7. Exercices

R is a commutative (unital) ring.

Exercice 1. Let M be a locally free R–module of rank $2n \geq 2$ equipped with a regular quadratic form q. Show that locally for the flat topology that (M,q) is hyperbolic. [Hint: one can deal first with the case of a local ring where 2 is invertible].

Exercice 2. Prove Lemma 3.1.

Exercice 3. Let R' be a finite locally free R–algebra. Let $r \geq 0$ be an integer.

(1) Show that the R–functor $S \mapsto \mathrm{End}_{S\otimes_R R'}\Big(S \otimes_R R')^r\Big)^\times$ is representable by an affine R–group scheme. We denoted it by $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ (the Weil restriction).
(2) Show that the category of $\widetilde{G}$–torsors is equivalent to the category of locally free R'-modules of rank r.
(3) Give an interpretation of the map $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ and show that this map is not in general injective nor surjective.
(4) We denote by $M = (R')^r$ the underlying R–module. Construct a map $\widetilde{G} \to \mathrm{GL}(M)$ and discuss the cohomological consequences.

Exercice 3. (Special case of Grothendieck-Margaux [26]) Let G be an affine R–group scheme of finite presentation and let $R_\infty = \varinjlim_\lambda R_\lambda$ be a colimit of R–algebras. For each λ, we put $G_\lambda = G \times_R R_\lambda$ and similarly $G_\infty = G \times_R R_\infty$.

Define a map $\varinjlim H^1_{fppf}(R_\lambda, G_\lambda) \to H^1_{fppf}(R_\infty, G_\infty)$ and show that it is bijective.

Exercice 4. Let $d \geq 1$ be an integer and let R' be a $\mathbb{Z}/d\mathbb{Z}$–Galois extension. We denote by σ the canonical generator of $\mathbb{Z}/d\mathbb{Z}$.

(1) Show that the formula $N(y) = y\,\sigma(y) \dots \sigma^{r-1}(y)$ defines a group scheme homomorphism $N : R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m$.
(2) Show that $1 \to \ker(N) \to R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m \to 1$ is an exact sequence of R–group schemes.
(3) Deduce an exact sequence involving $H^1(R, \ker(N))$.
(4) Show that the flat quotient of $R_{R'/R}(\mathbb{G}_m)$ by $\mathbb{G}_m$ exists in the category of schemes and is isomorphic to $\ker(N)$.
(5) Construct an exact sequence
$$R^\times \to (R')^\times \xrightarrow{\sigma-1} \ker(N)(R) \to \ker\big(\mathrm{Pic}(R) \to \mathrm{Pic}(R')\big).$$
(6) Discuss the case of the coordinate ring $A = R[\ker(N)]$ of $\ker(N)$.
(7) For $R = \mathbb{R}$ and $S = \mathbb{C}$, is the $\mathbb{G}_m$-torsor $R_{S/R}(\mathbb{G}_m) \to R_{S/R}(\mathbb{G}_m)/\mathbb{G}_m$ trivial?

Exercice 5. Let B be standard Borel R–subgroup of upper triangular matrices of $\mathrm{GL}_{2,R}$.

(1) Show that the flat quotient of $\mathrm{GL}_{2,R}$ by B exists in the category of R–schemes and is isomorphic to the projective line.

(2) Deduce an exact sequence of pointed sets
$$1 \to B(R) \to \mathrm{GL}_2(R) \to \mathbb{P}^1(R) \to H^1_{fppf}(R, B) \to H^1_{fppf}(R, \mathrm{GL}_2).$$

(3) For R local, show that $H^1_{fppf}(R, B) = 1$ and that $H^1_{fppf}(R, \mathbb{G}_a) = 1$.