IAS 2022 SUMMER COLLABORATORS RESEARCH REPORT: THE CENTRALIZER ALGEBRA OF A TENSOR POWER OF A TWO-DIMENSIONAL SIMPLE MODULE OVER THE DRINFELD DOUBLE OF THE TAFT ALGEBRA

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ABSTRACT. Over an algebraically closed field $\Bbbk$ of characteristic zero, the Drinfeld double D_n of the Taft algebra that is defined using a primitive n-th root of unity $q \in \Bbbk$ for $n \geq 2$ is a quasitriangular, non-semisimple Hopf algebra. For any $n \geq 2$, the R-matrix of D_n is used to construct an action of the Temperley-Lieb algebra $\mathsf{TL}_k(\xi)$, with $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$, on the k-fold tensor power $\mathsf{V}^{\otimes k}$ of any two-dimensional simple D_n-module V. This action is known to be faithful for arbitrary $k \geq 1$, and for $1 \leq k \leq 2n-2$ the Temperley-Lieb $\mathsf{TL}_k(\xi)$ is isomorphic to the centralizer algebra $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$. In our continuing work, we seek to give a full description of the centralizer algebra for $k > 2n-2$.

Throughout let $\Bbbk$ be an algebraically closed field of characteristic zero, A be a finite-dimensional Hopf algebra, and V be a finite-dimensional left A-module. The Hopf algebra A acts on $\mathsf{V}^{\otimes k}$, the k-fold tensor power of V, and the centralizer algebra $\mathsf{End}_{\mathsf{A}}(\mathsf{V}^{\otimes k})$ is the algebra of all endomorphisms of $\mathsf{V}^{\otimes k}$ that commute with the action of A on $\mathsf{V}^{\otimes k}$. Classically the actions of the general linear group GL_n and the symmetric group S_k were studied and shown to be commuting actions with "Schur-Weyl duality" holding, so that the representation theories of the two groups are related. Centralizer algebras of the Brauer algebra (and the orthogonal and symplectic groups) ([27, 20]), finite subgroups of SU_2 ([1]), partition algebras ([24, 15, 4]), and various q-analogues of enveloping algebras of Lie algebras ([23, 25, 20, 14, 21, 28, 29]), among other algebras, have been studied. Much of the computation of centralizer algebras has been for centralizers of tensor powers of modules over a semisimple Hopf algebra A.

For any integer $n \geq 2$, the Drinfeld double D_n of the Taft algebra is a $\Bbbk$-algebra of dimension n^4 generated by elements a, b, c, d that satisfy the following relations for q a primitive n-th root of unity:

$$ba = qab, \qquad db = qbd, \qquad bc = cb, \qquad ca = qac, \qquad dc = qcd, \qquad da - qad = 1 - bc,$$
$$a^n = 0 = d^n, \qquad b^n = 1 = c^n.$$

The coproduct, counit, and antipode of D_n are given by:

$$\Delta(a) = a \otimes b + 1 \otimes a, \quad \Delta(d) = d \otimes c + 1 \otimes d,$$
$$\Delta(b) = b \otimes b, \quad \Delta(c) = c \otimes c,$$
$$\varepsilon(a) = 0 = \varepsilon(d), \quad \varepsilon(b) = 1 = \varepsilon(c),$$
$$S(a) = -ab^{-1}, \ S(b) = b^{-1}, \quad S(c) = c^{-1}, \ S(d) = -dc^{-1}.$$

The Taft algebra occurs in a number of contexts, e.g. it is the quantum Borel of $\mathfrak{u}_q(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (see [22]). The Drinfeld double of the Taft algebra is a quasitriangular Hopf algebra that is ribbon when n is odd; it provides a context for exploring centralizer algebras of non-semisimple finite-dimensional Hopf algebras. The representation theory of D_n, including the fusion relations, has been studied in detail by H.-X. Chen and his collaborators (e.g. [6, 7, 8, 9, 10]).

Let V be a finite-dimensional module over a finite-dimensional Hopf algebra A. The *McKay matrix* M_{V} for tensoring with V has as its (i,j) entry $\mathsf{M}_{ij} = [\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j]$, the multiplicity of S_j as a composition factor of the A-module $\mathsf{S}_i \otimes \mathsf{V}$, or equivalently, the coefficient of $[\mathsf{S}_j]$ when $[\mathsf{S}_i \otimes \mathsf{V}]$ is

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expressed as a $\mathbb{Z}$-linear combination of the basis elements in the Grothendieck group $\mathsf{G}_0(\mathsf{A})$. In [2] we proved results about the eigenvalues and the right and left (generalized) eigenvectors of M_V by relating them to characters. We showed how the projective McKay matrix Q_V obtained by tensoring the projective indecomposable modules of A with V is related to the McKay matrix of the dual module of V. We illustrated these results for the Drinfeld double D_n of the Taft algebra when n is odd and V is a two-dimensional simple D_n-module by deriving expressions for the eigenvalues and eigenvectors of M_V and Q_V in terms of several kinds of Chebyshev polynomials.

