Visualizing Eigenpolytopes of Graphs

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Motivation

- **Eigenpolytopes** lie in the intersection between geometry, spectral theory, and combinatorics.
- By looking at a (simple) graph, we can try to deduce the properties of eigenpolytopes, and vice versa.
- For certain graphs (e.g. the 1-skeleta of regular polytopes), the correlation between graphs and their eigenpolytopes is strong. However, this is not well understood in general.
- A classification of edge-transitive polytopes is still open. But, eigenpolytopes have been used to provide a complete classification of a subclass of edge-transitive polytopes. [2]

Definition: Eigenpolytope

Definition (Godsil [1])

Choose some $\lambda \in \mathrm{Spec}(G)$ and a basis $\{u_1, \ldots, u_d\} \subset \mathbb{R}^n$ of the λ-eigenspace of $A(G)$. The **eigenpolytope matrix** $M(G, \lambda) \in \mathrm{Mat}_{d\times n}(\mathbb{R})$ has rows given by $u_1, \ldots, u_d \in \mathbb{R}^n$. Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be the columns of $M(G, \lambda)$. As such,

$$M(G,\lambda) = \begin{pmatrix} - & u_1 & - \\ & \vdots & \\ - & u_d & - \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

The λ-**eigenpolytope** of G is the convex hull

$$\mathscr{P}_G(\lambda) := \mathrm{conv}\{v_1, \ldots, v_n\} \subset \mathbb{R}^d.$$

The combinatorial type of an eigenpolytope is invariant under the choice of basis of the eigenspace.

Bipartite graphs - Symmetric Spectra

Proposition

A graph G is bipartite if and only if $A(G)$, the adjacency matrix of G, has the property that if λ is an eigenvalue of $A(G)$, then $-\lambda$ is an eigenvalue with the same multiplicity as λ.

Proof Sketch:

$$A(G) = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}, \; A \in M_{m \times n}(\mathbb{R}), \; x = \begin{bmatrix} v \\ w \end{bmatrix}, \; v \in \mathbb{R}^n, \; w \in \mathbb{R}^m,$$

$$A(G)x = \begin{bmatrix} Aw \\ A^T v \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix}, \quad A(G) \begin{bmatrix} v \\ -w \end{bmatrix} = \begin{bmatrix} -Aw \\ A^T v \end{bmatrix} = -\lambda \begin{bmatrix} v \\ -w \end{bmatrix}$$

Corollary

The $\pm\lambda_1$-eigenpolytope, where λ_1 is the largest eigenvalue of $A(G)$, of a connected bipartite graph G is at most one-dimensional. (by Perron-Frobenius [1907, 1912] and Proposition)

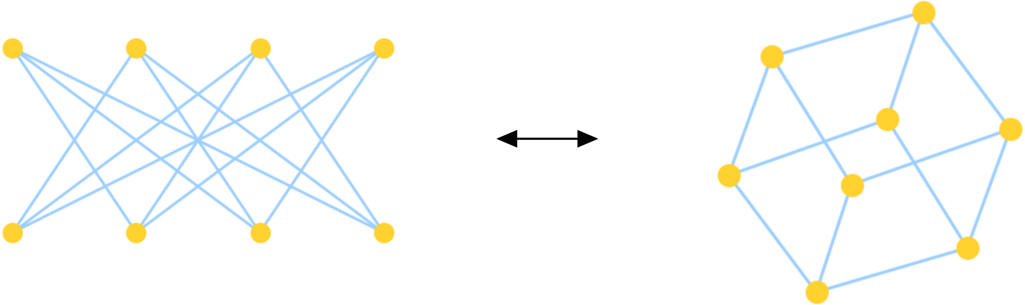
Initial Questions

Question

How do the $+\lambda$-eigenpolytope and $-\lambda$-eigenpolytope of bipartite graphs compare?

