as $|x_{12}|^{-4}$:

$$\text{OPE} \to \frac{\alpha_{eff}}{4\pi}\frac{I_{\mu\lambda}I_{\nu\rho} - I_{\nu\lambda}I_{\mu\rho}}{|x_{12}|^4}W_{pert}[0], \qquad I_{\mu\lambda} = \delta_{\mu\lambda} - 2\hat{x}_\mu\hat{x}_\lambda \tag{71}$$

- We isolate the non-perturbative dimension-4 contribution by contracting with the antisymmetric projector $\Pi_{\mu\nu\lambda\rho} = \frac{1}{2}(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\nu\lambda}\delta_{\mu\rho})$.

The OPE Cancellation at $D = 4$ and the Gluon Condensate

- **A Dimensional Miracle:** Contracting the projector with the leading conformal tensor structure in d dimensions yields exactly:

$$S(d) = \Pi_{\mu\nu\lambda\rho}(I_{\mu\lambda}I_{\nu\rho} - I_{\nu\lambda}I_{\mu\rho}) = (\mathrm{tr}\, I)^2 - \mathrm{tr}(I^2) = (d-1)(d-4) \tag{72}$$

- **In exactly $D = 4$, this is strictly zero!** The leading UV conformal divergence drops out completely under this projection.
- The surviving dimension-4 operator is the **Gluon Condensate**:

$$\Pi_{\mu\nu\lambda\rho} \langle \mathrm{tr}\, F(1)UF(2)U \rangle \to -\frac{g^2}{2N_c} \left\langle (G^a_{\mu\nu})^2 \right\rangle W_{pert}[0] \tag{73}$$

- Matching this to the second area derivative of our geometric zero mode ($W = W_{pert} \exp\left(-\kappa S[C]\right)$) on a small circle yields $16\kappa^2 W_{pert}[0]$.
- **The Result:** The macroscopic string tension κ is generated directly by the microscopic vacuum condensate!

$$\kappa^2 = \frac{\pi^2}{8N_c} \left\langle \frac{\alpha_s}{\pi} (G^a_{\mu\nu})^2 \right\rangle \tag{74}$$

The Physical Vacuum and the MLE Singularity

- **A Non-local Zero Mode in the MLE:** What does this geometric dressing factor imply for the momentum loop equation, and how is it related to the non-analyticity?
- Because this factor is a local multiplication in coordinate loop space, it becomes a **convolution** in momentum loop space:

$$W[P] = \int \mathcal{D}Q\, \mathcal{G}[P-Q] W_{\text{fluct}}[Q] \tag{75}$$

$$\mathcal{G}[Q] = \int \mathcal{D}C \, \exp\left(-\kappa S[C] + \mathrm{i} \oint d\theta\, C'_\mu(\theta) Q_\mu(\theta) \right), \quad \mathcal{L}_\nu(\mathcal{G})[Q] = 0 \tag{76}$$

- This representation is parametrically invariant, but generally **not analytic**. One cannot expand $\exp\left(\mathrm{i}\oint d\theta\, C'_\mu(\theta)(P_\mu(\theta) - Q_\mu(\theta))\right)$ in a functional Taylor series in $P(\theta)$, because the path-integral moments of $C'(\theta)$ **diverge for Brownian paths!**
- This convolution provides an explicit, non-local, and singular resolution to the $\mathcal{W}^{(8)}$ algebraic contradiction we discovered in Lecture II.
- The existence of an explicit analytic solution for the Hodge-dual minimal surface is **another mathematical miracle of 4D geometry, akin to instantons and twistors.**

Helicoid of the rigid string

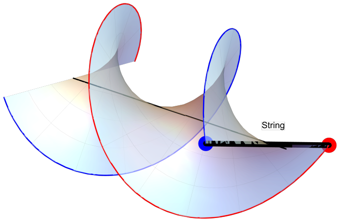


Figure: The helicoid spanned by rotating $\bar{q}q$ pair connected by a rigid stick (string). This minimal surface bounded by a double helix, was discovered by Meusnier in 1785.
In Minkowski space:

$$X(\tau, r) = (r\cos\omega\tau,\, r\sin\omega\tau,\, 0,\, \tau) \tag{77}$$

$$g_{ab} = \mathrm{diag}(1 - \omega^2 r^2,\, 1) \tag{78}$$

The holographic string and Regge trajectories

- This holographic area law offers the same phenomenological confinement mechanism via the linear potential $V = \sigma R$, with $\sigma = 2\sqrt{2}\kappa$, as the old string model.
- This Area in Minkowski space provides an effective action for the quark loop in the QCD vacuum:

$$S = -2\sigma \int_0^R dr \sqrt{1 - \omega^2 r^2} - 2m_q \sqrt{1 - \omega^2 R^2} \tag{79}$$

- The classical Lagrangian (with velocity $v = \omega R$) and angular momentum $J = \partial L / \partial \omega$ evaluate to:

$$L(v, \omega) = -\frac{\sigma}{\omega} \left(v\sqrt{1 - v^2} + \arcsin v \right) - 2m_q \sqrt{1 - v^2} \tag{80}$$

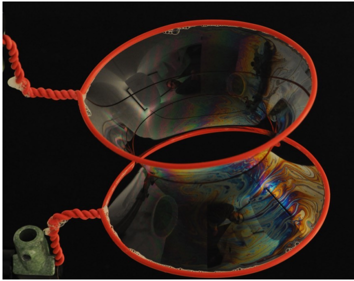
- Minimizing and resolving for R, v, we find that at large J and zero quark mass $m_q = 0$, the velocity $v \to 1$, and the **Regge trajectory is strictly linear**:

$$E^2 \to 2\pi\sigma J \tag{81}$$

- **The Crucial Distinction: There are no vibrational modes, this is a rigid stick rotation.**

Philosophical Discussion: Quantum Physics vs. Classical Models

- **The Classical Illusion:** Look at this catenoid. A classical soap film will always transition to a connected topology because it dynamically minimizes its physical surface energy.
- **The Quantum Reality:** But we must remember the fundamental physics—there is no literal "string" and no "surface energy" in QCD! Those were phenomenological models built decades ago to mimic confinement.
- **No Bulk Fluctuations:** Our physical requirement is minimizing the vacuum energy of the quantum gauge field. In our exact holographic solution, there are **no fluctuating surface degrees of freedom** in the bulk. There is no physical membrane exploring a configuration space to find a global minimum with cylindrical topology.



A classical mirage.

The Core Principle

Summary of Lecture III

1. **The Zero Mode:** The multi-instanton QCD vacuum is geometrically represented by a rigid surface with a (anti)self-dual area derivative.
2. **Goldschmidt Additivity:** To satisfy the non-linear factorization of the MM hierarchy, the functional must remain strictly additive at self-intersections (no topological phase transitions).
3. **Hodge-Dual Geometry:** We constructed this exactly by embedding a holomorphic map in $\mathbb{R}^3 \otimes \mathbb{R}^4$ via 't Hooft matrices.
4. **Active vs. Frozen Twistors:** The Hodge-dual solutions are fundamentally distinct from generic minimal surfaces due to the permanent freezing of one chiral twistor sector.
5. **The Gluon Condensate:** The $D = 4$ conformal OPE singularity exactly vanishes, tying the geometric string tension κ directly to the physical SVZ gluon condensate.

Next Lecture:

How to place Majorana fermions on this rigid surface to cancel non-planar intersections via the Pauli Principle and recover asymptotically free QCD.

Lecture IV: The Majorana fermions inducing planar QCD graphs

Discussion map: choose a fork

In Lecture III, we established the exact 4D rigid geometric background (the Hodge-dual minimal surface). Today, we place matter on this surface to dynamically induce Planar QCD:

Fork A: The Elfin Theory

- Majorana fermions on a surface
- Chiral bag boundary conditions
- Conformal gauge invariance

Fork B: Topological Stability

- Vdovichenko's Ising analogy
- Cancellation of intersections
- Planar graphs from trapped loops

Fork C: Recovering the MM Eq

- First and second area derivatives
- The Geodesic Inequality
- Reproducing Asymptotically Free QCD

(Let's build the fermion determinant step by step.)

The Elfin Theory on a Flat Surface

- We begin with the Elfin theory by itself. Consider a flat surface bounded by a contour C.
- We introduce the Elfin field as a Majorana bispinor: $\psi = \psi^{\alpha}_{\lambda}(\xi)$ where $\alpha = 1, 2$ and $\lambda = \pm 1$.
- The simplest form of the action depends on the conformal factor $e = \partial_+ X^i_\mu \partial_- X^i_\mu$:

$$A_0 = \int \mathrm{d}^2\xi \left(\bar{\psi}_\lambda \sigma^k \partial_k \psi_\lambda + \bar{\psi}_\lambda \psi_\lambda m \sqrt{e} \right) \tag{82}$$

- **Crucial Boundary Conditions:** At the boundary $\partial\Sigma$, we impose "chiral bag" conditions:

$$\sigma_3 \psi_\lambda = \lambda \psi_\lambda \tag{83}$$

This implies the left-hand states are occupied at the boundary, ensuring repulsion from the wall due to the Pauli principle!

