Operator instability from Voiculescu to Gromov-Lawson.

Some background for the talk of Dadarlat next week.

Thanks to Alex Lubotzky, Jianchao Wu, and Guoliang Yu

Philosophy: (Ulam). Is every "almost X" almost (an) X?

This is close to the issue of testability.

But, more generally, leads to lots of interesting math problems.

This seminar is devoted to the representation version of this.

From Ulam's anecdotal

Speaking of "epsilons," I want to mention a number of little amusements I have indulged in over the years concerning what I call "epsilon stability," not just of equations and their solutions, but more generally of mathematical properties.

As an example of this "epsilon stability," consider the simple functional equation: $f(x+y) = f(x) + f(y)$, i.e., the equation defining the automorphism of the group of real numbers under addition. The "epsilonic" analogue of this equation is $|g(x+y) - g(x) - g(y)| < \varepsilon$. The question is then: Is the solution g necessarily near some solution $\bar{f}$ of the strictly linear equation? As D. Hyers and I showed, the answer is yes. In fact, $|g - f| < \varepsilon$, with the same epsilon as above. This is not a very deep theorem. What about the more general case? Suppose I have a group for which I replace the group operation by one that is "close" to it in some appropriate sense. This of course requires a notion of distance in a group. The result of the replacement is an "almost endomorphism." Then we may ask: Is it of necessity "near" a strict endomorphism? The answer is not known in general, even for compact groups. Recently, D. Cenzer obtained an approximation result for some easy groups, e.g., the group of rotations on a circle.

In the same spirit, we may take the idea of a transformation which is an isometry, a transformation which preserves distances. What about transformations which do not exactly preserve distances but change them by very little, i.e., at most by a given $\varepsilon > 0$? Suppose I have a transformation of a Banach space or some space which transforms into itself, and where every distance is changed by less than a fixed ε. Is such a transformation "near" one which is a true isometry? Hyers and I proved, in a series of short papers, that this is true for Euclidean space, for Hilbert space, for the C space, and so forth. If you have such a transformation, it must then be within a fixed multiple of a strictly true isometry.

Recently I became more ambitious and looked at some other mathematical statements from this point of view. One could try to "epsilonize" in this sense theorems on projective geometry, on conics, and so on. More generally, take as

history of the Scottish Book.

an example some famous theorem like the theorem on functions with an algebraic addition. It is a well-known statement that the only functions which satisfy an algebraic addition theorem are, in addition to sine, cosine and elementary functions, the elliptic functions. One could ask (perhaps this question is not yet properly formulated): Is it true that a function which "almost" satisfies an algebraic addition theorem must be "almost" an elliptic function?

And in a similar vein: If we have a function which is differentiable, let us say five times, and its derivative vanishes and changes sign at a point, then any sufficiently differentiable function which is sufficiently close, in the sense of absolute value alone, must also have a vanishing fifth derivative at a nearby point. This is almost trivial to prove, though at first it seems false. Why is this true? Because the fifth derivative can be obtained by finite differences. This is all very nice and easy for functions of one variable. For functions of several variables the analog becomes interesting and not too well-known or established. The same is true, *mutatis mutandis*, for spaces of infinitely many dimensions, and is of possible interest to physicists as a general "stability" property.

Examples:

(1) In analysis.

Does every sequence where

$Af_i \longrightarrow 0$ converge to an

f where $Af = 0$?

e.g. Dirichlet principle.

- Many notions of convergence. (Pointwise versus Integral)
- Sometimes you exand the notion of an f.

Related to the problems of

invertibility (compact operators are small)

$$\begin{tikzcd} H_1 \arrow[r, bend left, "A"] & H_2 \arrow[l, dashed, bend left, "B"] \end{tikzcd}$$

so that $BA = I + \text{compact}$, i.e.
$AB = I + \text{compact}$

$A = \text{invertible} + \text{compact}$.

(Determined by Fredholm index.

$\operatorname{ind} A = \dim \ker A - \dim \operatorname{coker} A$)

(2) Kazhdan Property (T).

A group Γ has Property (T) if

for every unitary rep'n

$$\Gamma \times V \longrightarrow V$$

if $\|\gamma v - v\| < \varepsilon \|v\|$ $\gamma \in$ gen. set.

then v is close to v_∞

where $\gamma v_\infty = v_\infty$. (where close is linear in ε)

(3) Expander graphs.

Γ is an <u>expander</u> means that

if $A \subset \Gamma$ has

$$\# \partial A \approx 0$$

then $A \approx B$ with $\partial B = \phi$.

($\#$ is normalized by size of A, A^c).

(4) Ferry's theorem

<u>Theorem:</u> For every connected closed metric manifold M^n, there's an $\varepsilon > 0$, so that

$$f : M^n \longrightarrow N^n$$

$n \leq m$

M, N compact

can be moved a little to a homeo. if $\forall n \in N$, $\operatorname{diam} f^{-1}(n) < \varepsilon$.

f is homeo $\Leftrightarrow$ $\operatorname{diam}(f^{-1}(n)) = 0$ $\forall n \in N$

The analogue for $f: M \longrightarrow N \times F$ and you consider $f^{-1}\pi^{-1}(n)$: false for algebraic K-theory reasons. (Chapman - Ferry, Quinn)

(5) So now let's get to almost representations

Let Γ be generated by $\gamma_1, \dots, \gamma_k$, let G_i be a sequence of compact groups with bi-invariant norms $\| \ \|$

We want to know if $\cancel{\exists}\ \forall \delta > 0 \ \exists \varepsilon > 0$ so that every ε - almost representation

$$\rho : \Gamma \longrightarrow G$$

is within δ of representation $\hat{\rho}: \Gamma \to G$.

There are modifications, i.e. allowing post composition $G_i \to G_j$. When the G_i are nested.

We will have $G_i = U(i)$

and $\|A\|$ = operator norm

$$= \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}$$

For a fixed i this is trivial since $R(\Gamma) \subset U(i)^{\#\text{generators}}$ is compact.

Example $\Gamma = \mathbb{Z}/_k = \langle \gamma \rangle$

An almost representation of Γ

$\iff$ $A \subset U(d)$

such that $\| A^k - I \| < \varepsilon$

Of course A can be diagonalized

$$A \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}$$

and we are asking that $|\lambda_i^k - 1| < \varepsilon$

$\Rightarrow |\lambda_i - \text{root of unity}| < \frac{\varepsilon}{k}$

In general finite groups are stable.

This can be done by a cohomological argument or by using a "Hamel basis" + Shur's lemma argument.

Trivially, free groups are stable.

Voiculescu.

The next case to consider is $\mathbb{Z}^2$

closed loop.

$w(U,V) \in \mathbb{Z}$

Idea: $t \to \det((1-t)UV + tVU)$ has winding number 0 for commuting matrices.

around

constant

Exel–Loring's proof of Voiculescu. (Proc. AMS)

2. The Main Result. Fix once and for all an integer $n \geq 7$ and let $w_n = e^{2\pi i/n}$. Voiculescu's unitaries are defined by

X   Y

$$S_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Omega_n = \begin{pmatrix} \omega_n & & & & \\ & \omega_n^2 & & & \\ & & \omega_n^3 & & \\ & & & \ddots & \\ & & & & \omega_n^n \end{pmatrix}.$$

A_t = straight path from X to S_n

B_t straight path from Y to Ω_n

A few trivial and well known facts we shall need are in order

a) $\|\Omega_n S_n - S_n \Omega_n\| = |1 - \omega_n|$
b) $\det(\Omega_n) = \det(S_n) = (-1)^{n+1}$
c) $S_n \Omega_n S_n^* = \overline{\omega}_n \Omega_n$

By (a) we have that for n large, Ω_n and S_n are in fact almost commuting.

THEOREM. *If X and Y are commuting $n \times n$ complex matrices then*

$$\max\{\|X - \Omega_n\|, \|Y - S_n\|\} \geq \sqrt{2 - |1 - \omega_n|} - 1.$$

PROOF. Let X and Y be commuting $n \times n$ matrices and let $d = \max\{\|X - \Omega_n\|, \|Y - S_n\|\}$. Assume by way of contradiction that $d < \sqrt{2 - |1 - \omega_n|} - 1$.

