Torsors over affine curves

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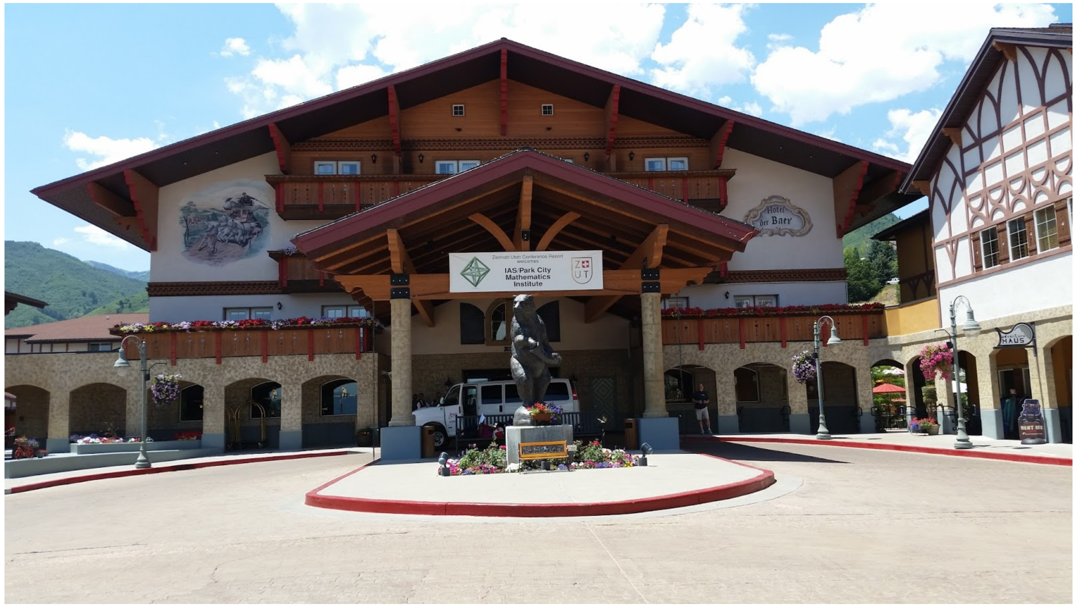

This is the first part

The Swan-Serre correspondence

- This is the correspondence between projective finite modules of finite rank and vector bundles, it arises from the case of a paracompact topological space [37].
  We explicit it in the setting of affine schemes following the book of Görtz-Wedhorn [18, ch. 11] up to slightly different conventions.

- **Vector group schemes.** Let R be a ring (commutative, unital).

  (a) Let M be an R–module. We denote by $\mathbf{V}(M)$ the affine R–scheme defined by $\mathbf{V}(M) = \mathrm{Spec}\big(\mathrm{Sym}^\bullet(M)\big)$; it is affine over R and represents the R–functor
  $S \mapsto \mathrm{Hom}_{S-mod}(M \otimes_R S, S) = \mathrm{Hom}_{R-mod}(M, S)$ [11, 9.4.9].

Vector group schemes

- $\mathbf{V}(M) = \mathsf{Spec}\big(\mathrm{Sym}^\bullet(M)\big)$;
- It is called the *vector group scheme* attached to M, this construction commutes with arbitrary base change of rings $R \to R'$.
- **Proposition** [32, I.4.6.1] The functor $M \to \mathbf{V}(M)$ induces an antiequivalence of categories between the category of R–modules and that of vector group schemes over R. An inverse functor is $\mathfrak{G} \mapsto \mathfrak{G}(R)$.

Vector group schemes II

- (b) We assume now that M is locally free of finite rank and denote by $M^\vee$ its dual. In this case $\mathrm{Sym}^\bullet(M)$ is of finite presentation (ibid, 9.4.11). Also the R–functor $S \mapsto M \otimes_R S$ is representable by the affine R–scheme $\mathbf{V}(M^\vee)$ which is also denoted by $\mathbf{W}(M)$ [32, I.4.6].
- *Remark*. Romagny has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathbf{W}(M)$ by a group scheme [29, th. 5.4.5].

Vector bundles

Let $r \geq 0$ be an integer.