Conformal Metric as a Local Gauge Parameter

- We want to move from a flat surface to our rigid minimal surface.
- Let the surface metric be $g_{ab} = \exp(2\rho)\,\delta_{ab}$. The general covariant action A_ρ includes the spinor connection and the Liouville kinetic term $-\frac{\partial_+\rho\partial_-\rho}{3\pi}$.
- **Theorem (Conformal Invariance):** The regularized functional integral is strictly invariant under local variations of ρ:

$$Z[S|e,\rho] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(A_\rho) \tag{84}$$

- **The Physics:** The variation of the Pauli-Villars regulator exactly cancels the variation of the Liouville term!
- Therefore, fermions on a curved minimal surface are mathematically equivalent to fermions on a flat surface with a variable mass $m\sqrt{e(\xi)}$.

The fermions on a curved surface

- The action of the general theory is given by

$$A_\rho = \int d^2\xi \left(e^{2\rho} \bar{\psi}_\lambda \sigma^k e^{-\rho} \nabla_k \psi_\lambda + m\sqrt{e} e^\rho \bar{\psi}_\lambda \psi_\lambda - \frac{\partial_+ \rho \partial_- \rho}{3\pi} \right); \quad (85)$$

- with the spinor connection ω_k:

$$\nabla_k = \partial_k + \frac{\mathrm{i}}{2} \omega_k \sigma^3, \quad \omega_k = \frac{1}{2} e_{kl} \partial_l \rho. \quad (86)$$

- vierbein and curvature

$$E_k^a = \delta_k^a e^\rho, \quad \hat{R}_{kl} = [\nabla_k, \nabla_l] = \mathrm{i}\sigma_3 e_{kl} R \det E; \quad R = -8e^{-2\rho} \partial_+ \partial_- \rho. \quad (87)$$

- At $\rho = 0$ this is the old action A_0, whereas at $\rho = \ln \sqrt{e}$ this is the action of the Dirac particle with constant mass at the surface S with the induced metric g_{ab}.

General Covariance and the 2D Spinor

- How is a 2D spinor on a curved surface described by this Dirac operator?
- In general relativity, coupling fermions to a curved background requires a local reference frame (a zweibein in 2D) E^a_k and an independent spin connection vector ω_k.
- However, any 2D metric can always be locally brought to a conformally flat form: $g_{kl} = e^{2\rho}\delta_{kl}$. Thus, the zweibein is simply diagonal: $E^a_k = e^{\rho}\delta^a_k$.
- **Why is a single scalar field ρ sufficient?** Because the zero-torsion condition of general relativity rigorously locks the spin connection to the derivatives of the zweibein.
- For our diagonal zweibein, solving the torsion-free equation explicitly collapses the connection vector into the orthogonal gradient of the scalar:

$$\omega_k = \frac{1}{2} e_{kl}\partial_l \rho \tag{88}$$

- Thus, general covariance is perfectly maintained by the covariant derivative $\nabla_k = \partial_k + \frac{\mathrm{i}}{2}\omega_k\sigma^3$, and all intrinsic geometry is captured by the single scalar field ρ.

Conformal Anomaly and the Gauge Variable ρ

- Why is ρ simply a dummy gauge variable? We require the physical string amplitudes to be independent of the internal conformal coordinate choice.
- Classically, massless fermions are conformally invariant. But quantum mechanically, the Dirac operator has anomalies.
- Under a local conformal variation, the Dirac operator transforms by left and right multiplication: $D \to e^{-\frac{3}{2}\delta\rho} D e^{-\frac{1}{2}\delta\rho}$
- The **naive Jacobian** from this left and right multiplication of the Dirac determinant would be: $\exp\left(-\int \mathrm{d}^2\xi\, e^{2\rho}\left(\frac{3}{2}+\frac{1}{2}\right)\delta\rho\right)$
- The operators must be disentangled. Regularizing these anomalies produces a curvature integral. Instead of the naive linear term, we find

$$\delta\,\mathrm{tr}(\ln D^2)_{reg} = -2\int \frac{\mathrm{d}^2\xi}{4\pi} e^{2\rho} 2\delta\rho[\,\textit{const}\; - R/12]$$

- This extra term exactly compensates the variation of the bare Liouville term in our action, rendering the regularized functional integral strictly invariant!
- This invariance will let us use the gauge $\rho = 0$ for proof of the loop equations, and then switch to $\rho = \log\sqrt{e}$ to go to the local limit and derive twistor string.

Fermion Determinants & Planar Topology

- We represent the Dirac determinant $\Delta = \Delta_+ \Delta_-$ as a path integral: a sum over closed vacuum loops on the surface.
- Each loop Γ carries a statistical weight:

$$A[\Gamma] = (-1)^{\nu} \exp\left(-m l_{\Gamma}\right) \tag{89}$$

where ν is the number of self-intersections.

- The extra negative sign $(-1)^{\nu}$ comes purely from Fermi statistics, linked to the rotation angle via the Gauss-Bonnet theorem:

$$\sum \Delta\theta = 2\pi(1 - \nu) \pmod{2\pi} \tag{90}$$

- This theorem only counts local rotations of the tangent vector of the curve, so it is valid in presence of holes on the disk.

Cancellation of Intersections: The Pauli Principle

- How do we get planar graphs from free intersecting loops? We follow the intuition of Vdovichenko's exact solution of the 2D Ising model.
- We rearrange the sum over loops into a sum over vertices.
- **The Miracle:** The non-planar vertex exactly cancels the first planar vertex!
- Terms with intersecting lines are algebraically annihilated by terms with parallel touching lines of opposite orientation.
- This is the Pauli Principle reformulated in path integrals: $(\bar{\psi}^R \psi^R)^2 = 0$.
- In case of the Ising model, these curves were phase boundaries between domains of spin $\pm 1/2$ (non-oriented curves). In our case, these are oriented curves tracking double line of gluon indices in planar graphs of QCD.

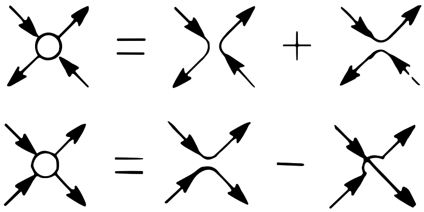


Figure: Types of line collision: two planar (upper) and one planar, another non-planar (lower).

Emergence of the Planar String

- By discarding all non-planar configurations, the functional integral reduces to a hierarchy of trapped loops moving in the free space left by others.
- A time slice of this picture (red line) describes a string with Elf pairs $(\psi, \bar{\psi})$ moving between neighbors.
- This exactly matches the topological structure of 't Hooft's planar diagram expansion, but rigorously preserves gauge invariance!
- This time slice also answers the question of the mediators of the forces between quark and antiquark: in perturbative QCD these were gluons, but here, on a minimal surface, these forces are mediated by fluctuations of elves, summing these planar QCD gluon graphs into fermion determinants.
- The projection of the gluon graphs from 4D space to the minimal surface makes the planar QCD solvable by powerful methods of two-dimensional field theory and conformal anomalies.

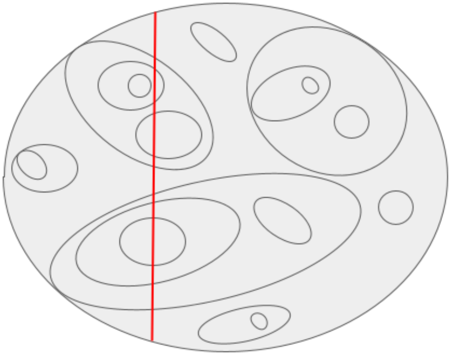


Figure: The hierarchical set of loops, some touching, but never intersecting each other nor self-intersecting. The time slice (red line) represents some number of Elf pairs.

Loop Equations on the Minimal Surface

- We now place this fermion determinant $Z[C]$ on the Rigid Hodge-Dual Surface $S_{min}[C]$.
- Use the $\rho = 0$ gauge and exact variation of Hodge-dual surface conformal metric

$$\frac{\delta e(x)}{\delta \sigma_{\mu\nu}(y)} = 2T_{\mu\nu}\delta^2(x-y) \tag{91}$$

- Now we can compute the first area derivative:

$$\delta Z[S]/\delta\sigma_{\mu\nu}(x_1) = -m\int \mathcal{D}\Gamma_{11} T_{\mu\nu}(1) Z[S_{in}]Z[S_{out}]e^{-ml_{11}}. \tag{92}$$

- This delta function in variation of conformal metric selects the Elf loop that touches the boundary, cutting

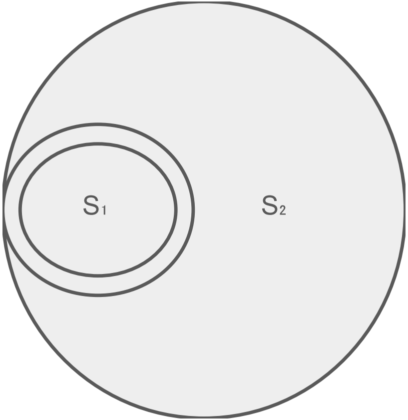


Figure: The first area derivative. An internal path Γ cuts the surface into S_{in} and S_{out}.