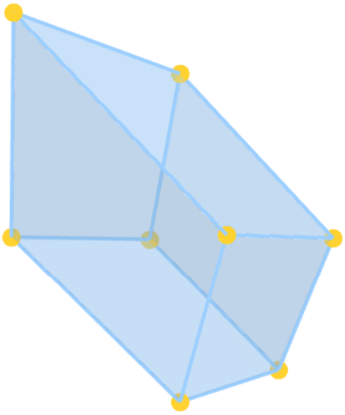
- This prompted us to study eigenpolytope pairs, besides the largest pair, of certain bipartite graphs, mainly eigenpolytope pairs of even cycles.

Pictures for $K_{4,4}$ with "vertical" edges deleted (Q_3)

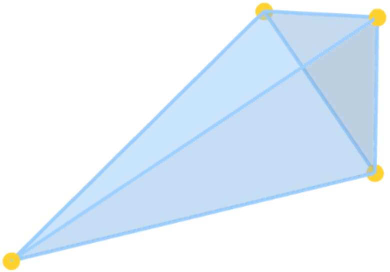


The next slide shows the eigenpolytopes of $K_{4,4}$ with "vertical" edges deleted when $\lambda = \pm 1$.

Eigenpolytopes of $K_{4,4}$ with "vertical" edges deleted (Q_3)

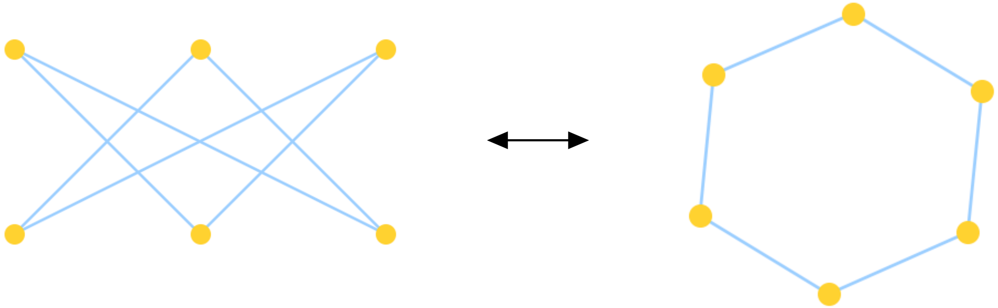


$\lambda_2 = 1$



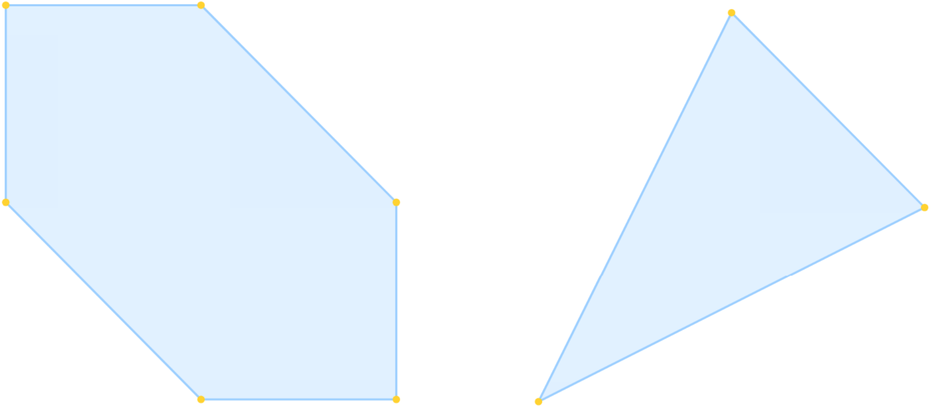
$-\lambda_2 = -1$

Pictures for $K_{3,3}$ with "vertical" edges deleted (C_6)



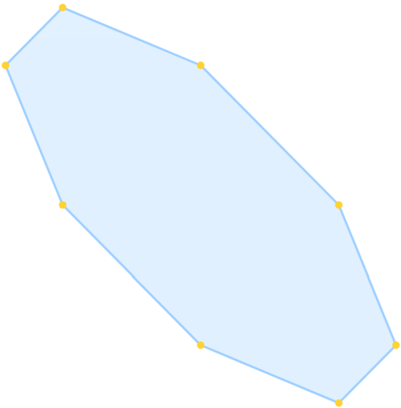
The next slide shows the eigenpolytopes of $K_{3,3}$ with "vertical" edges deleted when $\lambda = \pm 1$.