- **Definition.** A vector bundle of rank r over $\mathrm{Spec}(R)$ is an affine R-scheme X such that there exists a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : \mathbf{V}((R_{f_i})^r) \xrightarrow{\sim} X \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j : \mathbf{V}((R_{f_i f_j})^r) \xrightarrow{\sim} \mathbf{V}(R^r_{f_i f_j})$ is a linear automorphism of $\mathbf{V}((R_{f_i f_j})^r)$ for $i, j = 1, \ldots, n$.
- **Theorem** (Swan-Serre's correspondence) The above functor $M \mapsto \mathbf{V}(M)$ induces an equivalence of categories between the groupoid of locally free R–modules of rank r and the groupoid of vector bundles over $\mathrm{Spec}(R)$ of rank r.

Examples of vector bundles

- (a) Given a smooth map of affine schemes $X = \mathrm{Spec}(S) \to Y = \mathrm{Spec}(R)$ of relative dimension $r \geq 1$, the tangent bundle $T_{X/Y} = \mathbf{V}(\Omega^1_{S/R})$ is a vector bundle over $\mathrm{Spec}(S)$ of dimension r [12, 16.5.12].
- (b) The tangent bundle of the real sphere $Z = \mathrm{Spec}\Big(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2)\Big)$ is an example of vector bundle of dimension 2 which is not trivial. It can be proven by differential topology (hairy ball theorem) but there are also algebraic proofs, see for instance [38]. A consequence is that Z cannot be equipped with a structure of real algebraic group.

Linear groups

- Let M be a locally free R–module of finite rank. We consider the R–algebra $\mathrm{End}_R(M) = M^\vee \otimes_R M$.
- It is locally free R–module of finite rank. over S so that we can consider the vector R–group scheme $\mathbf{V}\big(\mathrm{End}_R(M)\big)$ which is an R–functor in associative and unital algebras [11, 9.6.2].
- We consider the R–functor $S \mapsto \mathrm{Aut}_S(M \otimes_R S)$. It is representable by an open R–subscheme of $\mathbf{W}\big(\mathrm{End}_R(M)\big)$ which is denoted by $\mathrm{GL}(M)$ (*loc. cit.*, 9.6.4). We bear in mind that the action of the group scheme $\mathrm{GL}(M)$ on $\mathbf{W}(M)$ (resp. $\mathbf{V}(M)$) is a left (resp. right) action.
- In particular, we denote by $\mathrm{GL}_r = \mathrm{Aut}(R^r)$.

Linear groups II

- *Remark*. For R noetherian, Nitsure has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathrm{GL}(M)$ by a group scheme [28].

  (c) If $\mathcal{B}$ is a locally free $\mathcal{O}_S$–algebra of finite rank, we recall that the functor of invertible elements of $\mathcal{B}$ is representable by an affine S-group scheme which is a principal open subset of $\mathbf{W}(\mathcal{B})$. It is denoted by $\mathrm{GL}_1(\mathcal{B})$ [6, 2.4.2.1].

Cocycles

- Let M be a locally free R–module of rank r. We consider a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : (R_{f_i})^r \xrightarrow{\sim} M \times_R R_{f_i}$.
- Then the $R_{f_i f_j}$–isomorphism $\phi_i^{-1}\phi_j : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is linear so defines an element $g_{i,j} \in \mathrm{GL}_r(R_{f_i f_j})$. More precisely we have $(\phi_i^{-1}\phi_j)(v) = g_{i,j} \,.\, v$ for each $v \in (R_{f_i f_j})^r$ (in other words, $(R_{f_i f_j})^r$ is seen as column vectors).
- **Lemma.** The element $g = (g_{i,j})$ is a 1–*cocycle*, that is, satisfies the relation

$$g_{i,j}\, g_{j,k} = g_{i,k} \in \mathrm{GL}_r(R_{f_i f_j f_k})$$

for all $i, j, k = 1, \ldots, n$.

Cocycles, II

- **Lemma** The element $g = (g_{i,j})$ is a 1–*cocycle*, that is, satisfies the relation

$$g_{i,j}\, g_{j,k} = g_{i,k} \in \mathrm{GL}_r(R_{f_i f_j f_k})$$

for all $i, j, k = 1, \ldots, n$.