For every t in $[0,1]$ let $A_t = \Omega_n + t(X - \Omega_n)$ and $B_t = S_n + t(Y - S_n)$ and define γ_t to be the closed complex curve given by

A_t takes A to Ω_n
B_t takes B to S_n

$$\gamma_t(r) = \det\big((1-r)A_t B_t + r B_t A_t\big), \qquad r \in [0,1].$$

For $t = 1$ we have that A_t and B_t commute so γ_1 is a constant curve. On the other hand for $t = 0$ we have $A_t = \Omega_n$ and $B_t = S_n$ hence

$$\gamma_0(r) = \det\big((1-r)\Omega_n S_n + r S_n \Omega_n\big) = \det\big((1-r)\Omega_n + r S_n \Omega_n S_n^*\big)\ \det(S_n) =$$

$$(-1)^{n+1}\det\big((1-r)\Omega_n + r\overline{\omega}_n \Omega_n\big) = (-1)^{n+1}(1 - r + r\overline{\omega}_n)^n\ \det(\Omega_n) = (1 - r + r\overline{\omega}_n)^n.$$

Note that as r goes from 0 to 1, $(1 - r + r\overline{\omega}_n)$ moves along the segment joining 1 to $\overline{\omega}_n$ in the complex plane. It follows that $\gamma_0(r)$ is never zero and that it winds around zero clockwise once.

Now, since the winding number is a homotopy invariant of closed curves in the complex plane with the origin removed we shall arrive at a contradiction as soon as we prove that $\gamma_t(r)$ is never zero. Equivalently it suffices to show that $(1-r)A_t B_t + r B_t A_t$ is invertible for all t and r which we do next by proving that the latter matrix is at a distance less than one from the unitary matrix $\Omega_n S_n$.

We have

$$\|(1-r)A_t B_t + r B_t A_t - \Omega_n S_n\| \leq (1-r)\|A_t B_t - \Omega_n S_n\| + r\|B_t A_t - \Omega_n S_n\| \leq$$

$$(1-r)\big(\|A_tB_t - A_tS_n\| + \|A_tS_n - \Omega_nS_n\|\big)+$$

$$r\big(\|B_tA_t - S_nA_t\| + \|S_nA_t - S_n\Omega_n\| + \|S_n\Omega_n - \Omega_nS_n\|\big) \le$$

$$(1-r)\big(\|A_t\|\|B_t - S_n\| + \|A_t - \Omega_n\|\|S_n\|\big)+$$

$$r\big(\|B_t - S_n\|\|A_t\| + \|S_n\|\|A_t - \Omega_n\| + |1-\omega_n|\big) \le$$

$$(1-r)\big((1+d)d + d\big) + r\big(d(1+d) + d + |1-\omega_n|\big) =$$

$$(1+d)d + d + r|1-\omega_n| \le d^2 + 2d + |1-\omega_n|.$$

Now, since $d < \sqrt{2 - |1-\omega_n|} - 1$ we have

$$d^2 + 2d + |1-\omega_n| < 1.$$

$\square$

We do not claim that our estimate is optimal. For example, estimating $\|(1-r)A_tB_t + rB_tA_t - \Omega_nS_n\|$ for $r \le 1/2$ and replacing Ω_nS_n by $S_n\Omega_n$ in a similar estimate for $r \ge 1/2$ we can prove that $d \ge \sqrt{2 - |1-\omega_n|/2} - 1$.

References

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Next week Dadarlat will explain how to prove (more than)

<u>Theorem</u> (D) If Γ is a linear group and some $\tilde{H}^{2i}(\Gamma;\mathbb{Q}) \neq 0$ then Γ is not operator stable.

My <u>main</u> goal is to explain this for Γ a uniform lattice in $O(2n,1)$ following an idea of Gromov, Lawson in the 80's

+ put this into a broader framework related to the other examples I mentioned earlier.

Subsidiary goals: (might or might not be achieved)

(a) The analogous orthogonal story.

(b) The role of elements of finite order to include groups which are concentrated in odd dimension e.g. for Garland reasons. (talking w/ Lubotzky, Yin ...)

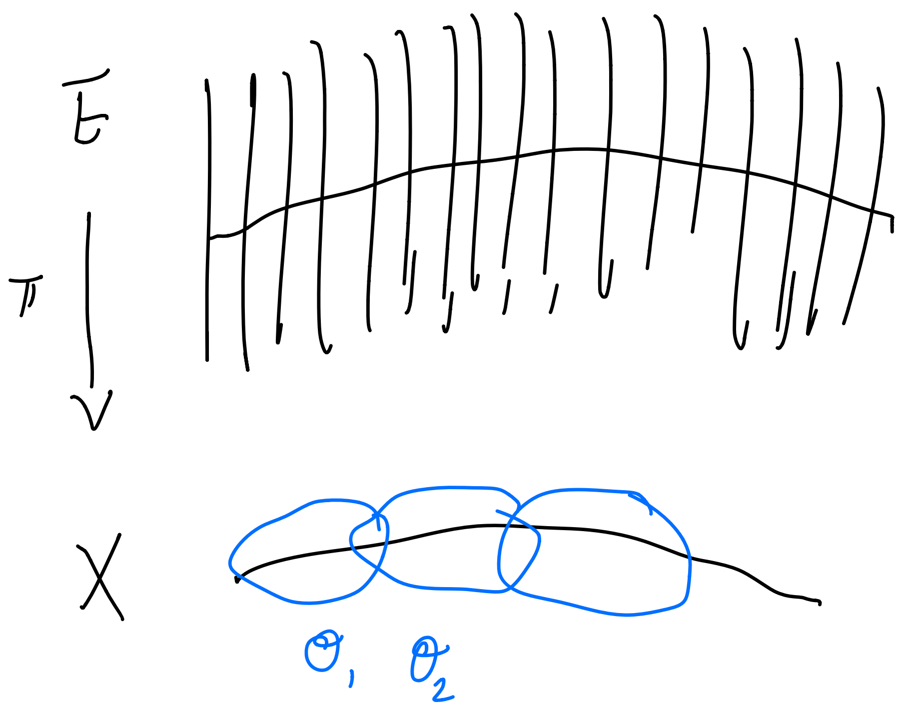
(c) Stability is pretty rare but valuable for proving non-approximability results - where might one look?

(d) Some quantitative questions, e.g. ε vs. d, How many "different" AR's are there?

Voiculescu's theorem is $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$.

$H^2(\Gamma ; \mathbb{Q}) \cong \mathbb{Q}$.

The relevant geometry is <u>bundle theory</u>.



(i)
$$\begin{tikzcd} \pi^{-1}(O_i) \arrow[r, "\cong", "\varphi_i"'] \arrow[dr, "\pi"'] & O_i \times V \arrow[d, "\mathrm{proj}"] \\ & O_i \end{tikzcd}$$

(iii) Over $\mathcal{O}_i \cap \mathcal{O}_j$ the maps

$$\begin{tikzcd}
(\mathcal{O}_i \cap \mathcal{O}_j) \times V \arrow[rr, shift left, "\varphi_j \cdot \varphi_i^{-1}"] \arrow[dr] & & (\mathcal{O}_i \cap \mathcal{O}_j) \times V \arrow[ll, shift left, "\varphi_i \cdot \varphi_j^{-1}"] \arrow[dl] \\
& \mathcal{O}_i \cap \mathcal{O}_j &
\end{tikzcd}$$

are fiberwise linear (isometries)*

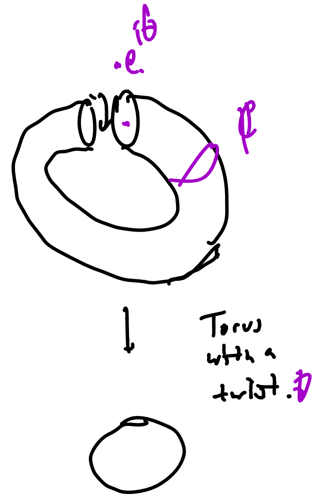
* Every $\mathbb{R}$-vector bundle can be given an inner product, and every $\mathbb{C}$ vector bundle can be given a fiberwise Hermitian product.

Examples:

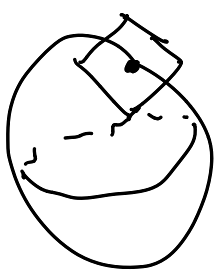
(a) Moebius band

$\downarrow$

S^1



(b)



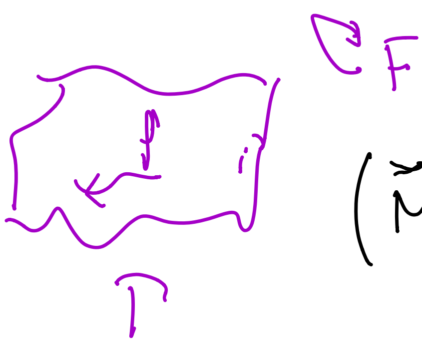
TS^2 can be made complex when you think of S^2 as the Riemann Sphere

(Hairy Ball Theorem).

