Exercises (recommended)

1. (a) Prove that if G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - s$ edges, then G can be made $(r-1)$-partite by deleting at most s edges.

   (b) Prove that if G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - s$ edges, then G can be made *complete* $(r-1)$-partite by adding or deleting at most $3s$ edges.

   ⋆(c) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds for all sufficiently large n. If G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - \delta n^2$ edges, then G can be turned into $T_{r-1}(n)$ by adding or deleting at most εn^2 edges.

2. On a previous homework, you might have proved the following statement: if an n-vertex directed graph has no copy of a cyclic triangle, then it has at most $\lfloor n^2/2 \rfloor$ edges. The extremal example is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, with all edges oriented in both directions.

   Prove that this extremal problem *does not* exhibit stability. Namely, find another directed graph with $\lfloor n^2/2 \rfloor - o(n^2)$ edges and no cyclic triangle, which cannot be turned into the extremal example above by adding/deleting $o(n^2)$ edges.

3. In this problem you'll prove lower bounds for the extremal numbers of cycles.

   (a) Let p be a prime, $2 \leqslant \ell \leqslant p$ a positive integer, and let $a_1, \ldots, a_\ell$ be ℓ distinct elements of $\mathbb{F}_p$. Prove that the vectors

   $$(1, a_1, a_1^2, \ldots, a_1^{\ell-1}), \qquad (1, a_2, a_2^2, \ldots, a_2^{\ell-1}), \qquad \ldots \qquad (1, a_\ell, a_\ell^2, \ldots, a_\ell^{\ell-1})$$

   are linearly independent in $\mathbb{F}_p^\ell$.

   (b) Let p and ℓ be as above, and consider the following bipartite graph G. Its two parts are X and Y, where $X = \mathbb{F}_p^\ell$ and Y consists of all lines in $\mathbb{F}_p^\ell$ of the form

   $$\{(b_1, \ldots, b_\ell) + t \cdot (1, a, a^2, \ldots, a^{\ell-1}) : t \in \mathbb{F}_p\}.$$

   Make $x \in X$ and $y \in Y$ adjacent in G if and only if the point x lies on the line y. Prove that G has $n = 2p^\ell$ vertices and $p^{\ell+1} = \Theta(n^{1+1/\ell})$ edges.

   ⋆(c) Prove that if $\ell \in \{2, 3, 5\}$, then G is $C_{2\ell}$-free. Conclude that $\mathrm{ex}(n, C_{2\ell}) = \Theta(n^{1+1/\ell})$.

   (d) What goes wrong if $\ell \notin \{2, 3, 5\}$?

   ?(e) Modify this construction to work for $\ell = 7$.

✣4. Recall that the *distance* between two vertices u, v in a graph G, denoted $d_G(u, v)$, is the number of edges in the shortest path connecting them.

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

(a) Prove that if H is a spanning subgraph of G (i.e. $V(H) = V(G)$ and $E(H) \subseteq E(G)$), then $d_G(u, v) \leqslant d_H(u, v)$ for all u, v.

(b) Given an integer k, a *k-spanner* of G is a subgraph $H \subseteq G$ for which

$$d_G(u, v) \leqslant d_H(u, v) \leqslant k \cdot d_G(u, v)$$

for all u, v. Prove[1] that every n-vertex graph G, regardless of how many edges it has, contains a $(2\ell - 1)$-spanner H with $e(H) \leqslant O(n^{1+1/\ell})$, for any $\ell \geqslant 1$.

Remark: Spanners are very important in computer science, as they allow us to approximate distances in G while using much less storage than it would take to store all of G. For example, even if G has $\Theta(n^2)$ edges, the result above shows that we can approximate distances in G up to a factor of 100 by storing only $O(n^{1.02})$ edges.

(c) Prove that this result is tight if $\ell \in \{2, 3, 5\}$. That is, there exists an n-vertex graph G containing no $(2\ell - 1)$-spanner with fewer than $cn^{1+1/\ell}$ edges, for some constant $c > 0$.

Problems (optional)

1. In this problem you'll see some variants of the supersaturation theorem for triangles.

   (a) Prove that if an n-vertex graph has $\lfloor n^2/4 \rfloor + 1$ edges, then it contains at least $\lfloor n/2 \rfloor$ triangles.

   (b) Prove that this bound is tight.

   ⋆⋆(c) Prove that if an n-vertex graph has $\lfloor n^2/4 \rfloor + 1$ edges, then it contains at least $\lfloor n/6 \rfloor$ triangles all sharing a single edge.

   ⋆(d) Prove that this bound is tight.

⋆2. Remove the minimum degree assumption from from the proof of Proposition 11.3, thus proving that $\mathrm{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ for all sufficiently large n.

⋆⋆3. Prove the following general stability theorem: for every H and every $\varepsilon > 0$, there exists $\delta > 0$ so that the following holds for all sufficiently large n. If G is an n-vertex K_r-free graph with at least $t_{\chi(H)-1}(n) - \delta n^2$ edges, then G can be made $(\chi(H) - 1)$-partite by deleting at most εn^2 edges.

⋆⋆4. Prove the following combination of the supersaturation and stability theorems. For every $r \geqslant 3$ and every $\varepsilon > 0$, there exist $\delta, \gamma > 0$ such that the following holds for all sufficiently large n. If G is an n-vertex graph with at most γn^r copies of K_r and minimum degree at least $(1 - \frac{1}{r-1} - \delta)n$, then G can be made $(r - 1)$-partite by deleting at most εn^2 edges.

[1] *Hint:* Greedily add edges to H while not creating a short cycle.

5. Let $\mathcal{F}$ be a finite collection of bipartite graphs, none of which is a forest. A famous conjecture of Erdős and Simonovits, called the *compactness conjecture*, asserts that there exists some $H \in \mathcal{F}$ such that

$$\mathrm{ex}(n, \mathcal{F}) \leqslant \mathrm{ex}(n, H) \leqslant C \cdot \mathrm{ex}(n, \mathcal{F}),$$

where $C > 0$ is an absolute constant, depending only on $\mathcal{F}$.

(a) Prove that the first inequality above holds for any $H \in \mathcal{F}$.

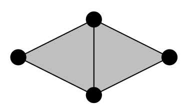
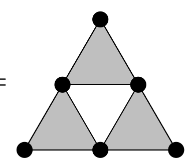
⋆ (b) Prove that the compactness conjecture can be false if we allow $\mathcal{F}$ to be infinite.

⋆ (c) Prove that the compactness conjecture can be false if we allow $\mathcal{F}$ to contain forests.

? (d) Prove or disprove the compactness conjecture.

⋆⋆ (e) The compactness conjecture is known to be false for hypergraphs! You'll see this in this part and the next.

Consider the following two 3-partite 3-graphs:

$K^{(3)}_{1,1,2} =$  $T =$ 

Prove that $\mathrm{ex}(n, K^{(3)}_{1,1,2}) = \Theta(n^2)$ and $\mathrm{ex}(n, T) = \Theta(n^2)$.

⋆⋆⋆⋆⋆ (f) Prove that $\mathrm{ex}(n, \{K^{(3)}_{1,1,2}, T\}) = o(n^2)$, thus disproving the compactness conjecture for hypergraphs.