- **Proof** Over $R_{i,j,k}$ we have $\phi_i^{-1}\phi_k = (\phi_i^{-1}\phi_j) \circ (\phi_j^{-1}\phi_k) = L_{g_{i,j}} \circ L_{g_{j,k}} = L_{g_{i,j}g_{j,k}}$.
- If we replace the ϕ_i's by the $\phi_i' = \phi_i \circ g_i$ for $g_i \in G(R_{f_i})$, we get $g_{i,j}' = g_i^{-1} g_{i,j} g_j$ and we say that $(g_{i,j}')$ is cohomologous to $(g_{i,j})$.
- We denote by $\mathscr{U} = (\mathrm{Spec}(R_{f_i})_{i=1,..,n}$ the affine cover of $\mathrm{Spec}(R)$, by $Z^1(\mathscr{U}/R, \mathrm{GL}_r)$ the set of 1-cocycles and by $H^1(\mathscr{U}/R, \mathrm{GL}_r) = Z^1(\mathscr{U}/R, \mathrm{GL}_r)/\sim$ the set of 1–cocycles modulo the cohomology relation. The set $H^1(\mathscr{U}/R, \mathrm{GL}_r)$ is called the pointed set of Čech cohomology with respect to $\mathscr{U}$.

Cocycles, III

- $Z^1(\mathscr{U}/R, \mathrm{GL}_r)$ is the set of 1-cocycles and $H^1(\mathscr{U}/R, \mathrm{GL}_r) = Z^1(\mathscr{U}/R, \mathrm{GL}_r)/\sim$ the set of 1–cocycles modulo the cohomology relation.
- Summarizing we attached to the vector bundle $\mathbf{V}(M)$ of rank r a class $\gamma(M) \in H^1(\mathscr{U}/R, \mathrm{GL}_r)$.
- Conversely by Zariski glueing, we can attach to a cocycle $(g_{i,j})$ a vector bundle $\mathbf{V}_g$ over R of rank r equipped with trivializations $\phi_i : \mathbf{V}(R^r_{f_i}) \xrightarrow{\sim} \mathbf{V}_g \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j = g_{i,j}$.

  Lemma The pointed set $H^1(\mathscr{U}/R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over Spec(R) which are trivialized by $\mathscr{U}$.

Functorialities

- We can pass the limit of this construction over all affine open subsets of X. We define the pointed set $\check{H}^1_{Zar}(R, \mathrm{GL}_r) = \varinjlim_{\mathscr{U}} H^1(\mathscr{U}/R, \mathrm{GL}_r)$ of Čech non–abelian cohomology of GL_n with respect to the Zariski topology of Spec(R). By passage to the limit, the preceding Lemma implies that $\check{H}^1_{Zar}(R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over Spec(R).
- The principle is that nice constructions for vector bundles arise from homomorphisms of group schemes. Given a map $f : \mathrm{GL}_r \to \mathrm{GL}_s$, we can attach to a vector bundle $\mathbf{V}_g$ of rank r the vector bundle $\mathbf{V}_{f(g)}$ of rank s. This extends to a functor $X \mapsto f_*(X)$ from vector bundles of rank r from vector bundles to rank s.

Examples

- (a) *Direct sum*. If $r = r_1 + r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \oplus A_2$. We have then $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \oplus \mathbf{V}_2$.
  Of course, it can be done with $r = r_1 + \cdots + r_l$, in particular we have in the case $r = 1 + \cdots + 1$ the diagonal map $(\mathbb{G}_m)^r \to \mathrm{GL}_r$ which leads to decomposable vector bundles, that is, direct sum of rank one vector bundles.
- (b) *Tensor product.* If $r = r_1\, r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \otimes A_2$ (called the Knonecker product). We have then $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \otimes \mathbf{V}_2$.
- (c) *Determinant*. We put $\det(\mathbf{V}) = \det_*(\mathbf{V})$, this is the determinant bundle.