(c) Suppose M is compact and

$$\begin{aligned} \rho : \pi_1 M &\longrightarrow O(d) \\ &\longrightarrow U(d) \end{aligned}$$

is a representation then



$$\left(\tilde{M} \times F^d\right)/_{\Gamma} \qquad F = \text{field.}$$

$$\big\downarrow$$

$$\tilde{M}/_{\Gamma} = M$$

is a bundle associated to ρ.

So you can study rep's via topology!

((a) is of this sort $\mathbb{Z} \to O(1)$.
(b) is not as $\pi_1(S^2) = 0$.)

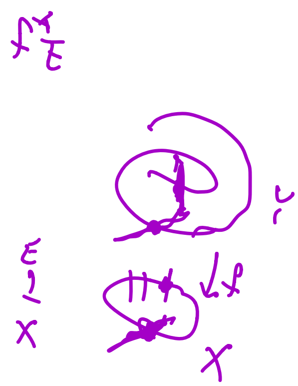
__Subexample__: $\rho : \mathbb{Z}_k \to U(d)$

$$\mathbb{Z}_k \times S^{2n-1} \circlearrowleft \text{ freely}$$

$$\begin{tikzcd} \mathbb{C}^d \arrow[r] & S^{2n-1} \times_\rho \mathbb{C}^d \arrow[d] \\ & S^{2n-1}/\mathbb{Z}_k \end{tikzcd}$$

$\leftarrow$ Lens space.

Constructions on bundles.



! (α) Pullback.

(β) Whitney Sum

(γ) $\otimes$, exterior powers, etc. i.e., all linear constructions.

(a) Enables, for us, two things:

- A best M to use for the construction associated to ρ.

$\pi_1 M = \mathbb{Z}$

$f^*C(\rho) \to C(\rho)$

$M \to S^1$

iso on π_1

$$B\Gamma \times_\rho \mathbb{C}^n \longrightarrow B\Gamma$$

so that for any M, $\pi_1 M = \Gamma$ $M \xrightarrow{\text{unique}} B\Gamma$ iso on π_1

Insert Here $B\Gamma$ (next slide).

- A classification of bundles.

<u>Theorem</u> d-dimensional bundles over X are in a 1-1 correspondence with

up to iso

$$X \xrightarrow{f} \text{Grassmannian of } F^d \in F^N$$

space of F^d's in F^N.

up to homotopy

(we'll always assume X is a finite complex to avoid technicalities).

$B\Gamma$ is a space (homotopy type) associated to Γ

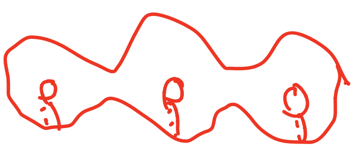
$$B\Gamma \longleftrightarrow \Gamma$$

as

$$S^1 \longleftrightarrow \mathbb{Z}$$

Example

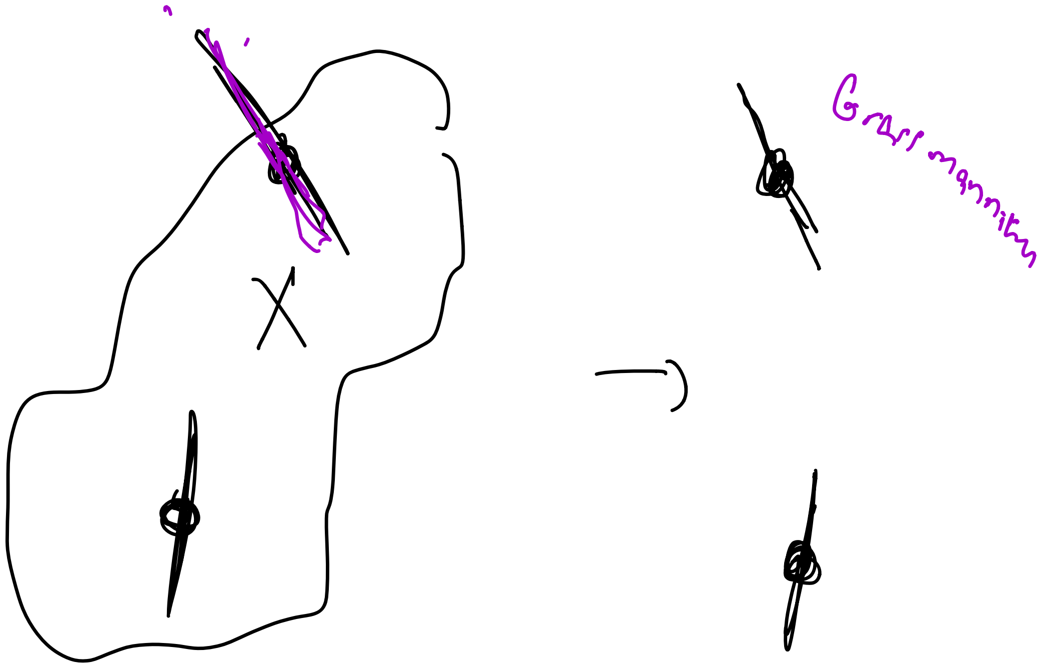
$$T^n \longleftrightarrow \mathbb{Z}^n$$



$$\Sigma_g \longleftrightarrow \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod [a_i, b_i] = e \rangle$$

etc.

$\boxed{\text{stick } \{ \subset F^N}$



points in X and the vector space above them

points in the Grassmannian

Note $Gr(F^d \subset F^N) \cong G(N)/_{G(d)\times G(N-d)}$

where $G = O$ or U.

So $H^*(Gr)$ can be studied by

classical methods (e.g. Cartan, Borel etc.)

We will let $N \to \infty$ and focus on $\mathbb{C}$.

Theorem: $H^*\left(U(N)/_{U(d)\times U(N-d)} \right)$ (is Grassmannian)

$\cong \mathbb{Z}[c_1, c_2 \cdots c_d]$

where $c_i \in H^{2i}$.

These are called <u>Chern classes</u>.

Definition: If $\pi : E \to X$ is a complex vector bundle then we can associate

$\varphi_E : X \to$ Grassmannian

$\varphi_E^*(c_i)$.

$$\overset{\shortparallel}{c_i(\pi)} \in H^{2i}(X;\mathbb{Z})$$

using the theorems of the previous two slides.

There are more direct definitions of $c_i(\pi)$ using other topology (e.g. Grothendieck) or geometry (Chern)

<u>Subexample:</u> $\rho : \mathbb{Z}_k \to U(d)$

Letting $n \to \infty$, then $H^{2i}(S^{2n-1}/\mathbb{Z}_k ; \mathbb{Z})$

$$\cong \mathbb{Z}_k$$

for each i.

$c_i(\rho) = i^{\text{th}}$ symmetric function of

the rotation numbers defining ρ.

So for k prime, the Chern classes

determine the representation, but for

composite numbers, not.

Concretely c_1 measures the following.

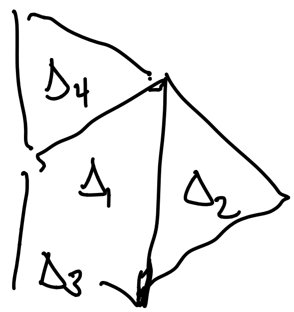
First for a line bundle.

$$\begin{tikzcd} \mathbb{C} \arrow[r] & E \arrow[d] \\ & X \arrow[u, dashed, bend left, "s"] \end{tikzcd}$$

choose a section that doesn't vanish more than it has to. At vertices no problem; at edges you can easily do it since $\mathbb{C}-\{0\}$ is connected. So for each 2-simplex of X we get

$\langle c_1, \Delta \rangle$ = winding number of the section around 0 in a local trivialization over Δ.

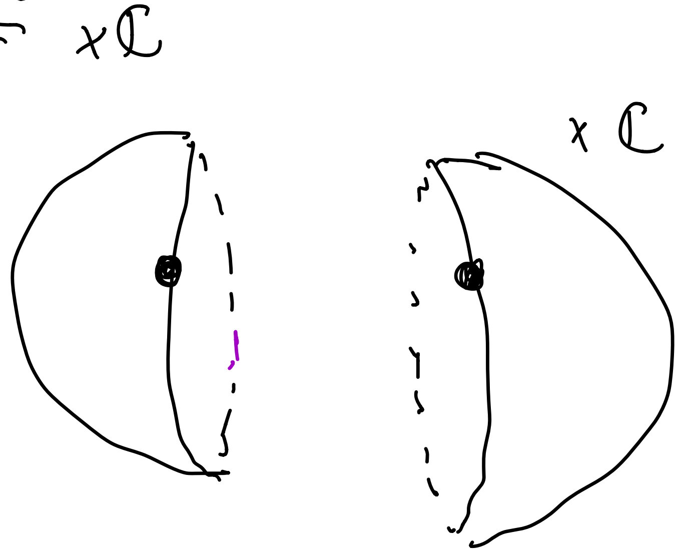
Note fixing over Δ_1 is likely to change the cochain over a neighboring Δ_j.