Loop Equations on the Minimal Surface

Taking the *second* area derivative produces two topological variations: The A-term (disconnected) and the B-term (connected).

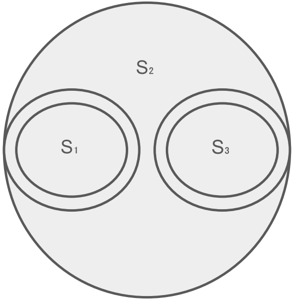


Figure: The A-term. Two closed loops Γ_l, Γ_r cut the surface into three pieces.

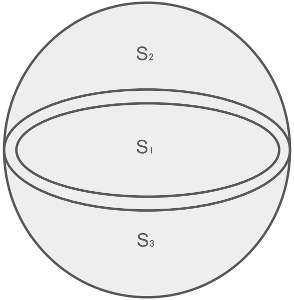


Figure: The B-term. One closed loop touching C at left and right points.

Large Fermion Mass Limit

- To compare the fermionic determinant with the MM loop equation we take the limit

$$m \gg \Lambda_{QCD} \tag{93}$$

- The Elf mass plays the role of a UV cutoff. Short fermionic loops dominate the path integral.
- The second area derivative of the determinant contains two contributions:

$$\frac{\delta^2 Z}{\delta\sigma_{\mu\nu}(l)\delta\sigma_{\alpha\beta}(r)} = A_{\mu\nu\alpha\beta} + B_{\mu\nu\alpha\beta} \tag{94}$$

- Physically:
  - A–term: two independent infinitesimal loops near l, r
  - B–term: one loop touching the boundary at both points
- In the heavy–mass limit the geometry becomes locally flat, so the path integrals reduce to universal constants.

The A–term: Vanishing by the Bianchi Identity

- In the large-m limit the loops Γ_l, Γ_r shrink to infinitesimal circles.
- The internal determinant factorizes $Z[S_{mid}] \to Z[C]$
- The path integrals over the small loops reduce to constants:

$$A_{\mu\nu\alpha\beta} \to T_{\mu\nu}(l) T_{\alpha\beta}(r)\, Z[C]\, a^2$$

- The tangent tensor is proportional to the area derivative of the minimal surface: $\frac{\delta S[C]}{\delta \sigma_{\mu\nu}} = -2T_{\mu\nu}$
- Therefore the A–term can be written as

$$A_{\mu\nu\alpha\beta} \propto \frac{\delta^2 S}{\delta\sigma_{\mu\nu}(l)\delta\sigma_{\alpha\beta}(r)} \exp\left(-\kappa S[C]\right) \tag{95}$$

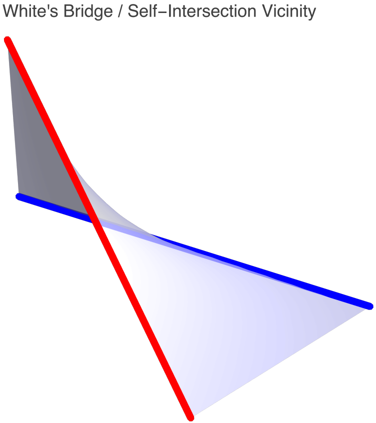
- Integrating over the loop operator gives (by translational invariance, with $dx_\beta \frac{\delta}{\delta\sigma_{\mu\beta}(x)} = \frac{\delta}{\delta x_\mu}$)

$$\int_{C_{ll}} dx_\beta \frac{\delta}{\delta\sigma_{\mu\beta}(x)} \frac{\delta S}{\delta\sigma_{\mu\nu}(l)} = -\partial_\mu(l) \frac{\delta S}{\delta\sigma_{\mu\nu}(l)} \tag{96}$$

- This is exactly zero by the Bianchi identity for the Hodge–dual surface.
- **Conclusion: the A–term cancels identically.**

White's Bridge and the Contact Term

- As the loop self-intersects, the additive Goldschmidt surface breaks into two pieces connected by an infinitesimal narrow bridge (White's bridge).



- White's bridge perfectly aligns the tangent tensors $T(l)$ and $T(r)$. The matrix product yields $-\frac{1}{4}\delta_{\alpha\nu}$.

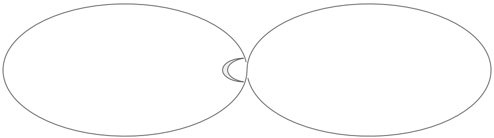


Figure: The two arcs bounding the crescent-shaped area S_{mid} collapsing.

The B–term: Generating the MM Contact Term

- The B–term corresponds to one loop touching the boundary at two points l, r.
- The tensor structure contains
$$T_{\alpha\mu}(r)T_{\mu\nu}(l)$$
- Near a self–intersection the minimal surface develops White's bridge connecting the two sheets.
- The tangent tensors align:
$$\vec{\tau}(l) \to \vec{\tau}(r)$$
- Using the algebra of the 't Hooft symbols we obtain
$$T_{\alpha\mu}(r)T_{\mu\nu}(l) \to -\frac{1}{4}\,\delta_{\alpha\nu}$$
- We see how the geometry of the Goldschmidt minimal surface together with Hodge duality work to produce universal $\delta_{\alpha\nu}$ term, necessary for the kernel in the MM equation.

Recovering the Makeenko-Migdal Equation

- For any minimal surface embedded in Euclidean space, the geodesic distance $L_{geo}(x, y)$ is strictly bounded by the Euclidean distance:

$$L_{geo}(x, y) \geq |x - y|_{\mathbb{R}^4}. \tag{97}$$

- The sum over the short paths Γ_{up}, Γ_{dn} weighted by $\exp\left(-ml\right)$ strictly localizes at the coordinate intersection, generating a smeared Dirac delta function:

$$\delta^4_m(x - y) = 4m^4 \exp\left(-2m\left|x - y\right|\right) \tag{98}$$

- As $m \to \infty$, we exactly recover the Makeenko-Migdal Loop Equation:

$$\mathcal{L}_\nu W[C] = \lambda \int_C \mathrm{d}y_\nu \delta^4(x - y) W[C_{xy}] W[C_{yx}] \tag{99}$$

  with the positive 't Hooft coupling $\lambda \propto Z[\mathbf{1}]^2$.
- Two properties of the Hodge-dual minimal surface were instrumental in establishing the equivalence of Elf determionant to the planar QCD: (i) The self-duality was used to nullify the unwanted $A-$ term in the second area derivative contribution to the loop equation, (ii) the identity for the tangent tensors, which follows from Hodge duality, was used to reduce the $B-$ term to the universal kernel in the MM equation.

Inducing Asymptotically Free QCD

- By tuning $Z[\mathbf{1}]^2 \to 0$ as $m \to \infty$, we induce an asymptotically free coupling $\lambda \to 0$.

- **The Bootstrap Equation:** We can invert the point derivative ∂_μ in the MM equation to rewrite it as a path integral over Brownian paths Γ_{xz} in 4D space.

- Iterating this equation systematically with "glassed windows" $(W \to 1)$ flawlessly reproduces the standard planar Feynman graphs, complete with gluon propagators and Faddeev-Popov ghosts.

- These planar graphs, when summed up by RG, produce the famous formula for the physical mass scale in terms of the UV cutoff m and the bare coupling $\lambda \propto Z[\mathbf{1}]^2$:

$$\Lambda_{QCD} = m\lambda^{-\frac{51}{121}} \exp\left(-\frac{24\pi^2}{11\lambda}\right); \qquad (100)$$

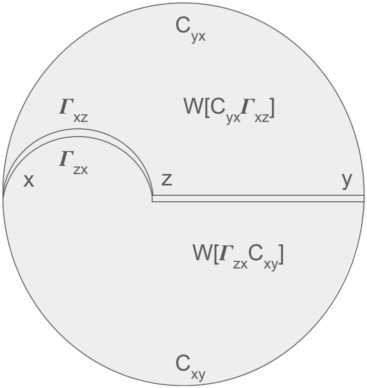


Figure: The Bootstrap path integral with the delta function represented by a straight double line, and Brownian paths by a crescent.