Dedekind ring

Let R be a Dedekind ring, that is, a noetherian domain such that the localization at each maximal ideal is a discrete valuation ring. The next result is a classical fact of commutative algebra, see [20, II.4, th. 13].

- **Theorem.** A locally free R–module of rank $r \geq 1$ is isomorphic to $R^{r-1} \oplus I$ for I an invertible R–module which is unique up to isomorphism.
- Since I is the determinant of $R^{r-1} \oplus I$, the last assertion is clear. Our goal is to provide a geometric proof of this statement.
- Firstly it states that vector bundles over R are decomposable and secondly that vector bundles over R are classified by their determinant. We limit ourself to prove the following corollary.
- **Corollary.** A locally free R–module of rank $r \geq 1$ is trivial if and only its determinant is trivial.

Dedekind rings, II

- **Corollary.** A locally free R–module of rank $r \geq 1$ is trivial if and only its determinant is trivial.
- We are given a vector bundle $\mathbf{V}(M)$. It trivializes over an open affine subset $\mathrm{Spec}(R_f)$ and we put $\Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_c\}$ where the $\mathfrak{p}_j$'s are maximal ideals of R. Let $\widehat{R}_{\mathfrak{p}_j}$ be the completion of the DVR $R_{\mathfrak{p}_i}$ and denote by $\widehat{K}_{\mathfrak{p}_j} = K \otimes_R \widehat{R}_{f_i}$ its fraction field.
- According to Nakayama lemma, the $\widehat{R}_{\mathfrak{p}_i}$–module $M \otimes_R \widehat{R}_{\mathfrak{p}_i}$ is free so we pick a trivialization $\phi_i : (\widehat{R}_{\mathfrak{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{R}_{\mathfrak{p}_i}$.
- On the other hand, let $\phi_f : (R_f)^r \xrightarrow{\sim} M \times_R R_f$ a trivialization. The linear map $\phi_f^{-1} \widehat{\phi}_i : (\widehat{K}_{\mathfrak{p}_i})^r \to (\widehat{K}_{\mathfrak{p}_i})^r$ gives rise to an element $g_i \in \mathrm{GL}_r(\widehat{K}_{\mathfrak{p}_i})$. Taking into account the choices, we attached to M an element of the double coset

$$c_\Sigma(R, \mathrm{GL}_r) := \mathrm{GL}_r(R_f) \backslash \prod_{j=1,\dots,c} \mathrm{GL}_r(\widehat{K}_{\mathfrak{p}_i}) / \mathrm{GL}_r(\widehat{R}_{\mathfrak{p}_i}).$$

- The map
$$\ker\Big(H^1(R,\mathrm{GL}_r)\to H^1(R_f,\mathrm{GL}_r)\Big)\to c_\Sigma(R,\mathrm{GL}_r)$$
is injective.

- We assume now that the determinant of $\mathbf{V}(M)$ is trivial so that (g_i) belongs by functoriality to the kernel of the map $\det_* : c_\Sigma(R,\mathrm{GL}_r)\to c_\Sigma(R,\mathbb{G}_m)=R_f^\times\backslash \prod_{j=1,\dots,c}(\widehat{K}_{\mathfrak{p}_i}^\times/\widehat{R}_{\mathfrak{p}_i}^\times)$.

- Up to change the trivializations we can then assume that $g_i\in\mathrm{SL}_n(\widehat{K}_{\mathfrak{p}_i})$ for $i=1,\dots,c$. Since $\mathrm{SL}_n(\widehat{K}_{\mathfrak{p}_i})$ is generated by elementary matrices and since R_f is dense in $\prod_i \widehat{K}_{\mathfrak{p}_i}$, it follows that $\mathrm{SL}_r(R_f)$ is dense in $\prod_{i=1,\dots,c}\mathrm{SL}_r(\widehat{K}_{\mathfrak{p}_i})$.

- On the other hand, each group $\mathrm{SL}_n(\widehat{R}_{\mathfrak{p}_i})$ is open (actually clopen) in $\mathrm{SL}_r(\widehat{K}_{\mathfrak{p}_i})$ so that $c_\Sigma(R,\mathrm{SL}_r)=1$. The injectivity fact enables us to conclude that $\mathbf{V}(M)$ is a trivial vector bundle.