$[c_1] \in H^2(X;\mathbb{Z})$ is well defined.

In general take $E|_{X^2}$ and use $s_1 \cdots s_{n-1}$ globally defined and take $s_n \perp \langle s_1 \cdots s_{n-1} \rangle$ and take the winding number.

Example.



At point $z \in S^1$ glue $(z_L \times \mathbb{C})$ to $(z_R \times \mathbb{C})$ sending $(z, \vec{v})$ to $(z, z\vec{v})$

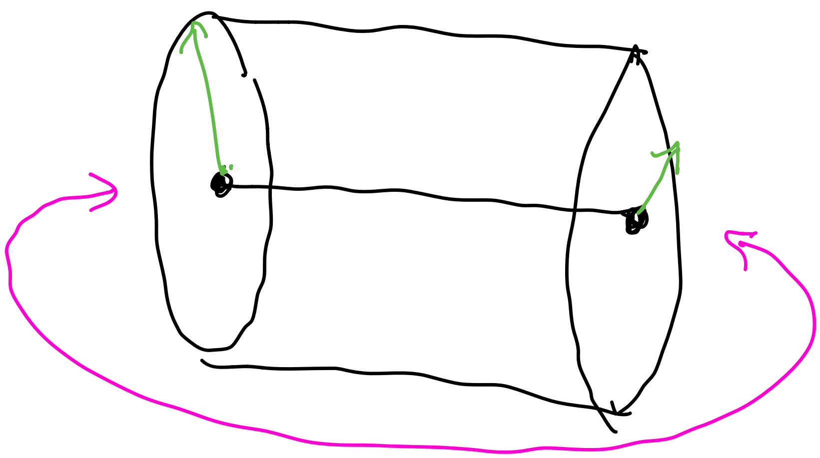
has $c_1 = 1 \in H^2(S^2) = \mathbb{Z}$.

- For higher dimensional fiber you get
  $S^1 \to U(d) \xrightarrow{\det} S^1$ and take winding number.

- Our next goal is to understand a role of almost representations in bundle theory and what is special about their Chern classes.

(J) Bundle associated to AR.

We would like to generalize the construction of the bundle over S^1 from a representation $\rho: \mathbb{Z} \to U(d)$ which only uses $\rho(1)$.



Bundle over

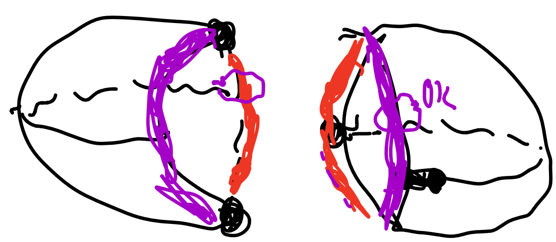
$\downarrow$

$B\Gamma$

(2) ~~…~~ Connections on bundles

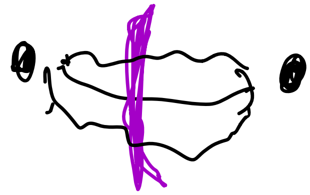
Flatness

A connection is a way to define transport from fiber to fiber. I won't give the axioms but let's consider an example.



$\times \mathbb{C}$

id on (z^2 on)

If you go from 

you always move the fibres in the same way by  it really depend on where you go through.

For flat bundles (i.e. bundles coming from representations) nearby curves transport the same way

but in general there might

not be any way to define a locally well defined fiber transport.

③ (Curvature and Chern classes)

These local deviations are measured by a 2-form Ω (with values in $\mathrm{End}(V)$)

$$c_n(\xi) = \frac{1}{(4\pi i)^n} \operatorname{tr}(\Lambda^n \Omega)$$

(This is a hint about the symmetric functions in $\mathbb{Z}_k \to U(d)$.)

It turns out $[c_n] \in H^{2n}(\,;\mathbb{Z})$!

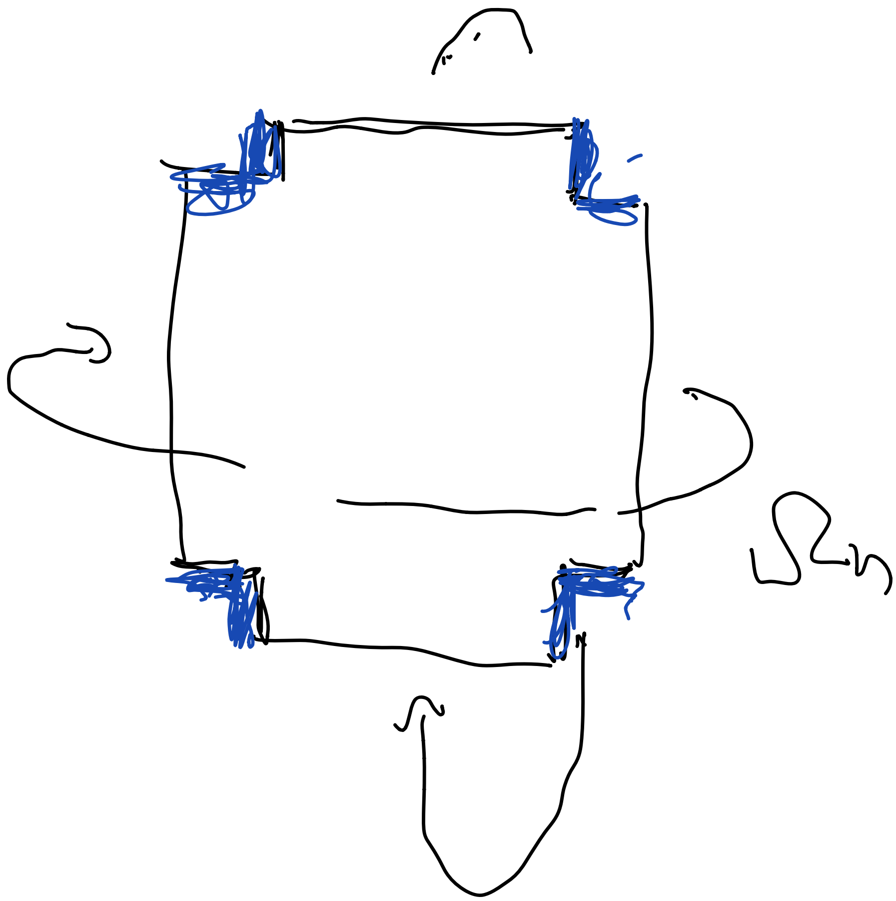
but not by definition.

So if curvature is small

$[c_n] = 0 \in H^{2n}(\; ; \mathbb{C})$.

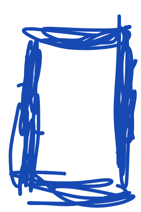
(and therefore $H^{2n}(\; ; \mathbb{Q})$).

We are now ready to give a more sophisticated view of Voiculescu example.



S_n

What do you do over



$S_n \Omega_n S_n^{-1} \Omega_n^{-1}$

close to id n

glue in

$\blacksquare \times \mathbb{C}^n$

④ There's a correspondence

Flat bundles over X $\longleftrightarrow$ Bundles with 0 curvature,

which generalizes to

"Almost flat" bundles over X $\longleftrightarrow$ Bundles with small curvature.

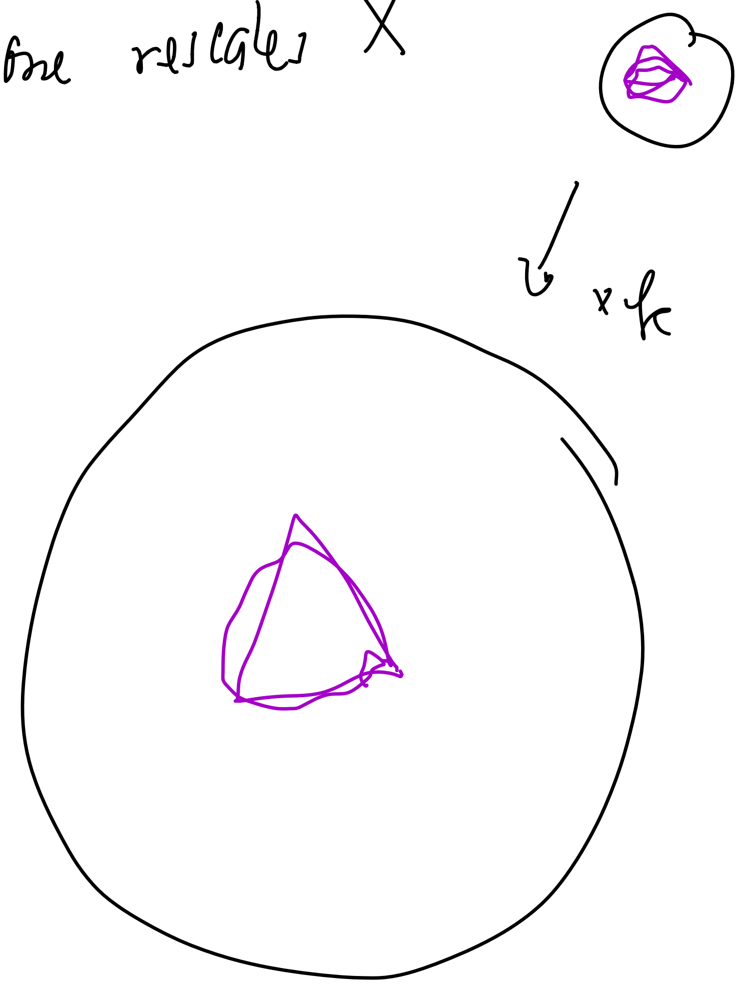
(Connes-Gromov-Moscovici, Carrion-Dadarlat-Hunger)

Key lemma: An almost flat bundle over a simply connected space is always topologically trivial.

