Inhomogeneous Branching Brownian Motion

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In our collaborative visit, we studied a branching Brownian motion (BBM) with space-time dependent compactly-supported branching. We consider a collection of particles that undergo independent d-dimensional branching Brownian motion. Let $\mathcal{D}_0$ be a compact set in $\mathbb{R}^d$, $\mathbf{v}$ be a unit vector in $\mathbb{R}^d$, $\rho \geq 0$ be a constant, and define g to be a nonnegative function supported on $\mathcal{D}_0$. A particle at position x at time t undergoes binary branching at exponential rate $g(x - \rho\mathbf{v}t)$. Thus, in this inhomogeneous BBM model, the branching rate at time t is supported on the compact set

$$\mathcal{D}_t := \{x + \rho\mathbf{v}t : x \in \mathcal{D}_0\}. \tag{1}$$

We are interested in understanding how ρ, the speed at which the branching support moves, influences existence and speed of a linear front. In the case when $\rho = 0$, the compact branching support does not move, and the existence and speed of the front, as well as the structure of the population inside the front location, is well-understood (see [3] and references therein). In this case, the process is characterized by the parabolic PDE

$$\partial_t u = \frac{1}{2}\Delta u + g(x)u. \tag{2}$$

When the top eigenvalue, $\lambda(0)$, is positive, there exists a linear front located at $\sqrt{\lambda(0)/2}$. When the top eigenvalue is nonpositive, there is no linear front. The moving support model ($\rho > 0$) introduces an additional consideration to the long-term spread of the process and has not been studied before. This was the focus of our two week collaboration at IAS.

Our goal was to develop intuition about the process, formulate precise conjectures about the speed of the process, and to begin to prove these statements. The intuition can be obtained by considering known speed results for BBM with constant branch rate and BBM with compactly branching support that does not move. When the compact support moves at a linear speed, only "fast-moving" particles in the process have the opportunity to reproduce. Depending on the speed ρ of the support, this can lead to a linear front that is faster than the linear front of the associated BBM with $\rho = 0$. On the other hand, for ρ beyond a threshold ρ_c, even the leaders cannot keep up with the location of the moving branching domain, and hence there is no linear front for the process. Thus, we describe a phase transition: for ρ sufficiently small, we see a linearly-growing front and when ρ is too large, the system has no front.

Questions of front speed for a BBM have two perspectives which can be used to approach the analysis, PDE techniques and probabilistic techniques, and we employ both in the study of the process. The connection to PDEs comes through the McKean representation, which tells us that the density of particles at $y \in \mathbb{R}^d$, denoted by $u(t, x, y)$, satisfies the equation

$$\frac{\partial u(t,x,y)}{\partial t} = \frac{1}{2}\Delta u(t,x,y) + g(x - \rho\mathbf{v}t)u(t,x,y), \quad u(0,x,y) = \delta_y(x), \tag{3}$$

where the Laplace operator is applied in the x variables and y is treated as a parameter. When $\rho = 0$, precise global asymptotics for the above PDE are known from [4], which help in analyzing the location

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of the front and distribution of particles inside the front. No such asymptotic is available *a priori* for $\rho > 0$. However, if we view this problem in a moving frame with a drift ρ in direction $-\mathbf{v}$, then we see a new form of this PDE, given by

$$\frac{\partial w(t,x,y)}{\partial t} = \frac{1}{2}\Delta w(t,x,y) + \rho \mathbf{v} \cdot \nabla w + g(x)w(t,x,y), \quad w(0,x,y) = \delta_y(x). \tag{4}$$

From the perspective of the particle system, the moving frame system is equivalent to considering branching Brownian motions with constant drift ρ in the direction $-\mathbf{v}$ that experience a positive branch rate in the set $\mathcal{D}_0$. We define the operator $\mathcal{L}_\rho$ to be

$$\mathcal{L}_\rho w := \frac{1}{2}\Delta w + \rho \mathbf{v} \cdot \nabla w + g(x)w. \tag{5}$$

and call the top eigenvalue of this operator $\lambda(\rho)$. As before, the behavior of the front will depend on whether $\lambda(\rho)$ is positive or not. During our stay at IAS, we quantified the relationship between the spectrum of the operators $\mathcal{L}_0$ and $\mathcal{L}_\rho$ by considering eigenfunctions of $\mathcal{L}_\rho$ obtained by exponentially tilting the measure associated with the eigenfunction of $\mathcal{L}_0$.