For any quasitriangular Hopf algebra H and any finite-dimensional H-module V, the R-matrix $\mathcal{R}$ of H can be used to construct elements of $\mathsf{End}_\mathsf{H}(\mathsf{V}^{\otimes k})$ that admit an action of the braid group, which sometimes factors through an action of the Iwahori-Hecke algebra. Leduc and Ram [20] described a general recipe for constructing such an action on the H-module $\mathsf{V}^{\otimes k}$. Each generator s_i, $1 \le i \le k-1$, of the Iwahori-Hecke algebra acts by applying a scalar multiple of $\mathcal{R}$ to positions i and $i+1$ of $\mathsf{V}^{\otimes k}$ followed by switching those two tensor factors. The action of the Iwahori-Hecke algebra on $\mathsf{V}^{\otimes k}$ is not faithful in general.

The Temperley-Lieb algebra is a quotient of the Iwahori-Hecke algebra, and it has generators t_i, $1 \le i \le k-1$, and a parameter ξ. It was introduced in [26] to study the partition function of the Potts model of interacting spins in statistical mechanics and later shown to have applications in the study of von Neumann algebras and subfactors [16], tensor categories [12], canonical bases of the quantum group $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ for q generic [13], $\mathfrak{sl}_2$-tensor invariants and webs [11], and countless other topics in mathematics and physics [5, 17, 18, 19]. The Temperley-Lieb algebra arises as a subalgebra of several centralizer algebras. For example, the centralizer algebra $\mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k})$ of the k-th tensor power of the SU_2-module $\mathsf{V} = \mathbb{k}^2$ is the Temperley-Lieb algebra $\mathsf{TL}_k(\xi)$, and so $\mathsf{TL}_k(\xi)$ is contained in the centralizer algebra $\mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$ of $\mathsf{V} = \mathbb{k}^2$ for any finite subgroup G of SU_2, a fact that was used to compute the centralizer algebra over G, see e.g. [1]. The vector space dimension of $\mathsf{TL}_k(\xi)$ is the k-th Catalan number $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$.

In [3], we explicitly determined the ribbon element of D_n, for n odd. We computed the Bratteli diagram for tensoring with any two-dimensional simple D_n-module, and compared this Bratteli diagram with the Bratteli diagram for tensoring with the simple module $\mathbb{k}^2$ for the quantum group $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ at a generic value of $\mathsf{q} \in \mathbb{k}$. In the $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$-case, nodes in the k-th row of the Bratteli diagram can be labeled by partitions β of k with at most two parts. The comparison enabled us to compute the vector space dimension of the centralizer algebra $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$ for any two-dimensional simple D_n-module V [3, Theorem 5.4]. One important difference between the Bratteli diagrams for $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ and D_n is that in the D_n diagram the modules in the decomposition can have multiplicities bigger than 1. For any $n \ge 2$, we used the R-matrix of D_n to construct an action of the Temperley-Lieb algebra $\mathsf{TL}_k(\xi)$ with $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ on the k-fold tensor power of any two-dimensional simple D_n-module V. We showed that this action on $\mathsf{V}^{\otimes k}$ is faithful for arbitrary $k \ge 1$. By comparing the vector space dimension of $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$ with the Catalan number $\mathcal{C}_k$ we showed that $\mathsf{TL}_k(\xi)$ is isomorphic to the centralizer algebra $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$ for $1 \le k \le 2n-2$, but not for $k > 2n-2$. Note that $\mathsf{TL}_k(\xi)$ with $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ is not semisimple when $k \ge n$, see e.g. [14].

We are grateful for the support of the IAS during the Summer Collaborators Program in June 2022. While at IAS, we studied the previous work on centralizer algebras. Using the Bratteli diagram computed in [3], we worked to explicitly compute the centralizer algebra of D_3 acting on the self-dual simple module $\mathsf{V} = V(2,1)$ and the action of the proper subalgebras $\mathsf{TL}_k(\xi)$ on $\mathsf{V}^{\otimes k}$ for $k \ge 5$, with the goal of describing the centralizer algebra in terms of generators and relations or as a diagram algebra. In doing so, we developed an algorithm to describe a D_3-basis of $V(\ell, r) \otimes \mathsf{V}$ inductively, where $V(\ell, r)$ is any simple D_3-module which occurs in the decomposition of $\mathsf{V}^{\otimes k}$. This example should provide new techniques for computing centralizer algebras. We are continuing to work on this problem, and the two weeks at IAS have given us a good start on this project.

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