Quantum Query Complexity
PCMI Graduate Summer School 2023
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Course page: https://yassine-hamoudi.github.io/pcmi2023/

Problem Session 4
The adversary method and its dual

Problem 1 (Simplifying the dual adversary)

Recall the dual formulation of the adversary method:

\[
\text{Adv}(f) = \min_{\{w(x,i)\}} \max_x \sum_i \|w(x,i)\|^2 \\
\text{s.t. } \sum_{i:x_i \neq y_i} \langle w(x,i) | w(y,i) \rangle = 1 \forall x, y
\]

The goal of this problem is to show that the above program is equivalent to the next one (which may come in handy in Problems 3 and 4):

\[
\text{Adv}^*(f) = \min_{\{w(x,i)\}} \sqrt{C_0 C_1} \\
\text{s.t. } C_0 = \max_{x:f(x)=0} \sum_i \|w(x,i)\|^2 \\
C_1 = \max_{x:f(x)=1} \sum_i \|w(x,i)\|^2 \\
\sum_{i:x_i \neq y_i} \langle w(x,i) | w(y,i) \rangle = 1 \forall x, y, f(x) \neq f(y)
\]

**Question 1.** Show that \(\text{Adv}^*(f) \leq \text{Adv}(f)\).

**Question 2.** Let \(\{w(x,i)\}\) be a feasible solution to the first program. Define \(C_0 = \max_{x:f(x)=0} \sum_i \|w(x,i)\|^2\) and \(C_1 = \max_{x:f(x)=1} \sum_i \|w(x,i)\|^2\) (the solution has value \(\max\{C_0, C_1\}\)). Show that there exists another feasible solution to the same program of value \(\sqrt{C_0 C_1}\).

**Question 3.** Let \(\{w(x,i)\}\) be a feasible solution to the second program. Define \(|w(x,i)| = |w(x,i)|_x \oplus f(x)|. Show that it satisfies \(\sum_{i:x_i \neq y_i} \langle w(x,i) | w(y,i) \rangle = 1 f(x) \neq f(y)\) for all \(x, y\).

**Question 4.** Conclude that \(\text{Adv}^*(f) = \text{Adv}(f)\).

Problem 2 (Connectivity)

Consider the function \(\text{Connectivity} : \{0, 1\}^2 \rightarrow \{0, 1\}\) whose quantum query complexity was shown to be \(\Omega(n^{3/2})\) in the last problem session. The goal of the next questions is to give a matching upper bound by constructing a feasible solution to the dual adversary program.

Define \(G_0\) as the set of input graphs consisting of exactly two connected components and \(G_1\) as the set of inputs consisting of \(n\)-cycle graphs. For each edge query \(\{i, j\} \in \binom{n}{2}\), define the vector \(|w(x,\{i,j\})\rangle \in \text{span}\{|k\rangle : 1 \leq k \leq n\}\) as follows.
If \( x \in G_0 \) then:
\[
|w^{(x,\{i,j\})}\rangle = \begin{cases} 
0 & \text{if } i \text{ and } j \text{ are in the same connected component of the graph } x \\
|i\rangle - |j\rangle & \text{if } 1 \text{ and } i \text{ are in the same connected component, but not } i \text{ and } j \\
|j\rangle - |i\rangle & \text{if } 1 \text{ and } j \text{ are in the same connected component, but not } i \text{ and } j 
\end{cases}
\]

If \( x \in G_1 \) then fix any orientation of the cycle and set
\[
|w^{(x,\{i,j\})}\rangle = \begin{cases} 
0 & \text{if } \{i,j\} \text{ is not an edge of the graph } x \\
|i\rangle & \text{if } \{i,j\} \text{ is an edge and its orientation is } i \rightarrow j \\
|j\rangle & \text{if } \{i,j\} \text{ is an edge and its orientation is } j \rightarrow i 
\end{cases}
\]

**Question 1.** Show that for all \( x \in G_0 \), \( y \in G_1 \) we have
\[
\sum_{(i,j):x(i,j)\neq y(i,j)} \langle w^{(x,\{i,j\})}|w^{(y,\{i,j\})}\rangle = 0.
\]

**Question 2.** Modify the above construction such that it becomes a feasible solution to the dual adversary and show that \( Q(\text{Connectivity}) = O(n^{3/2}) \).

The above algorithm uses only \( O(\log n) \) qubits of memory\(^1\). This is in contrast to an older quantum algorithm\(^2\) that required \( O(n \log n) \) space. This result can be adapted to other graph problems, such as \( st\)-\text{Connectivity}.

**Problem 3** (Composition)

Given two functions \( f : \{0,1\}^n \rightarrow \{0,1\} \) and \( g : \{0,1\}^m \rightarrow \{0,1\} \), define their composition \( f \circ g : \{0,1\}^{n \times m} \rightarrow \{0,1\} \) as \( f \circ g(X) = f(g(X_{1,1}, \ldots, X_{1,m}), \ldots, g(X_{n,1}, \ldots, X_{n,m})) \).

A striking property of the adversary method is that \( \text{Adv}(f \circ g) = \text{Adv}(f) \text{Adv}(g) \). This problem studies some parts of the proof of this result.

**Question 1.** Show that \( \text{Adv}(f \circ g) \leq \text{Adv}(f) \text{Adv}(g) \).

*Hint:* Take any dual adversary solutions \( \{w^{(x,i)}_f\} \) and \( \{w^{(x,j)}_g\} \) for \( f \) and \( g \) respectively, and consider \( |w^{(X,\{i,j\})}_{f \circ g}| = |w^{(g(X_{1,1}, \ldots, g(X_{n,1}, \ldots, X_{n,m}))}_{f}|w^{(X,\{i,j\})}_g\rangle \).

**Question 2.** Suppose that \( f \) is the OR function. Show that \( \text{Adv}(f \circ g) \geq \sqrt{n} \cdot \text{Adv}(g) \).

*Hint:* Start from a primal adversary solution \( \Gamma \) for \( g \) and construct a primal adversary solution for \( f \circ g \).

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