Eigenpolytopes of C_6

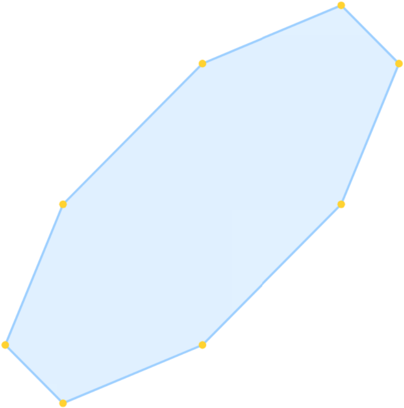


$\lambda_2 = 1$ $\qquad$ $-\lambda_2 = -1$

Eigenpolytopes of C_8

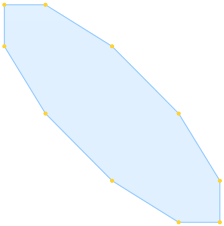


$\lambda_2 = 1.414$

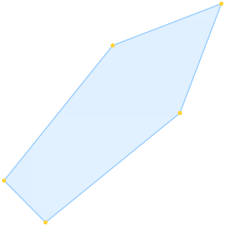


$-\lambda_2 = -1.414$

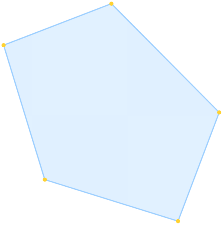
Eigenpolytopes of C_{10}



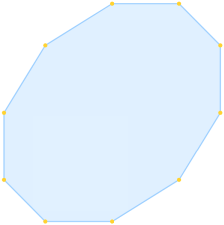
$\lambda_2 = 1.618$



$-\lambda_2 = -1.618$

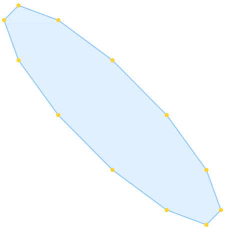


$\lambda_3 = 0.618$

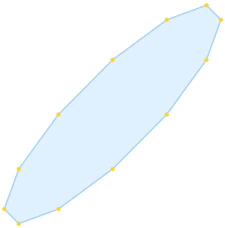


$-\lambda_3 = -0.618$

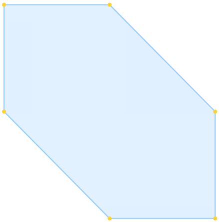
Eigenpolytopes of C_{12}



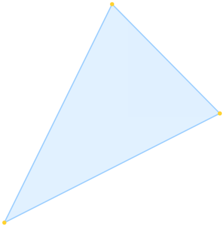
$\lambda_2 = 1.732$



$-\lambda_2 = -1.732$



$\lambda_3 = 1$



$-\lambda_3 = -1$

Observation

Let $G = C_{2n}$ and let λ_2 be the second largest eigenvalue of $A(G)$. In the above examples,

- *when n is* **odd**, *$\mathscr{P}_G(\lambda_2)$ is a $2n$-gon, while $\mathscr{P}_G(-\lambda_2)$ is an n-gon,*
- *and when n is* **even**, *both $\mathscr{P}_G(\lambda_2), \mathscr{P}_G(-\lambda_2)$ are $2n$-gons.*

Question

Given an eigenvalue λ of $A(C_n)$, what is the dimension of $\mathscr{P}_{C_n}(\lambda)$ and (if it is 2-dimensional) how many sides does the λ-eigenpolytope have?

Adjacency Matrices of Cycle Graphs

Consider the adjacency matrix of a cycle graph C_n (with rows/columns indexed by the vertices 0 through $n-1$):

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

This is a **circulant matrix**, a matrix whose eigenvalues and eigenvectors are well understood.