(If $\varepsilon < \varepsilon_X$: ε_X might be tiny.)

Warning: ε does depend on X

If one rescales X



curvature is multiplied by k^{-2}

so you can always uselessly make curvature small! We need a fixed X.

The upshot is that we have the following:

(i) For a sufficiently good AR of Γ we get a bundle over $B\Gamma$.

(ii) For sufficiently close AR's the bundles are topologically isomorphic

(iii) The rational Chern classes of any flat bundle (= bundle associated to a presentation) vanish.

This explains the Voiculescu example.

For later developments it's important to know that

CLASSICAL FACT:

For every n there is a complex vector bundle $\xi \downarrow S^{2n}$ with $c_n(\xi) \neq 0 \in H^{2n}(S^{2n})$.

In fact:

<u>Bott's theorem:</u>

Complex vector bundles over S^{2k+1} are all stably trivial and over S^{2n} are in a 1-1 correspondence with the multiples of $(n-1)!$ in $H^{2n}(S^{2n}) \cong \mathbb{Z}$

This says $\pi_{2k+1}(U(n)) = \mathbb{Z}$ $k > n$

and $\pi_{2k}(U(n)) = 0$ $k > n$

Which is (the beginnings of) why this is called Bott's periodicity theorem.

It's at the foundations of K-theory.

<u>Definition:</u>

$$K(X) = \left\{ \begin{array}{c} \text{complex vector} \\ \text{bundles } \xi \downarrow X \end{array} \middle| \begin{array}{l} [\xi] = [\eta] \text{ if} \\ \text{there are trivial} \\ \text{bundles of the same} \\ \text{dimension so that} \\ \xi \oplus \varepsilon \cong \eta \oplus \varepsilon . \end{array} \right\}$$

(One also adds in formally negative dimensional trivial bundles

We don't need K-theory yet, but last week already we saw a functor with the same letter play a role in proving positive stability results.

Theorem (Atiyah-Hirzebruch) 1950's

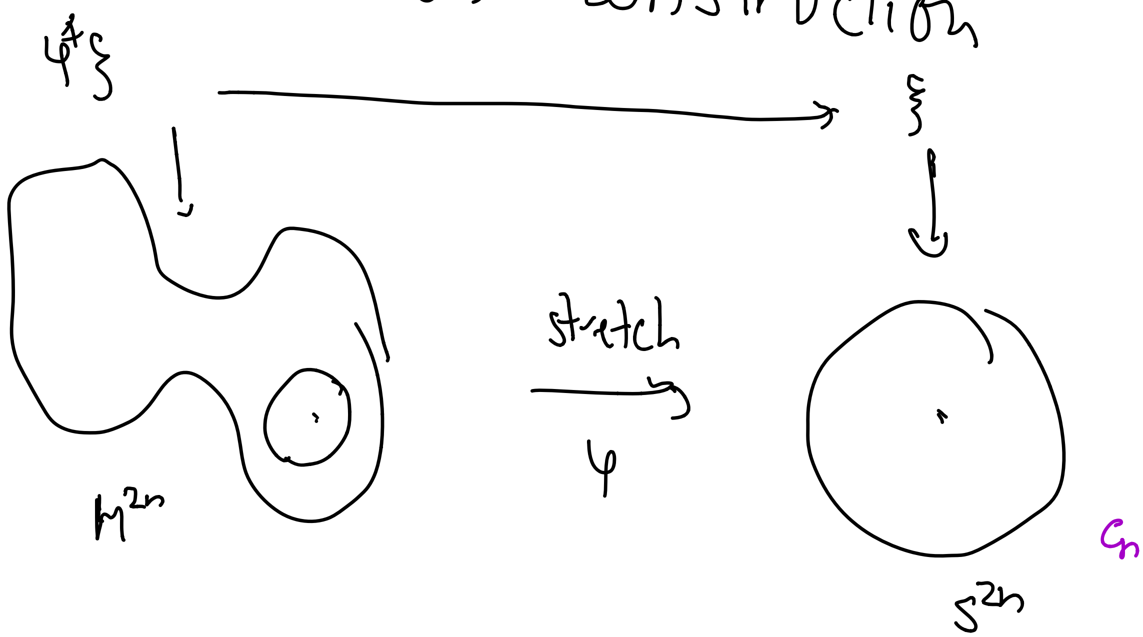
$$K(X) \otimes \mathbb{Q} \cong \bigoplus H^{2i}(X; \mathbb{Q})$$

When X is a finite complex K(X) is finitely generated, so we obtain

Cor: A flat bundle over a finite complex (e.g. compact manifold) always gives an element of finite order in K-theory.

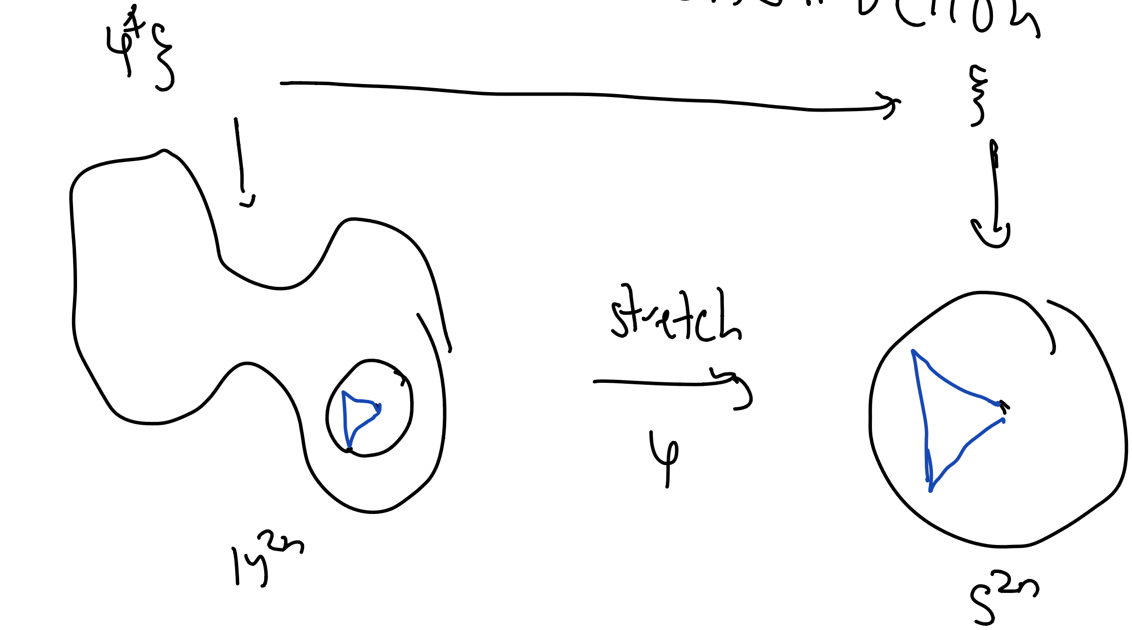
Not true for ∞-complexes or for almost representations. (Hence K-theoretic obstructions to stability.)