Summary of Lecture IV

- **The Elfin Theory:** Majorana fermions on a Hodge dual minimal surface precisely mimic the gauge invariant loop dynamics of Planar QCD.
- **Topological Cancellation:** The Fermi statistics natively enforce the Pauli principle, annihilating non-planar intersections to leave only planar diagrams.
- **Geodesic Localization:** In the large mass limit, White's bridge and the geodesic inequality dynamically generate the exact $\delta^4(x-y)$ contact term of the loop equations.
- **Induced QCD:** This theory serves as a globally regularized definition of Planar QCD, in a spirit of Induced QCD. Iterating its bootstrap equation successfully yields asymptotically free perturbation theory.
- To answer the physical argument by Nima at the previous lecture: In relativistic theory, there is no such thing as instantaneous action at a distance. So, there must be some particles propagating between quarks and mediating this force. My answer is yes; there are such particles—these are tiny elves living on this minimal surface.
- And the Casimir effect $-\pi/(12R)$ in the quark potential is coming from the zero-point fluctuations of two components of elves, not from two transverse components of the vibrating string (it does not vibrate in my theory).
- **Replacement of vibrations by elves is not an arbitrary choice of string model-builder; this is a necessity dictated by the topology of planar graphs and the instability of fluctuating surfaces in 4 dimensions.**

Next Lecture: Transitioning from Phase space of Quark loop to Twistor Space and extracting the exact Geometric Mass Spectrum via Twistor Monodromies and Catastrophe Theory.

Lecture V: From Twistor String Monodromies to the Exact Meson Spectrum

Discussion map: A straight path to the QCD Mass Spectrum

In Lecture IV, we established the exact 4D rigid minimal surface with internal Majorana fermions (Elves), inducing Planar QCD in the limit of large Elf mass. Today, we traverse the final path from the quark phase space to the physical meson spectrum:

Fork A: Quark Loop Phase Space

- Phase space quark paths & the Diffeomorphism gauge fixing
- Null twistor parametrization
- Holographic Liouville field and exact scale invariance

Fork B: WKB & Monodromy

- Minkowski target space and light-cone twistors
- Periodic orbits & Exact WKB quantization
- Generating mass poles via twistor monodromy

Fork C: The Exact Spectrum

- Twisted boundaries (the Helicoid)
- Exact parametric Regge trajectories
- Global geometric fit to PDG experimental data

(We will keep the intricate algebraic details on a general level today. If the public demands, a special seminar can be given later on the details of the twistor measure and complex Langevin.)

Twistor Parametrization of the Loop Space Measure

- We start with the linear phase space measure over loops $p(\theta), x(\theta)$ with path derivative $v(\theta) = x'(\theta)$:

$$\mathcal{D}P\mathcal{D}X \propto \delta^4\left(\oint v(\theta)\mathrm{d}\theta\right)\prod_\theta \mathrm{d}^4 p(\theta)\mathrm{d}^4 v(\theta). \tag{101}$$

- To fix reparametrization invariance $(\mathrm{Diff}(S^1))$, we enforce the Virasoro constraint $(f'_\mu)^2 = 0$ by factorizing the holomorphic tangent vector into boundary null twistors:

$$f'_\alpha(z)\sigma_\alpha = \lambda(z) \otimes \mu(z) \tag{102}$$

This converts the loop velocity $v_\alpha(\theta)$ algebraically into twistor bilinears:

$$v_\alpha(\theta) = 2\,\mathrm{Re}\left(\mathrm{i}z\lambda\sigma_\alpha\mu\right)_{z=e^{\mathrm{i}\theta}} \tag{103}$$

- The entire path integral in the quark amplitude is reduced to a 1D boundary integration over holomorphic spinor fields.
- The residual local real scaling symmetry $\lambda \Rightarrow \lambda e^r, \mu \Rightarrow \mu e^{-r}$ is fixed via the Faddeev-Popov constraint $(\bar{\lambda}\lambda = \bar{\mu}\mu)_{|z|=1}$.

The measure change from loop velocity to twistors

- This projection induces a metric on the tangent space of the twistors. The norm of a variation in loop space is:

$$\|\delta v\|^2 = \oint d\theta (\delta v_\alpha)^2 \propto \oint d\theta \left| z\delta\lambda\sigma_\alpha\mu + z\lambda\sigma_\alpha\delta\mu - \text{c.c.}\right|^2 \tag{104}$$

- Expanding the square and using Fierz identities, this norm can be written in a quadratic form acting on the spinor variations $\delta\Lambda = (\delta\lambda, \delta\mu)^T$:

$$\|\delta v\|^2 \propto \oint d\theta \, \delta\Lambda^\dagger \cdot \hat{Q} \cdot \delta\Lambda \tag{105}$$

- The kernel is the 4×4 block matrix (in spinor indices):

$$\hat{Q} = \begin{pmatrix} (\bar{\mu}\mu)\mathbb{I}_2 & \lambda\bar{\mu} \\ \mu\bar{\lambda} & (\bar{\lambda}\lambda)\mathbb{I}_2 \end{pmatrix} \tag{106}$$

where $\lambda\bar{\mu}$ denotes the dyadic product (rank-1 matrix).

The local Jacobian

- To determine the measure, we analyze the eigenvalues of $\hat{Q}$. Let $\Lambda = \bar{\lambda}\lambda = \bar{\mu}\mu$ on the constraint surface. The eigenvalues are:
  1. $\omega_1 = 0$: The **zero mode**, corresponding to the gauge transformation $\delta\lambda = \epsilon\lambda, \delta\mu = -\epsilon\mu$.
  2. $\omega_2 = 2\Lambda$: The dilatation mode $\delta\lambda = \lambda, \delta\mu = \mu$.
  3. $\omega_{3,4} = \Lambda$: Rotation modes orthogonal to the spinors.
- The determinant of $\hat{Q}$ vanishes due to the zero mode, which reflects the redundancy fixed by the constraint. Following the Faddeev-Popov procedure, the correct Jacobian is the square root of the determinant restricted to the physical subspace (orthogonal to the gauge orbit):

$$\mathrm{d}^4 v(\theta) = \sqrt{\det{}' Q}\, d\Omega_{FP}(\lambda, \mu) \tag{107}$$

- Evaluating the determinant on the physical subspace yields the factor:

$$\sqrt{\det{}' Q} = \sqrt{2\Lambda^3} \tag{108}$$

Elimination of the zero mode

- The zero mode corresponds to the generator of the real rescaling symmetry. The variation along this gauge orbit is given by:

$$\delta\lambda = \delta r \lambda, \quad \delta\mu = -\delta r \mu \tag{109}$$

where δr is the infinitesimal parameter of the transformation.

- The norm of this tangent vector in the flat spinor space is:

$$\|\delta_{\text{gauge}}\|^2 = |\delta\lambda|^2 + |\delta\mu|^2 = (\delta r)^2 \bar{\lambda}\lambda + (\delta r)^2 \bar{\mu}\mu \tag{110}$$

- Imposing the constraint $\bar{\lambda}\lambda = \bar{\mu}\mu = \Lambda$, this simplifies to:

$$\|\delta_{\text{gauge}}\|^2 = 2\Lambda(\delta r)^2; \tag{111}$$

$$\sqrt{\det G_{\|}} = \sqrt{2\Lambda} \tag{112}$$

- The total measure on the spinor space is the product of the measure on the physical subspace and the measure along the gauge orbit.

$$J_{\text{total}} = \sqrt{\det{}' Q} \times \sqrt{\det G_{\|}} = \left(\sqrt{2}\Lambda^{3/2}\right) \times \left(\sqrt{2\Lambda}\right) = 2\Lambda^2 \tag{113}$$

Summary and Interpretation of Twistor parametrization of Quark loop

- The original linear measure $\mathcal{D}C$ is overparameterized due to reparameterization invariance.
- Gauge fixing via the Virasoro constraint and twistor parametrization reduces the integration to the physical moduli space.
- The physically correct path integral measure is defined by the Faddeev-Popov Jacobian for the null twistor constraint. This ensures a unique counting of twistor states.
- Additionally, the change of variables from velocity $v(\theta)$ to twistors introduces the Jacobian, representing the volume of the physical tangent space.
- While this measure covers the moduli space uniformly, there remains a local $U(1)$ gauge invariance: $(\lambda, \mu) \Rightarrow (e^{\mathrm{i}\phi}\lambda, e^{-\mathrm{i}\phi}\mu)$. The corresponding volume $|U(1)| = 2\pi$ will be simply factored out of the measure for the normalized spinors $\xi, \eta, \bar{\xi}\xi = \bar{\eta}\eta = 1$. These spinors vary on $S^3 \times S^3$, therefore, the space becomes $S^3 \times S^3/S^1$ after this final factorization.