Eigenvectors of Cycle Graphs

The eigenvectors of $A(G)$ in $\mathbb{C}$ can be given by the roots of unity $\zeta_n^j := e^{\left(j\cdot\frac{2\pi}{n}\right)i}$:

$$w_j := (1, \zeta_n^j, \zeta_n^{2j}, \cdots \zeta_n^{(n-1)j}),$$

with corresponding eigenvalues

$$\mu_j = \zeta_n^{-j} + \zeta_n^{-(n-1)j} = \zeta_n^j + \zeta_n^{-j} = 2\cos\left(j\cdot\frac{2\pi}{n}\right).$$

Then it is clear that $\mu_j = \mu_{-j} = \mu_{n-j}$, so w_j, w_{-j} form a basis of the μ_j-eigenspace, and thus we obtain the following $\mathbb{R}$-basis:

$$u_{j+} := \frac{w_j + w_{-j}}{2} = \left(1, \cos\left(j\cdot\frac{2\pi}{n}\right), \cos\left(2j\cdot\frac{2\pi}{n}\right), \cdots, \cos\left((n-1)j\cdot\frac{2\pi}{n}\right)\right),$$

$$u_{j-} := \frac{w_j - w_{-j}}{2i} = \left(0, \sin\left(j\cdot\frac{2\pi}{n}\right), \sin\left(2j\cdot\frac{2\pi}{n}\right), \cdots, \sin\left((n-1)j\cdot\frac{2\pi}{n}\right)\right).$$

Characterization of the Eigenpolytopes of C_n

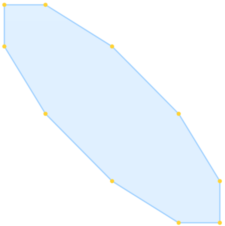
Theorem

Let $\mu_0 > \mu_1 > \cdots > \mu_\ell$ be the distinct eigenvalues of the adjacency matrix $A(C_n)$ of some cycle graph C_n in decreasing order. We have the following:

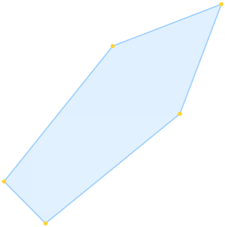
- *For $\mu_k \neq \pm 2$, the eigenpolytope $\mathscr{P}_{C_n}(\lambda_k)$ is a $\frac{n}{\gcd(k,n)}$-gon in $\mathbb{R}^2$.*
- *We have $\mu_0 = 2$, and the corresponding eigenpolytope $\mathscr{P}_{C_n}(2)$ is a point in $\mathbb{R}$.*
- *When $\mu_\ell = -2$ (i.e. when n is even) $\mathscr{P}_{C_n}(-2)$ is a segment in $\mathbb{R}$.*

- The first conclusion follows from the fact that the vertices of $\mathscr{P}_{C_n}(\mu_j)$ are points on the unit circle generated by ζ_n^j, so answering this reduces to asking what the size of $\langle \zeta_n^j \rangle$ is.