Gromov-Lawson Construction

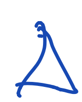


Curvature of $\varphi^* \xi$ is related to curvature of ξ and $\mathrm{Lip}(\varphi)$

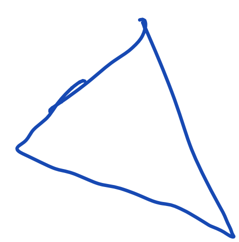
Gromov-Lawson Construction



Deviation from invariance of transport accumulated over

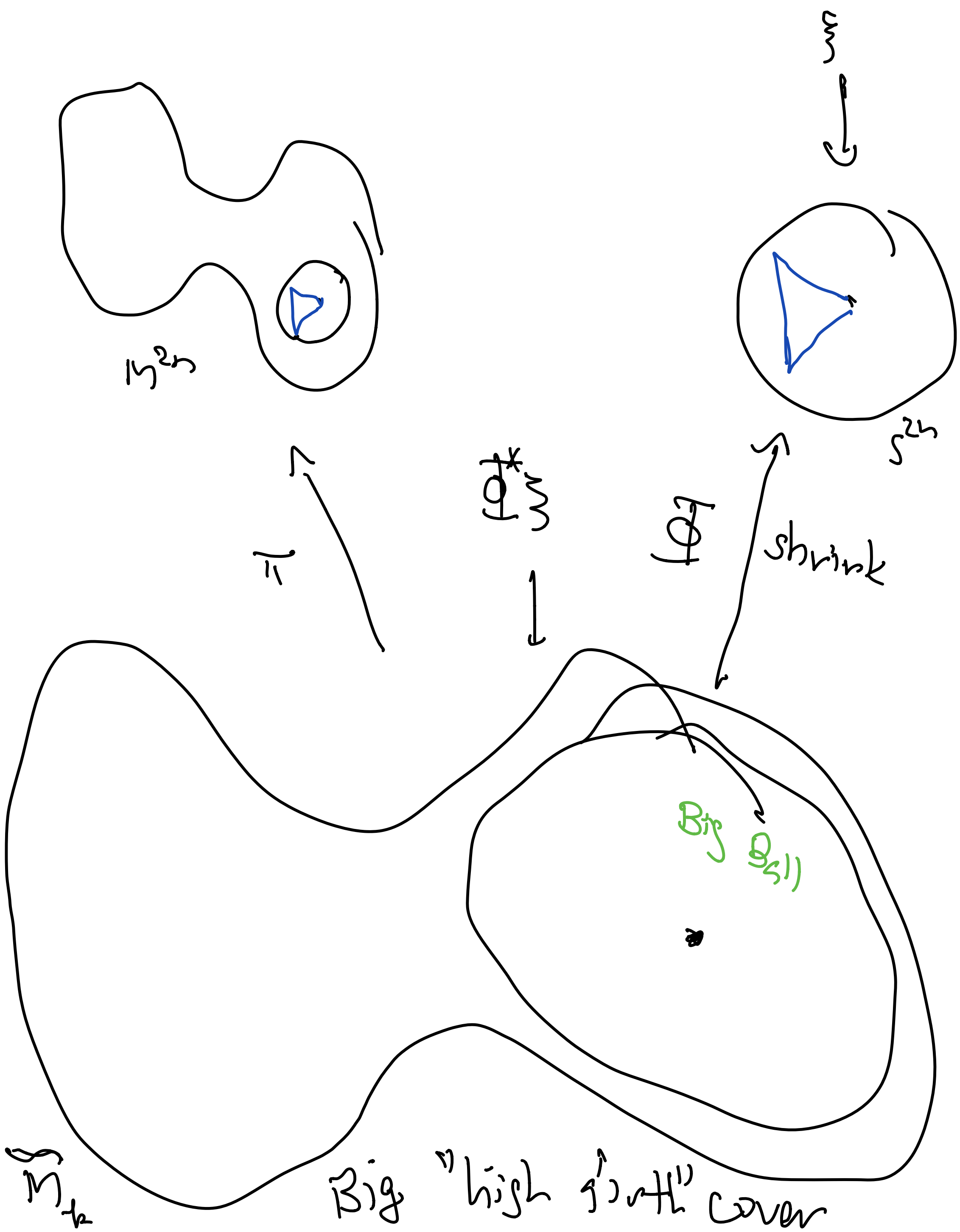


= that accumulated over

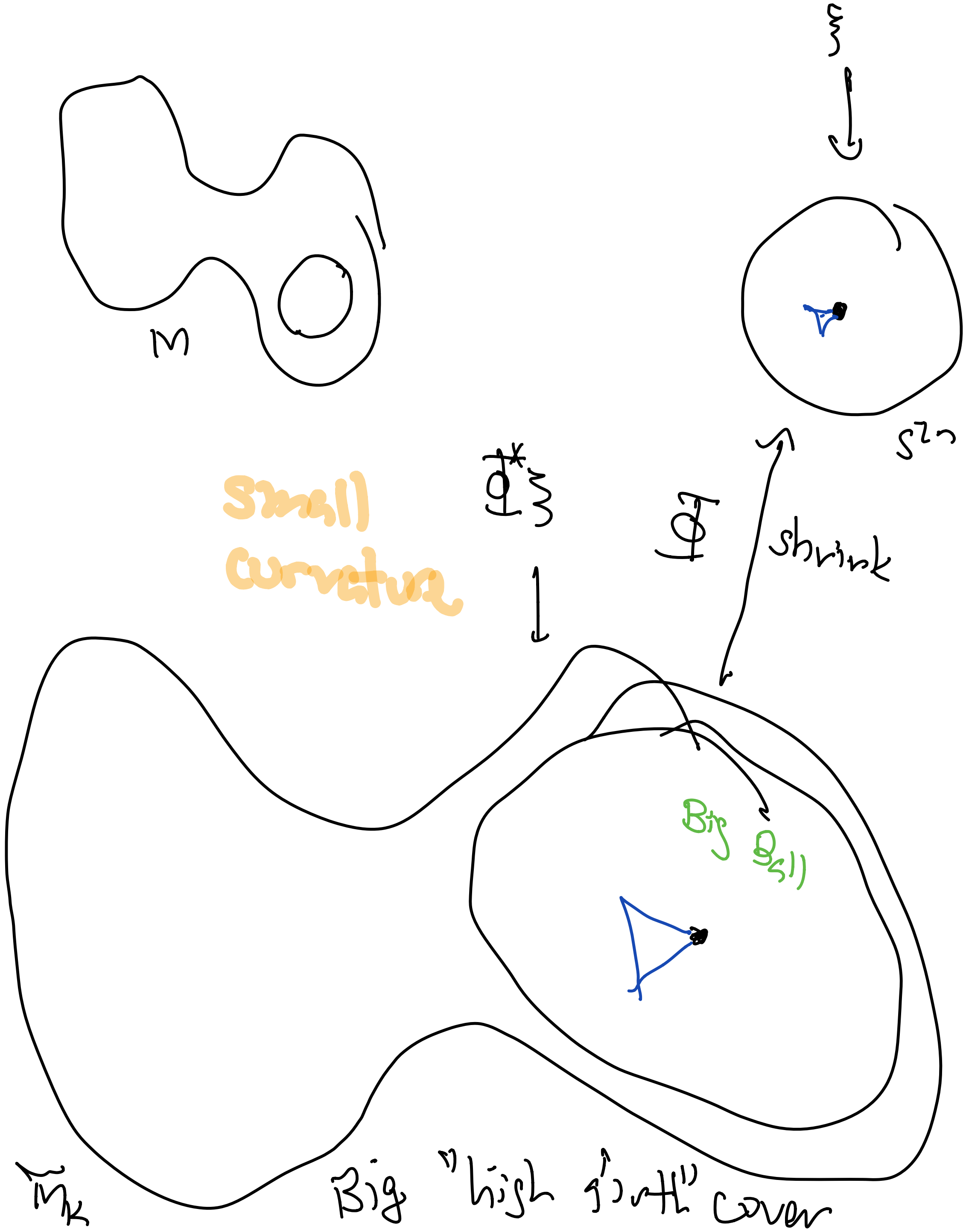


So curvature gets mult by $\mathrm{Lip}(\varphi)^2$

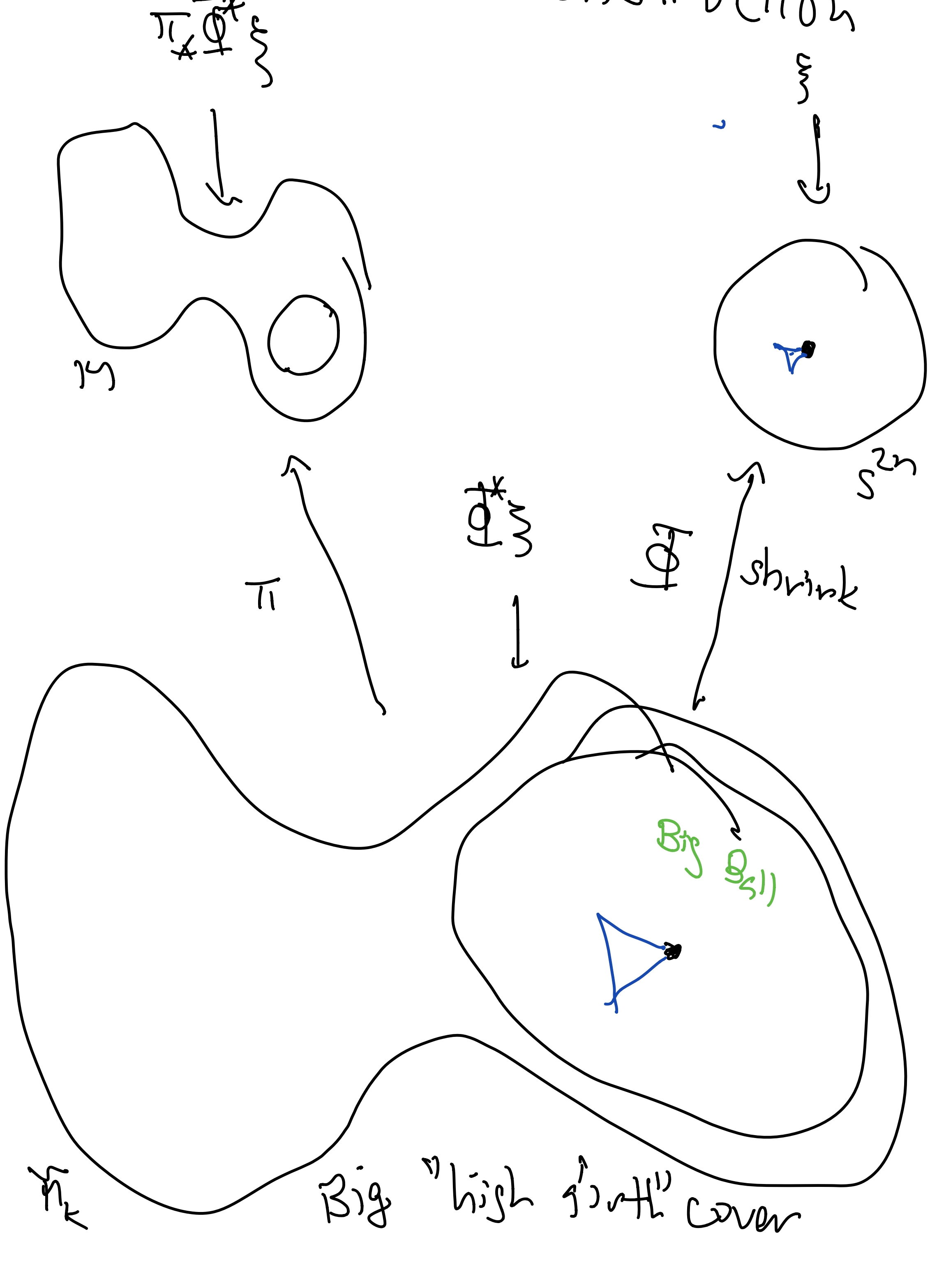
Gromov-Lawson Construction



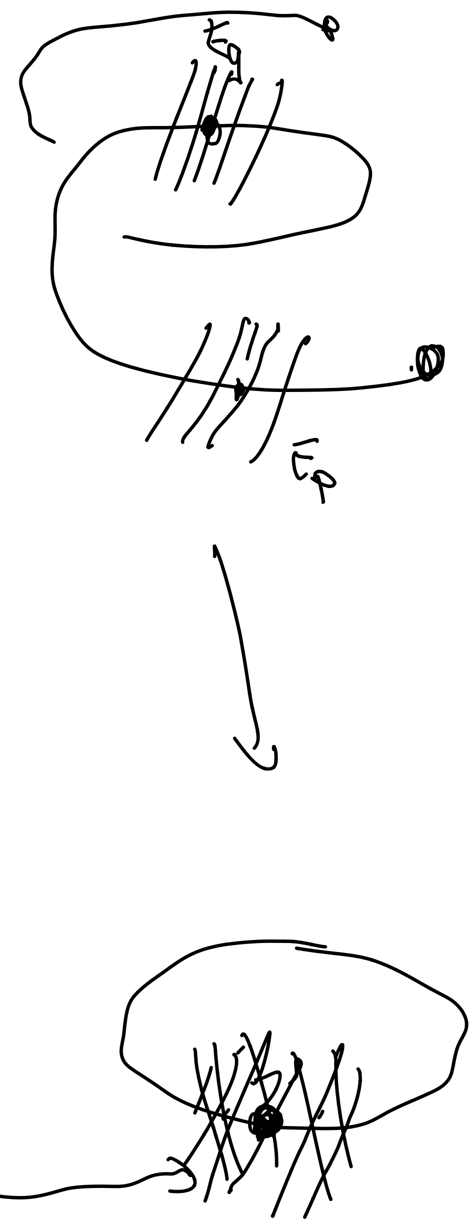
Gromov-Lawson Construction



Gromov-Lawson Construction



Picture of π_*



$E_p \oplus E_q$

π_* for flat bundles corresponds to

$$\mathrm{ind} : \mathrm{Rep}(\Delta) \longrightarrow \mathrm{Rep}(\Gamma)$$