Local limit of $W[P]$ using effective action of Elfin theory

- With the non-singular functional $W[P]$, we can utilize the local limit of our continuum solution. Parametrizing the loop C by twistors λ, μ, we compute the Fourier integral for $W[\hat{P}]$ (with $\hat{P} = P_\alpha \sigma_\alpha^\dagger$):

$$W[P] = \int \mathcal{D}\lambda \mathcal{D}\mu \, 2\Lambda^3 \, \delta\left(\bar{\lambda}\lambda - \bar{\mu}\mu\right) \exp\left(-\int d^2 z L(z, \bar{z}) - m_q \oint \mathrm{d}\theta \Lambda\right)$$
$$\times \int d^4 k \exp\left(-2\mathrm{i} \oint \mathrm{d}\theta \, \mathrm{Im}\left(z\lambda(\hat{P} + \hat{k})\mu\right)\right)_{z=e^{\mathrm{i}\theta}} \tag{114}$$

- Here the combined measure factor $2\Lambda^2 \cdot \Lambda \sim \Lambda^3$ (depending on normalization conventions) is absorbed into the effective Lagrangian:

$$L(z, \bar{z}) = \frac{\sigma}{2}\Lambda^2 + \frac{1}{12\pi} \left|\partial_z \left(\log \Lambda\right)\right|^2 \tag{115}$$

- The extra integration over the "zero mode" k_μ reproduces the delta function $\delta\left(\oint v \, d\theta\right)$ for the periodicity of the loop ($C(\infty) = C(-\infty)$ in the upper plane map). This results in a nonlinear theory involving spinor fields $\lambda(z), \mu(z)$ holomorphic inside the unit circle.

Polar Coordinate Parametrization of the Null Twistor Measure

- Consider the twistor variables $(\lambda_\alpha, \mu^{\dot\alpha})$ on a unit circle $|z| = 1$, subject to the null constraint:

$$\bar\lambda^\alpha \lambda_\alpha = \bar\mu_{\dot\alpha} \mu^{\dot\alpha} \tag{116}$$

- We parametrize the spinors in polar coordinates as:

$$\lambda_\alpha = u\,\xi_\alpha, \mu^{\dot\alpha} = v\,\eta^{\dot\alpha} \tag{117}$$

- where $u, v \in \mathbb{R}_+$ are positive real scales, and ξ, η are normalized spinors:

$$\bar\xi^\alpha \xi_\alpha = \bar\eta_{\dot\alpha} \eta^{\dot\alpha} = 1 \tag{118}$$

- The constraint implies $u = v$. The measure on the null twistor space in these coordinates transforms as:

$$\mathcal{D}\lambda\,\mathcal{D}\mu\,\delta(\bar\lambda\lambda - \bar\mu\mu) \propto u^5\,du\,d\Omega_\xi\,d\Omega_\eta \tag{119}$$

- where $d\Omega_\xi$ and $d\Omega_\eta$ are the Haar measures on S^3.
- The local gauge $U(1)$ transformation $(\xi, \eta) \Rightarrow e^{\pm \mathrm{i}\alpha}(\xi, \eta)$ leaves the physical variables invariant. This $U(1)$ subgroup acts diagonally on the spinors and should be factored out, defining the angular manifold as the coset space:

$$\xi, \eta \in \frac{S^3 \times S^3}{U(1)}; \quad \Omega_{\xi\eta} = \frac{d\Omega_\xi d\Omega_\eta}{2\pi} \tag{120}$$

Reduction to u, ξ, η Variables

- The path integral over null twistors reduces to:

$$\int \mathcal{D}\lambda\,\mathcal{D}\mu\,\delta(\bar{\lambda}\lambda - \bar{\mu}\mu)\,F(\lambda,\mu) = \int_0^\infty u^5 du \int_{(S^3\times S^3)/U(1)} d\Omega_{\xi\eta}\,F(u\xi, u\eta) \tag{121}$$

- We define the Liouville field ρ by the logarithmic map:

$$\rho = 2\log u \qquad \Rightarrow \qquad u = e^{\rho/2} \tag{122}$$

- The radial measure transforms as:

$$u^5 du = e^{5\rho/2}(e^{\rho/2}d\rho) = e^{3\rho}d\rho \tag{123}$$

- Consequently, the path integral measure becomes:

$$\int \mathcal{D}\lambda\,\mathcal{D}\mu\,\cdots = \int_{-\infty}^{\infty} d\rho\,e^{3\rho}\int d\Omega_{\xi\eta}\,\ldots \tag{124}$$

Phase space path integral for quark loop amplitudes

- The physical amplitudes require one final functional integration: the path integral over the momentum field $P_\mu(\theta)$ (defined on the boundary).
- To restore parametric invariance in the effective theory, we introduce an einbein $e(\theta) > 0$ multiplying the Dirac operator:

$$\mathcal{D}e\mathcal{D}P \,\mathrm{tr}\,\mathcal{T}\exp\left(-\oint \mathrm{d}\theta\, e(\theta)(\mathrm{i}\gamma_\mu P_\mu(\theta))\right) = \int \prod_\theta d^4P(\theta) de(\theta) \exp\left(-\oint \mathrm{d}\theta\, e(\theta)(\mathrm{i}\gamma P(\theta))\right) \tag{125}$$

- While the conventional gauge choice is $e(\theta) = \mathrm{const}$, we have already fixed the reparametrization gauge via the Virasoro constraint on the holomorphic vector $f_\mu(z)$.
- Since the reparametrization acts simultaneously on P and C, and we have factored out the volume of Diff(S^1), there are no more gauge fixing to be done. We are left with integration over the arbitrary positive local einbein field $u = e(\theta)d\theta$.
- This integration yields the inverse matrix structure.

$$\int_0^\infty \mathrm{d}u\, \exp\left(-u\left(\mathrm{i}\gamma_\mu P_\mu\right)\right) = \frac{1}{\mathrm{i}\gamma_\mu P_\mu(\theta)} \tag{126}$$

Phase space path integral for quark loop amplitudes

- We compute the integral over the local momentum variable $q_\mu(\theta) = P_\mu(\theta) d\theta$:

$$I(\tau) = \int d^4 q \, \frac{\exp\left(\mathrm{i} q_\alpha \tau_\alpha\right)}{\mathrm{i}\gamma \cdot q}, \quad \tau_\alpha = \mathrm{Re}\left(\mathrm{i} z \xi \sigma_\alpha \eta\right) e^\rho \tag{127}$$

- This Fourier integral is calculable. Up to normalization, we obtain:

$$I(\tau) \propto \frac{\mathrm{i}\gamma_\mu \tau_\mu}{(\tau^2)^2} \tag{128}$$

- Using the normalization of the spinors ξ, η, and the Fiertz identity, the square of the source vector simplifies:

$$\tau^2 = \left(\mathrm{Re}\left(\mathrm{i} z \xi \sigma_\alpha \eta\right)\right)^2 e^{2\rho} = \frac{-1}{4}\left(z\xi\sigma\eta - \bar{z}\overline{\xi\sigma\eta}\right)^2 e^{2\rho} = \frac{1}{4}(4) e^{2\rho} = e^{2\rho} \tag{129}$$

- Substituting this back into the result:

$$I(\tau) \propto \frac{\mathrm{i}\gamma_\alpha \, \mathrm{Re}\left(\xi \sigma_\alpha \eta\right) e^\rho}{(e^{2\rho})^2} = \exp\left(-3\rho\right)\left[\mathrm{i}\gamma_\alpha \, \mathrm{Im}\left(z \xi \sigma_\alpha \eta\right)\right] \tag{130}$$

Holographic Liouville Theory and Exact Scale Invariance

- Remarkably, the factor $e^{-3\rho}$ from the momentum integration precisely cancels the measure factor $e^{3\rho}$ derived from the twistor transformation, rendering the effective local volume element scale-invariant.
- Decomposing the twistors into polar components $(\lambda = u\xi, \mu = u\eta)$ allows us to map the radial scale into a boundary Liouville field: $\rho = 2\log u$.
- The transformation from loop coordinates to twistors generates a local Jacobian: $e^{3\rho}$.
- However, integrating out the conjugate loop momentum P_μ in the phase space integral with the Dirac amplitude factor yields a source term inversion, producing the opposite factor $e^{-3\rho}$.
- The Invariance of Phase Space Measure: The factor $e^{-3\rho}$ from the momentum integration exactly cancels the measure factor $e^{3\rho}$ from the twistor transformation.
- This ensures the effective local volume element $dP \wedge dC \sim \hbar$ remains scale-invariant. The effective action becomes a boundary 1D Sigma Model coupled to a rigidly induced Liouville anomaly.