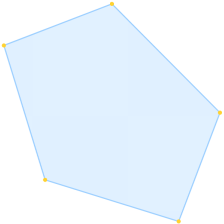
Eigenpolytopes of C_{10}



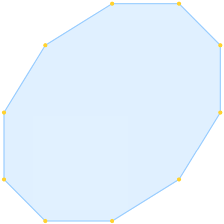
$\mu_1, \;\; \gcd(1, 10) = 1$



$\mu_4, \;\; \gcd(4, 10) = 2$

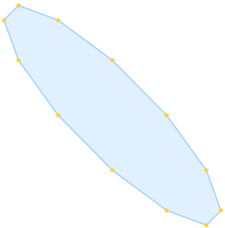


$\mu_2, \;\; \gcd(2, 10) = 2$

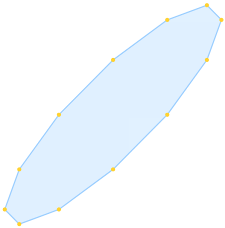


$\mu_3, \;\; \gcd(3, 10) = 1$

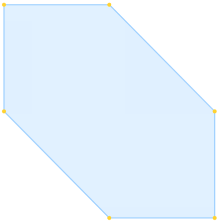
Eigenpolytopes of C_{12}



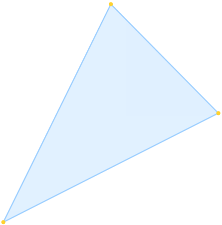
$\mu_1,\ \text{gcd}(1,12) = 1$



$\mu_5,\ \text{gcd}(5,12) = 1$



$\mu_2,\ \text{gcd}(2,12) = 2$

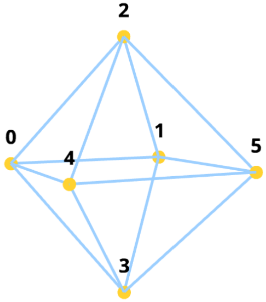


$\mu_4,\ \text{gcd}(4,12) = 4$

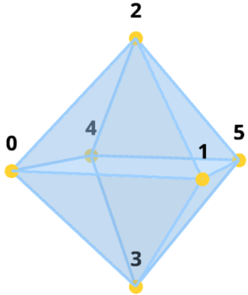
Winter's Question: Are Edges "preserved" in the λ_2-Eigenpolytope?

Question (Winter [2], Question 6.3)

Given an edge $\{i, j\} \in E(G)$, if v_i and v_j are distinct vertices of the λ_2-eigenpolytope $\mathscr{P}_G(\lambda_2)$, must conv$\{v_i, v_j\}$ also be an edge of $\mathscr{P}_G(\lambda_2)$?

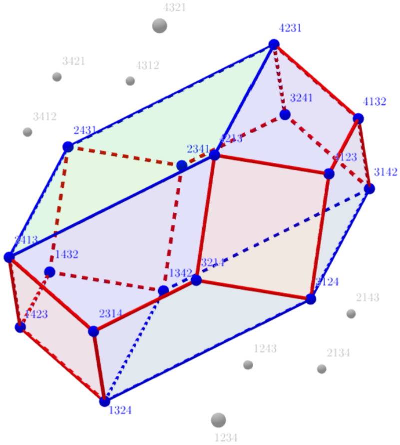


The Octahedral **Graph** $G = K_{2,2,2}$



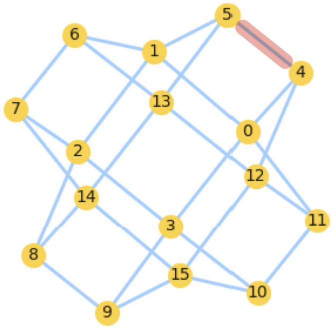
Eigenpolytope $\mathscr{P}_G(\lambda_2)$ for $\lambda_2 = 0$

Consider the following 3-dimensional (Bruhat interval) polytope $Q_{[1324,4231]} \subset \mathbb{R}^4$, constructed from the interval $[1324, 4231] \subset S_n$ (in the Bruhat Order):

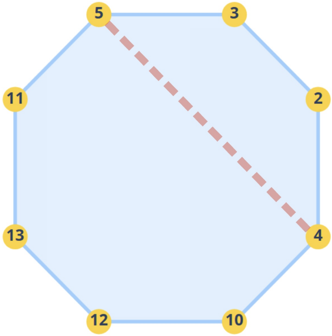


An Answer to Winter's Question - Continued

Taking G to be the 1-skeleton of $Q_{[1324,4231]}$, we obtain the following λ_2-eigenpolytope: ($\lambda_2 = 2.236$)



Graph G



Eigenpolytope $\mathscr{P}_G(\lambda_2)$ for $\lambda_2 = 2.236$

Note that the edge $\{4, 5\}$ is not an edge of the eigenpolytope. Thus, we now know the phenomenon noted by Winter is **not** true for all λ_2-eigenpolytopes.

Future Directions

1. Generalize the results about eigenpolytopes of cycle graphs to *circulant graphs*.
    - Given a *circulant graph* G and an eigenvalue λ, could we at least determine the *dimension* of $\mathscr{P}_G(\lambda)$ using properties of roots of unity?
2. Investigate how eigenpolytopes change under certain graph operations such as squaring and line graphs.

Acknowledgments

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