Our second focus during our IAS visit was to prove a precise statement about the position of the leading particle. Let $\mathcal{N}_t(\rho)$ be the number of particles alive in the system at time t. We will index the particles by the integers $1, \ldots, \mathcal{N}_t$ and denote the position of particle k at time t by $X_k(t)$. We define the leading particle to be

$$R_t(\rho) = \max_{k \in [\mathcal{N}_t]} \|X_u(t)\|$$

When $\rho = 0$, it is known that

$$\lim_{t\to\infty} \frac{R_t(0)}{t} = \sqrt{\frac{\lambda(0)}{2}}$$

where $\lambda(0)$ is defined, as above, to be the top eigenvalue of the associated operator. Our main result proves that when $0 < \rho < \rho_c$, the leading particle moves as $t \to \infty$ with a speed faster than ρ, almost surely. We obtain the formula for the speed of this rightmost particle, which depends on ρ and the top eigenvalue $\lambda(0)$.

To analyze the leading particle location, we adapt the probabilistic method developed in [2] and [1], and define advantageous paths particles may take to get to positions $x = c\mathbf{w}t$ at time t, where $\mathbf{w}$ is a unit vector in $\mathbb{R}^d$, $c > 0$. These paths the particles take depend on c and ρ.

By large deviation theory, the number of particles along a trajectory $F(s)$, in expectation, is given by

$$\exp\left(\int_0^T g(F(s) - \rho \mathbf{v} s)\, ds - \frac{1}{2}\int_0^T \|\nabla F(s)\|^2\, ds\right) \tag{6}$$

for $F : [0,T] \to \mathbb{R}^d$ suitably 'nice' with $F(0) = 0$ and T large. Trajectories that have an $O(1)$ number of particles will balance the growth component and the penalty component of the exponent. This leads us to guess that the leading particle boundary satisfies the balance

$$\int_0^T g(F(s) - \rho \mathbf{v} s)\, ds - \frac{1}{2}\int_0^T \|\nabla F(s)\|^2\, ds = 0. \tag{7}$$

Since g is supported on $\mathcal{D}_0$, we focus our analysis on piecewise linear functions of the form

$$F(s) = \begin{cases} \rho \mathbf{v} s & s \in [0, \alpha T] \\ \frac{cs}{1-\alpha}\mathbf{w} - \frac{\rho \alpha s}{1-\alpha}\mathbf{v} + C(T) & s \in [\alpha T, T] \end{cases}$$

where $C(T) = \frac{\alpha T}{1-\alpha}(\rho \mathbf{v} - c\mathbf{w})$. By finding a balancing α and, in turn, finding an optimal c, we can find the precise location of the leading particle in this inhomogeneous BBM process.

Future Work

Going forward, we will complete the proof of the asymptotic speed of $R_t(\rho)$. In addition to computing the asymptotic speed of the leading particle almost surely, we will continue to study the paths followed by particles in the inhomogeneous BBM in more detail. By using a class of paths in $C[0,1]$ and then scaling the space-time domain, we can bound probabilities of large deviation events where particles in this inhomogeneous BBM follow unlikely paths. This will allow us to prove a precise growth rate of the total number of particles along paths in the specified class.

By thinking of this problem in a moving frame, we were able to formulate a parabolic PDE with drift $\rho > 0$ and compactly-supported linear potential. Our next step in this direction is to obtain the precise global asymptotics for this parabolic PDE with drift, and thus obtain a precise asymptotic for the density of our inhomogeneous BBM, similar to those obtained in [4]. From this, we can obtain asymptotics of all higher moments and the limiting distribution of particles inside the front, as well as the structure of the population all the way up to the front. These results will give us another way of obtaining information about the population.

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References

[1] Julien Berestycki, Éric Brunet, John W. Harris, Simon C. Harris, and Matthew I. Roberts. Growth rates of the population in a branching Brownian motion with an inhomogeneous breeding potential. *Stochastic Process. Appl.*, 125(5):2096–2145, 2015.

[2] J. W. Harris and S. C. Harris. Branching Brownian motion with an inhomogeneous breeding potential. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(3):793–801, 2009.

[3] Pratima Hebbar and Leonid Koralov. Distribution of particles near the front in supercritical branching brownian motion with compactly supported branching, 2026.

[4] L. Koralov and B. Vainberg. Global limit theorem for parabolic equations with a potential. *SIAM J. Math. Anal.*, 54(2):2097–2113, 2022.