Twistor Holography vs. Topological Twistor String

- The effective action reduces to a Liouville-type term for the field ρ:

$$S_{\text{Liouville}} = \int d^2 z \left(\frac{1}{3\pi} |\partial_z \rho|^2 + \sigma e^{2\rho} \right) + m_q \oint d\theta e^{\rho(z=e^{i\theta})}, \quad \rho = \frac{1}{2} \log(\bar{\lambda}\lambda) + \frac{1}{2} \log(\bar{\mu}\mu) \quad (131)$$

- By expressing the minimal surface purely through holomorphic boundary twistors, the 4D bulk dynamics are projected completely onto the 1D boundary.
- Note that we bypassed the unsolved problem in the theory of minimal surfaces: how to invert the twistor parametrization of the boundary loop C and compute $W[C]$ as an explicit functional of C.
- Instead of computing the Wilson loop functional (which does not even have a continuum limit), we are computing the phase space path integral in the physical amplitudes involving $W[C]$.
- Unlike AdS/CFT, this "Twistor Holography" is pure geometry: there is no fluctuating worldsheet metric and no gravitational bulk.
- Crucial distinction from Witten's Twistor String: Our bulk functional is not a topological Wess-Zumino-Witten (WZW) term.
- The bulk geometry depends explicitly on the unique analytic continuation of the twistors into the unit disk, encoding the geometric rigidity of the minimal surface (akin to Penrose's Non-Linear Graviton).

Path integral in Minkowski space and the Exact-WKB Valley

- The Euclidean Obstruction: In Euclidean space, the oscillatory path integral forces the saddle points deep into the complex domain of twistor parameters. This obscures the physical geometry and removes the mathematical guarantee of a strictly real energy spectrum.
- The Minkowski Resolution: By formulating the boundary data natively in Minkowski spacetime, the path integral returns to its Feynman-Dirac roots: a quantum interference sum over real physical histories (=steepest descent path) weighted by $\exp(iS/\hbar)$. Minimal area is replaced by extremal Minkowski area.
- Restoration of Physical Reality: The exact-WKB saddle points now manifest as purely real classical Minkowski trajectories. This mathematically guarantees a strictly real mass spectrum, intrinsically forbids tachyons, and naturally restricts physical states to positive angular momenta $(J \geq 0)$.
- The twistor path integral is highly oscillatory at large windings, so we evaluate it by saddle point in Minkowski space.
- Physical bound states occur when the complexified effective action develops a *one-dimensional monodromy valley*, parametrized by the branch-point degree a (equivalently, the twist phase $\alpha = 4\pi a$).
- This valley is flat only in that single direction (string rotation). In all directions normal to it, the action becomes asymptotically steep in the large-winding sector.
- Therefore, the infinite sum over winding sectors freezes the transverse quantum fluctuations and localizes the path integral onto the monodromy valley.

Twisted Boundaries: The Helicoid

- In our twistor string formulation, we project this twisted kinematics directly into the boundary twistor data.

$$\hat{P}_J A = \sum_k \oint d\alpha \exp\left(-\mathrm{i}k\alpha J\right) A_k(\alpha) \quad (132)$$

- Here $A_k(\alpha)$ is the amplitude after k winding around the quark loop, with the boundary condition that the rotation angle α is introduced at every full circle around the quark loop.
- When the macroscopic boundary of the string rotates by an angle α, the boundary twistors must geometrically twist.
- The coordinate differential $d\hat{X} = \sum \sigma_\alpha dX_\alpha$ of the string must transform covariantly, with $SU(2)$ spinor rotation matrix $M(\alpha)$

$$d\hat{X} \Rightarrow M(\alpha) d\hat{X} M^\dagger(\alpha) \quad (133)$$

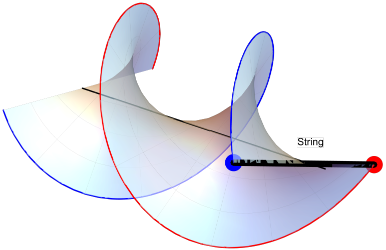


Figure: The helicoid spanned by a rotating $\bar{q}q$ pair connected by a rigid string. Discovered by Meusnier (1785), providing macroscopic physical motivation for twisted boundaries.

Twistor in Minkowski space and twisted boundary conditions

- In Minkowski (pseudo-Euclidean) space $R^{1,3}$, the target-space coordinate differential is parameterized by an additive sum of two independent 2-component complex spinors:

$$dX = \phi(\xi)\phi^{\dagger}(\xi)d\xi + \psi(\eta)\psi^{\dagger}(\eta)d\eta, \\ \xi, \eta = \tau \pm \theta. \tag{134}$$

- To incorporate the rotation of the meson, we require the coordinate matrix $d\mathbf{X}$ to be unitarily transformed by a spatial rotation matrix when the time coordinate winds around its fundamental cycle on the torus ($\tau \to \tau + 2\pi$).
- Because the left-moving ($\xi = \tau + \theta$) and right-moving ($\eta = \tau - \theta$) dependencies are decoupled, we achieve this target-space monodromy by defining the component spinors ϕ and ψ as complex eigenvectors of the $SU(2)$ rotation group.

$$\phi(\xi) = \sqrt{\frac{R}{2}} \begin{pmatrix} e^{i(a\xi+\pi/4)} \\ e^{-i(a\xi+\pi/4)} \end{pmatrix}, \\ \psi(\eta) = \sqrt{\frac{R}{2}} \begin{pmatrix} e^{i(a\eta-\pi/4)} \\ e^{-i(a\eta-\pi/4)} \end{pmatrix}. \tag{135}$$

Twistor monodromies

- In terms of the holomorphic functions of Euclidean space where $z \Rightarrow e^{\mathrm{i}\xi}, \bar{z} \Rightarrow e^{\mathrm{i}\eta}$, this parameter a corresponds to the degree of a branch point at $z = 0$

$$e^{\mathrm{i}a\xi} \Rightarrow z^a, \quad e^{\mathrm{i}a\eta} \Rightarrow \bar{z}^a \tag{136}$$

- A single period $\tau \Rightarrow \tau + 2\pi$ in Minkowski space would correspond to a trajectory rotating around the branch point on the Riemann surface, never closing on itself and eventually covering the torus at irrational a.
- Under one such winding, the arguments shift as $\xi \to \xi + 2\pi$ and $\eta \to \eta + 2\pi$. The column vectors are left-multiplied by the diagonal rotation matrix (monodromy):

$$\phi(\xi + 2\pi) = M(2\pi a)\phi(\xi), \quad \psi(\eta + 2\pi) = M(2\pi a)\psi(\eta), \quad \text{where} \quad M(2\pi a) = \begin{pmatrix} e^{2\pi ia} & 0 \\ 0 & e^{-2\pi ia} \end{pmatrix}. \tag{137}$$

- Substituting this into the additive factorization, the target-space coordinate differential rotates as:

$$d\mathbf{X}(\tau + 2\pi) = M(2\pi a)\, d\mathbf{X}(\tau)\, M^{\dagger}(2\pi a). \tag{138}$$

- This twisted boundary condition will allow us to project the quark loop amplitude to a fixed angular momentum J by means of the projection Fourier integral with $\alpha = 4\pi a$.

The metric and its light cone boundaries

- The induced metric on the worldsheet is determined by the determinant of the target-space coordinate differentials.

$$\Omega^2 = |\phi_1 \psi_2 - \phi_2 \psi_1|^2. \tag{139}$$

- Evaluating this for the given complex eigenvectors:

$$\begin{aligned} \phi_1 \psi_2 - \phi_2 \psi_1 &= \frac{R}{2}\left(e^{ia(\xi-\eta)+i\pi/2} - e^{-ia(\xi-\eta)-i\pi/2}\right) \\ &= \frac{R}{2}\left(ie^{2ia\theta} + ie^{-2ia\theta}\right) = iR\cos(2a\theta). \end{aligned} \tag{140}$$

- Taking the absolute square $\Omega^2 = R^2 \cos^2(2a\theta)$, the induced metric on the worldsheet is:

$$ds^2 = R^2 \cos^2(2a\theta)(d\tau^2 - d\theta^2). \tag{141}$$

- This metric vanishes at the roots $\theta = \pm\frac{\pi}{4a}$, which correspond to the string endpoints moving at the speed of light ($ds^2 = 0$). For massive endpoint quarks ($m_q > 0$), the physical string is truncated at boundaries $\pm\theta_b$ strictly inside this null radius ($0 \leq \theta_b < \frac{\pi}{4a}$), ensuring a positive proper time.

Minkowski Action monodromy

We evaluate the classical WKB action components integrated over the fundamental torus temporal period and the bounded spatial domain. We define the boundary phase parameter $\beta = 2a\theta_b$.