for a finite index subgroup of Γ.

<u>Note</u>: The curvature doesn't change with π_*

(Operator norm is "non-Archimedian")

i.e. $$\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\| = \| A \oplus B \|$$

$$= \sup(\|A\|, \|B\|).$$

(and small Δ's in M lifted to the same small Δ's in $\widetilde{M}_2$ just a lot of them).

So we have

Theorem of Gromov - Lawson

If $M^{2n} = K\backslash G / \Gamma$

G real semisimple, Γ a uniform lattice, then M has a family of almost flat bundles which are not close to flat bundles.

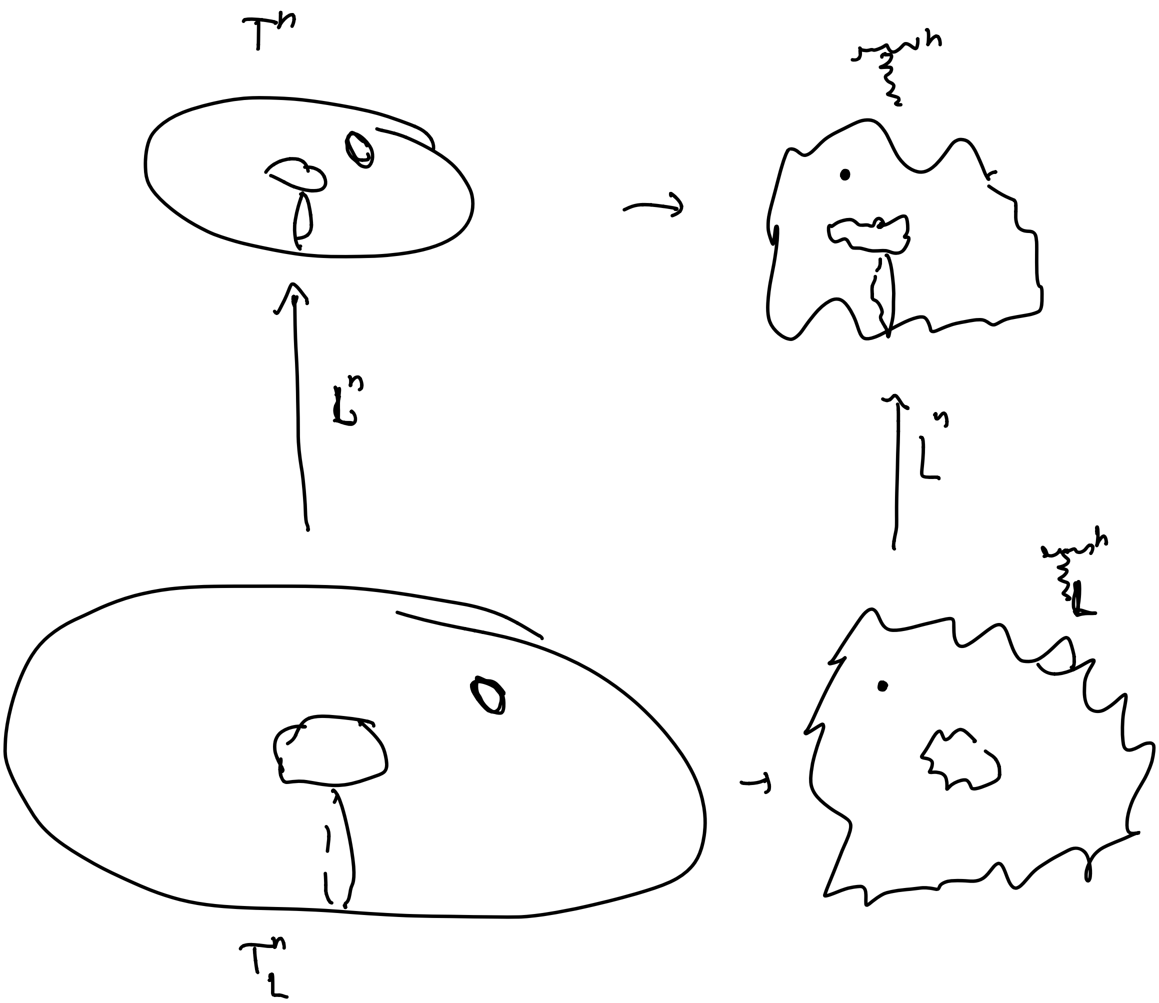
Remark: Typically, e.g. Γ is a high rank group the dimension of the bundle is

$$d \approx \exp\left(\frac{1}{\varepsilon}\right)$$

The number that can be produced that are different from one another is d^{α} for some $\alpha < 1$.