- The Target Time Boundary Term:

$$E\Delta X^0 = E\int_0^{2\pi} R\,d\tau = 2\pi E R \tag{142}$$

- The Minimal Area Term:

$$\Delta S_{Area} = \sigma R^2 \int_0^{2\pi} d\tau \int_{-\theta_b}^{\theta_b} \cos^2(2a\theta)\,d\theta = \frac{\pi\sigma R^2}{a}\left(\beta + \frac{1}{2}\sin(2\beta)\right) \tag{143}$$

- The Liouville term:

$$\Delta S_{\text{Liouv}} = \frac{1}{12\pi}\int d\tau d\theta\left((\partial_\theta\rho)^2 - (\partial_\tau\rho)^2\right) = -\frac{2a}{3}(\tan\beta - \beta). \tag{144}$$

- Proper Mass Integral:

$$\Delta S_{mass} = 2m_q\int_0^{2\pi} R\cos(\beta)\,d\tau = 4\pi m_q R\cos(\beta). \tag{145}$$

The Minkowski Minimal Surface Action

We evaluate the classical WKB action components on the Minkowski minimal surface, bounded by string boundaries $\pm\theta_b$ parameterizing the angular parameter $\beta = 2a\theta_b$, and $K = \sqrt{\sigma}R/a$. $\Delta S = a\Phi(E, K, \beta)$, where $\Phi(E, J, K, \beta) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5$, with all masses and energy in the units of string tension $\sigma \equiv 1$:

- **Target Time Boundary Term:** $\Phi_1 = 2\pi EK$
- **Minimal Area Term:** $\Phi_2 = -\pi K^2 \left(\beta + \frac{1}{2}\sin(2\beta)\right)$
- **Proper Boundary Mass Integral:** $\Phi_3 = -4\pi m_q K \cos(\beta)$
- **Spin Projection:** $\Phi_4 = -4\pi \left(J + \frac{q}{2}\right)$
- **Liouville Anomaly Defect:** Integrating the purely spatial conformal anomaly $-(\partial_\theta \rho)^2/(12\pi)$ yields: $\Phi_5 = -\frac{2}{3}(\tan\beta - \beta)$

- Because ΔS depends linearly on a at fixed K, β, extremizing the action leads to $\Phi(E, K, \beta) = 0$ at its extremum. Since $\partial_K \Phi(E, K, \beta) = \partial_\beta \Phi(E, K, \beta) = 0$, the first equation mandates that all higher Bohr-Sommerfeld levels vanish ($n = 0$ only).
- This angular variable β parametrizes the speed of the quark at the end of the string: $\beta = 0$ is the non-relativistic limit relevant to heavy quarks, and $\beta \to \frac{\pi}{2}$ corresponds to a massless quark moving with the speed of light.

Stationary metric, linear a dependence and exact spin projection.

- We introduced the Fourier projector of our twisted amplitude $A(\alpha)$ onto the given J state (with the Fourier term $-\mathrm{i}k\alpha J$ in the exponential included in effective action monodromy $k\Delta S = ka\Phi$).
- In a classical approximation, neglecting quantum fluctuations of the twistor field:

$$A_J^{cl} = \int_0^{2\pi} d\alpha \sum_k e^{-\mathrm{i}k\alpha J} A_k(\alpha) = \sum_{k=1}^{\infty} \int_0^{2\pi} d\alpha e^{\mathrm{i}k\alpha\Phi_\star(E,J)/(4\pi)}$$

$$\propto \frac{4\pi\mathrm{i}}{\Phi_\star(E,J)} \sum_{k=1}^{\infty} \frac{1 - \exp\left(\mathrm{i}k\Phi_\star(E,J)/2\right)}{k}; \tag{146}$$

- Here $\Phi_\star(E,J)$ is the value of $\Phi(E,J,\mathcal{K},\beta)$ at its extremum as described by above saddle point equations for moduli $\mathcal{K},\beta$.
- The bare winding sum is logarithmically divergent; after including the transverse fluctuation determinant, this divergence is strengthened as discussed on the next slide, while the simple pole in $1/\Phi_\star(E,J)$ is unchanged.

$$A_J^{cl} \propto \frac{4\pi\mathrm{i}\log k_{\text{max}}}{\Phi_\star(E,J)} \tag{147}$$

Steep walls of the flat valley and Hessian determinant

- A more accurate estimate of the divergent sum over windings must account for the transverse quantum fluctuations of the trajectory around this flat classical valley. Because the classical orbit length scales linearly with the winding number k, the stability matrix (the Hessian H) is directly proportional to k.
- The scaling factor for the integration measure can be extracted via the ζ-regularization of the functional determinant of this Hessian. Using the canonical value $\zeta(0) = -1/2$ for the trace over the 1D loop boundary modes, this yields the fluctuation weight:

$$\frac{1}{\sqrt{\det(kH)}} \propto \exp\left(-\frac{1}{2}\operatorname{tr}\log(kH)\right) \propto k^{-\frac{1}{2}\zeta(0)} = k^{\frac{1}{4}} \tag{148}$$

- Combining the $1/k$ kinematic factor from $\alpha integration$ with the $k^{1/4}$ fluctuation weight, the terms in the macroscopic winding sum scale as $k^{1/4}/k = k^{-3/4}$. Summing this up to a macroscopic cutoff k_{max} makes the divergence algebraically stronger than a logarithm, but rigorously preserves the fundamental geometric existence of the simple mass pole:

$$A_J^{quant} \propto \frac{1}{\Phi_\star(E,J)} \sum_{k=1}^{k_{\text{max}}} k^{-\frac{3}{4}} \propto \frac{(k_{\text{max}})^{\frac{1}{4}}}{\Phi_\star(E,J)} \tag{149}$$

- **Status of the exact-WKB argument:** the physical mechanism is clear—one flat monodromy direction and frozen transverse fluctuations at large winding. A fully rigorous proof would require a dedicated analysis of the transverse Hessian near the twistor branch-point saddle.

The Parametric Regge Trajectories

- Introducing the dynamically dressed dimensionless constituent quark mass $\bar{x} = x + \sqrt{x^2 - \frac{1}{3\pi}}$ (where $x = m_q/\sqrt{\sigma}$), the saddle point equations collapse to a unified parametric system for Energy E and Spin J

$$\frac{E(\beta)}{\sqrt{\sigma}} = \mathcal{K}(\beta, x)\left(\beta + \frac{\sin 2\beta}{2}\right) + 2x\cos\beta \tag{150}$$

$$J(\beta) = \frac{\mathcal{K}(\beta, x)^2}{4}\left(\beta + \frac{\sin 2\beta}{2}\right) - \frac{1}{6\pi}(\tan\beta - \beta) - \frac{q}{2} \tag{151}$$

$$\mathcal{K}(\beta, x) = \frac{\sin\beta}{\cos^2\beta}\bar{x} \tag{152}$$

- The Chiral Bound: The Liouville integration rigidly restricts the discriminant to $x \geq 1/\sqrt{3\pi}$. The geometric pressure dynamically generates an effective constituent mass gap.
- Topological Shift: The trace over the path-ordered Dirac holonomy introduces the topological vector index q. $q = 0$ for Pseudoscalars $(\pi, K, \dots)$, $q = -1$ for Vectors $(\rho, K^*, \dots)$.

Asymptotics: The Relativistic Drag and Non-Relativistic Hook

Ultra-Relativistic Asymptote (Large J)

- As endpoints approach the speed of light ($\beta \to \pi/2$), the Liouville anomaly term acts as an inertial drag.
- Expanding the parameters yields:

$$J(E) \approx \frac{E^2}{2\pi\sigma} - \left(x\bar{x} - \frac{\bar{x}^2}{6} + \frac{1}{6\pi}\right)\sqrt{\frac{2E}{\pi\bar{x}\sqrt{\sigma}}} + \frac{1}{12} - \frac{q}{2} \tag{153}$$

- The negative $\mathcal{O}(\sqrt{E})$ curvature reproduces physical slope-softening observed in heavy resonances.

Non-Relativistic Hook (Small J)

- Near the constituent mass threshold ($J \to 0, \beta \to 0$), the string is short and slow.
- Kinetic energy scales quadratically (β^2) but angular momentum scales cubically (β^3).
- Inverting this yields a fractional power law:

$$J(E) \approx \left(\frac{\bar{x}^2}{2} - \frac{1}{18\pi}\right) \times \left(\frac{E - 2m}{\sqrt{\sigma}(2\bar{x} - x)}\right)^{3/2} - \frac{q}{2} \tag{154}$$

- The derivative $dJ/dE \to 0$ at the threshold creates a distinct geometric hook into the linear regime.

Global Fit: Comparison with Experimental Meson Spectrum

- We perform a global joint geometric optimization $(\min \sum(\Delta S)^2 = 0)$ over **36 states** spanning **13 meson families** $(\pi, \rho, K, K^*, D, D^*, D_s, D_s^*, B, B^*, B_s, B_s^*, B_c)$.
- **Single Universal String Tension:** $\sqrt{\sigma} \approx 417$ MeV.
- The light sector saturates the chiral bound, dynamically generating an effective constituent boundary mass $m_{u,d} \approx 136$ MeV.
- The 5D optimization dynamically extracts physical constituent mass scales: $m_s \approx 219$ MeV, $m_c \approx 1.60$ GeV, and $m_b \approx 5.07$ **GeV**.
- **Parameter-Free Predictions:** Asymmetric heavy trajectories mathematically utilize the arithmetic mean of these scales. Therefore, the D_s, B_s, and B_c families are predicted *without any new free parameters*.

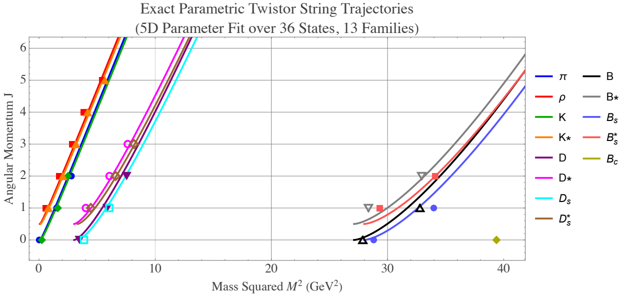


Figure: Exact Parametric Twistor String Trajectories plotted against the 36-state PDG meson spectrum.

A Unified Spectrum: Linear for Light, Curved for Heavy

Linear Asymptote for Light Quarks

- The π and ρ trajectories are highly relativistic and quickly approach the linear Regge asymptote.
- At high spins, they exhibit a slight $\mathcal{O}(\sqrt{E})$ curvature driven by the Liouville geometric drag, reproducing the slope-softening observed in physical high-spin resonances.

Non-Relativistic Hook for Heavy Quarks

- The heavy charmed (D, D^*, D_s, D_s^*) and bottom $(B, B^*, B_s, B_s^*, B_c)$ states sit deep in the non-relativistic regime for low spins.
- The twistor geometry natively predicts a sharp geometric "hook" $(dJ/dE \to 0)$ near the mass threshold before accelerating into the linear regime.

Global Planar Universality

- **Universal Kinematics:** The exact same symmetric string equations seamlessly bridge the 140 MeV ultra-relativistic pion and the **6.27 GeV charmed-bottom B_c resonance** using a single, universal string tension.
- **Planar Accuracy:** Standard string models draw straight lines that crash through heavy quark thresholds. The twistor string mathematically derives this non-relativistic bend.
- Experimental variations in macroscopic slopes $(\lesssim 10\%)$ are naturally absorbed by the expected $\mathcal{O}(1/N_c^2)$ non-planar corrections, yielding an $\approx 8.1\%$ average relative mass error.

Fit Accuracy: Physical and Geometric Metrics

- We do not perform an arbitrary least-squares mass regression; we mathematically minimize the variance of the fundamental geometric action constraint $\Phi = \Delta S/a = 0$.

- **1. Relative Mass Error (δm_{RMS}):**
  - Excluding the 140 MeV pion (which acts as an explicit pseudo-Goldstone boson and sits naturally below the geometric chiral threshold $2m_u \approx 271$ MeV), the RMS of the relative mass error across the remaining 35 massive states is $\approx \mathbf{8.1}\%$.
  - This firmly aligns with the expected $\mathcal{O}(1/N_c^2 \approx 10\%)$ theoretical variance of planar dynamics at $N_c = 3$.

- **2. Regge Trajectory Accuracy (ΔJ_{RMS}):**
  - Evaluating the theoretical spin at the exact experimental masses yields the absolute vertical deviation: $\Delta J_{\mathrm{RMS}} \equiv \sqrt{\langle (J_{\mathrm{exp}} - \alpha(M_{\mathrm{exp}}^2))^2 \rangle} \approx \mathbf{0.18}$.
  - Since physical Regge states are separated by exactly $\Delta J = 1$, this confirms the twistor geometry threads the mass spectrum to within **less than one-fifth of an orbital quantum step**.

- **3. Action Monodromy Variance (Φ_{RMS}):**
  - Sequential states in the Bohr-Sommerfeld quantization are separated by exactly $\Delta S_{\mathrm{gap}}/a = 2\pi$.
  - Evaluating the root-mean-square of the action residual across all 36 states yields $\mathrm{RMS}(\Delta\Phi/2\pi) \approx \mathbf{0.36}$.
  - The geometric phase error is merely $\sim 1/3$ of a single quantum level, ensuring the physical states remain safely isolated from higher excitations.

Conclusion: The QCD String is Derived, Not Postulated

- **Start from the loop equations, not from an invented string model:** In momentum loop space the MM equations are finite, and the breakdown of the Taylor–Magnus expansion at $W^{(8)}$ shows that a purely one-dimensional scalar loop functional cannot solve the full vector equations.
- **The missing structure is 4D rigid geometry:** the quark-loop phase space is parameterized by boundary values of holomorphic twistors of the Hodge-dual minimal surface. The twistor string is therefore *derived* from planar QCD as a classical Master field of $N_c = \infty$ gauge theory, solving MM loop equations.
- **Worldsheet Majorana fermions induce planar QCD:** they enforce planar factorization at intersections, while their conformal anomaly generates the Liouville term, which acts as a geometric drag on the rigid twistor boundary data.
- **Meson masses come from twistor monodromy:** after analytic continuation to Minkowski space, S-matrix poles are associated with winding sectors around twistor branch points. In the one-branch-point sector, the spectrum reduces by *exact WKB* to parametric Regge trajectories.
- **Global meson-spectrum description:** one universal string tension and four constituent boundary masses describe **13 meson families** and **36 states**, with an RMS relative mass error of about 8.1% excluding the pion.
- **The Chiral Mass Gap is a Geometric Reality**: The absence of a massless Goldstone mode is a physical reality of the isolated geometric string, not a mathematical flaw. The bare quark mass is inevitably additively renormalized by the macroscopic perimeter term $\oint d\theta \Lambda$ of the effective action. Even in the strict chiral limit ($m_{bare} \to 0$), the geometric boundary dynamics generate a strictly positive constituent mass gap. The anomalously light pion therefore requires the non-geometric multi-string chiral condensate, operating outside the $N_c = \infty$ limit of Planar QCD.

Outlook: The Loop Calculus as a Skeleton Key to Nonlinear Flows

- **Planar QCD Observables:** The framework naturally sets the stage to analytically extract the exact planar Glueball spectrum and, via multi-boundary Y-junction surfaces, compute exact planar Baryon masses and crossing-symmetric scattering amplitudes.
- **Fluid Dynamics and Macroscopic Turbulence:** The MM loop equations are not limited to gauge theory—they are the fundamental language of nonlinear flows in classical and quantum physics.
    - Exact analytic solutions to 3D Navier-Stokes HD turbulence (verified by K.R. Sreenivasan's DNS, to appear in *Phil. Trans. A*).
    - Solutions to MHD turbulence in plasma physics, revealing a first-order phase transition at $Pr = 1$, with possible implications for astrophysics and the fusion problem.
    - Solutions to turbulent mixing, revealing a singular concentric shell hierarchy in the passive scalar density related to Euler's totient summatory function.
- **A Skeleton Key vs. a Master Key:**
    - We call our Loop Calculus a **skeleton key**, as opposed to the *master key* suggested by Polyakov in his 1981 paper *"Quantum geometry of bosonic strings."* A master key is complex—it has a set of levers designed to match the specific wards of every lock in a building.
    - A skeleton key is profoundly simple—the wards are filed away, leaving only the necessary geometric minimum required to open every lock. The Loop Calculus is exactly this: a universal, basic solution stripped of unnecessary coordinate-space complications.

Thank you for your attention over this lecture series!

References and Supplementary Material I

- **Geometric QCD (Published in Nuclear Physics B):**

  - A. Migdal, *Geometric QCD I: The Hodge-Dual Surface and Quark Confinement*. Nuclear Physics B, Article 117380. https://doi.org/10.1016/j.nuclphysb.2026.117380
  - A. Migdal, *Geometric QCD II: The Confining Twistor String and Meson Spectrum*. Nuclear Physics B, Article 117424. https://doi.org/10.1016/j.nuclphysb.2026.117424

- **Macroscopic Turbulence:**

  - A. Migdal, *Geometric Solution of Turbulence as Diffusion in Loop space*. https://arxiv.org/abs/2511.02165v3

- **Interactive Mathematica Source Code (Exact 5D Mass Fit):**

References and Supplementary Material II

- A. Migdal, *Twistor String With Quark Masses* (March 2026). Published on Wolfram Cloud: https://www.wolframcloud.com/obj/sasha.migdal/Published/TwistorStringWithQuarkMasses.nb

- **Website and Blog:**
  - *Turbulence: Harmony of Primes under the Mask of Chaos*: https://alexandermigdal.com/turbulence-harmony-of-primes-under-the-mask-of-chaos/
  - Main Website: https://alexandermigdal.com