Does this mean anything?

Let's use the trick again.



Corollary: Any homotopy torus has a cover that's a torus. Indeed ALL large enough covers.

[Aside: This is circular reasoning, but it illustrates an argument that works]

Corollary: The tangent bundle of T^n is trivial.

i.e. We controlled an arbitrary element of

$K(T^n)$ via this trick

Note: $\mathrm{Rank}(K(T^n)) = 2^{n-1}$

Issues involved with topological versus differentiable which I must ignore.

Also $\tau_M \in KO(M)$ not $K(M)$.

KO = K-theory based on real (orthogonal) vector bundles.

There are maps

$$\begin{tikzcd} KO(M) \arrow[r, "\otimes_{\mathbb{R}} \mathbb{C}"'] \arrow[rr, bend left, "\times 2"] & K(M) \arrow[r, "\text{forget}"'] & KO(M) \end{tikzcd}$$

So $KO(M) \otimes \mathbb{Z}[\frac{1}{2}]$ is a summand of $K(M) \otimes \mathbb{Z}[\frac{1}{2}]$

<u>Corollary</u> of GL: For $M^{4k} = K \backslash G / \Gamma$

One does not have orthogonal group stability.

$$KO(M) \otimes \mathbb{Q} \cong \bigoplus H^{4i}(M; \mathbb{Q}).$$

Note: For $\mathbb{Z}^2$ one does have orthogonal group stability (Loring-Sorenson 2014)

Speculation: For π_1 (hyperbolic 3-manifold) - orthogonal stability holds.

Gromov and Lawson were interested in "controlling" all of K-theory

(to understand the geometry of M's with

$$M \xrightarrow{\varphi} B\Gamma$$

having $\varphi_*([M]) \neq 0 \in H_*(B\Gamma; \mathbb{Q})$)

but for their purpose they
didn't need to worry about
odd dimensions or even
anything below the top.

Alas we must skip the rest
of this story. However
we will say that

thru the work of

Higson-Kasparov -

Skandalis - Tu - Yu

- Guentner - Higson - W

one sees that for Γ

a torsion free linear group

generalisation of the Atiyah-Singer index theorem.

$$K(B\Gamma) \longrightarrow K(C^*_{max}\Gamma)$$

is split injective.

Gromov and Lawson essentially

found a map

$$K(C^*_{max}\Gamma) \longrightarrow \mathbb{R}$$

associated to a weak * limit

of almost flat bundles.

Dadarlat uses more sophisticated

C^*-algebra techniques

to get <u>all</u> homomorphisms

from the image

$$K(B\Gamma) \xrightarrow{\mu} K(C^*_{max}\Gamma)$$

to come from almost flat

bundles. (assuming an approx. condition on Γ and the splitting is good)

It is related to a non-commutative Fredholm index.

Deduolat obtains.

<u>Theorem:</u> If Γ is a linear group and for some $i > 0$ $H^{2i}(B\Gamma; \mathbb{Q}) \neq 0$ then Γ is not uniformly stable.

(Uniform lattice result of GL is a very special case.)

Similarly if some $H^{4i}(B\Gamma;\mathbb{Q}) \neq 0$

then Γ is not orthogonally stable.

FINAL REMARKS.

① There's more K-theory

② There's more than K-theory.

1. <u>More in K-theory.</u>

a) Obviously one can use torsion in $K(B\Gamma)$ and $KO(B\Gamma)$ to work. But it's not so direct. (because of torsion that $\boxed{BS(1,n)}$ does come from representations)

b) We didn't take the torsion in Γ seriously enough.

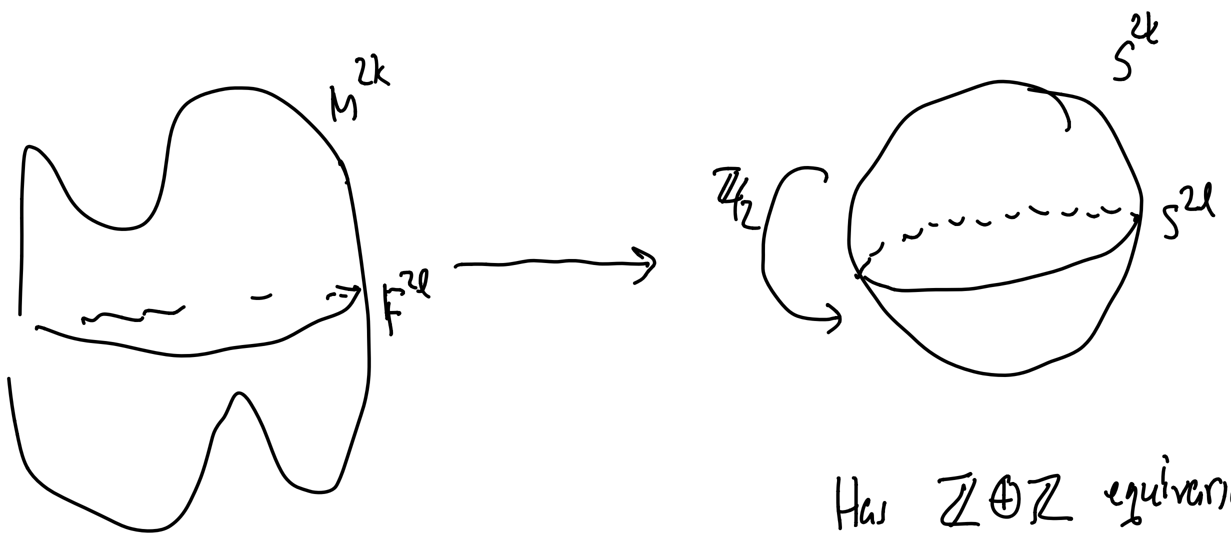
<u>Example</u> $\Delta \times \mathbb{Z}_2$ gets twice as many AR's as Δ does.

So if Γ has a $\Delta \times \mathbb{Z}_k$ inside

one can potentially use $H^{2i}(\Delta;\mathbb{Q})$

to detect almost representations

GL-example.



Has $\mathbb{Z}\oplus\mathbb{Z}$ equivariant vector bundles on $\mathcal{H}$.

and then play all the same games.

- The Gromov-Lawson types of geometry have been geometrised by a bunch of people and part of the Baum-Connes conjecture.

- An unconditional result is that if Γ contains torsion then $\tilde{K}(C^*_{max}\Gamma)$ (modulo the part coming from $B\Gamma$) is always an infinite group (has a homo to $\mathbb{R}/\mathbb{Z}$ ... still mysterious)

$\Rightarrow$ Some lattices with only odd cohomology are still not uniformly stable.

2. <u>More than K-theory.</u>

$$BS(2,3) = \langle a, b \mid a^{-1} b^2 a = b^3 \rangle$$

One can calculate that $\mathbb{Z} \subset BS(2,3)$
$g \longrightarrow b$

is a cohomological isomorphism (& ∴ K-theory isomorphism) so one would guess that it's stable. But it's <u>not</u>

<u>Theorem</u>: (Deeley et al) $BS(m,n)$ is never stable when not residually finite. In particular $BS(2,3)$.

The proof is similar to the de Chiffre - Glebsky - Lubotzky - Thom argument. (which shows non-Frobenius approximability from non-residual finiteness and Frobenius stability)

He uses residual amenability to get Matrix-approximability* so he deduces non-stability from non-residual finiteness.

*[Tikuisis-White-Winter]

$$\langle a, b \mid a^{-1} b^2 a = b^3 \rangle$$

$$\phi : \begin{matrix} a \to a \\ b \to b^2 \end{matrix} \; : BS(2,3) \to BS(2,3)$$

is surjective and has

$b^{-1} a^{-1} b a b^{-1} a^{-1} b a b$ in the

kernel.

THE END.