Zariski topology is not fine enough

- We start with rhe example of quadratic bundles.
- A quadratic form over an R–module M is a map $q : M \longrightarrow R$ which satisfies

  (i) $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in R$, $x \in M$.

  (ii) The form $M \times M \longrightarrow R$,
  $(x, y) \mapsto b_q(x, y) = q(x + y) - q(x) - q(y)$ is (symmetric) bilinear.
- This is stable by arbitrary base change. The form q is *regular* if b_q induces an isomorphism $M \xrightarrow{\sim} M^\vee$. A fundamental example is the hyperbolic form $(V \oplus V^\vee, hyp)$ attached to a locally free R–module of finite rank defined by $hyp(v, \phi) \rightarrow \phi(v)$.

Quadratic bundles

- We are given a regular quadratic form (M, q) where M is locally free of rank r. It is tempting to make analogies with vector bundles and to use the orthogonal group scheme $O(q, M)$ which a closed subgroup scheme of $\mathrm{GL}(M)$.
- For an open cover $\mathscr{U}$ of R as above we define similarly $Z^1(\mathscr{U}/R, O(q, M))$ and $H^1(\mathscr{U}/R, O(q, M))$ (it makes sense for any R–group scheme). What we get is the following.
- **Lemma** The set $H^1(\mathscr{U}/R, O(q, M))$ classifies the isomorphism of regular quadratic forms (q', M') which are locally isomorphic over $\mathscr{U}$ to (q, M).
- This is nice but the point is that regular quadratic forms over R of dimension r have no reason to be locally isomorphic to (M, q) (e.g. this occurs already with $R = \mathbb{R}$, the field of real numbers). So the set $H^1_{Zar}(R, O(q, M))$ is only a piece of what we would like to obtain.

Functoriality issues

- If we have a map $f : G \to H$ of group schemes, we would like to have some control on the map $f_* : H^1_{Zar}(R, G) \to H^1_{Zar}(R, H)$.
- A basic example is the Kummer map $f_d : \mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ for an integer d. It gives rise to the multiplication by d mapping on the Picard group $\mathrm{Pic}(R)$. In terms of invertible modules, it corresponds to the map $M \mapsto M^{\otimes d}$.
- We would like to understand its kernel and its image. We can already say something about the kernel. Given $[M] \in \ker\big(\mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)\big)$, then there exists a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$.
- We define then the commutative group $A_d(R)$ of isomorphism classes of couples (M, θ) where M is an invertible R–module equipped with a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$.

Kummer sequence

- The group $A_d(R)$ is the group of isomorphism classes of couples (M,θ) where M is an invertible R–module equipped with a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$.
- We have a forgetful map $A(R) \to \mathrm{Pic}(R)$ and we claim that we have an exact sequence

$$R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)$$

with $\phi(r) = [(R,\theta_d)]$ where $\theta_d : R \xrightarrow{\sim} R^{\otimes d} = R$, $x \mapsto x^d$. We let this as exercise to the reader. We will see later that we can provide a cohomological meaning to the group $A_d(R)$.

Čech non-abelian cohomology

- Grothendieck-Serre's idea is to extend the notion of covers in algebraic geometry. They did it originally with étale covers but it turns out that the flat cover setting is simpler in a first approach (this is that of the book by Demazure-Gabriel [10, §III], there are other variants).
- **Definition.** A flat (or fppf= fidèlement plat de présentation finie) cover of R is a finite collection $(S_i)_{i\in I}$ of R–rings satisfying

  (i) S_i is a flat R–algebra of finite presentation for $i = 1, \dots, c$;

  (ii) $\mathrm{Spec}(R) = \bigcup_{i\in I} \mathrm{Im}\Big(\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\Big)$
- If we put $S = \prod_{i\in I} S_i$, the conditions rephrase by saying that S is a faithfully flat R–algebra of finite presentation. We can then always deal with a unique ring.
- Remark : for a partition $1 = f_1 + \dots + f_n$, then $(R_{f_j})_{j=1,\dots,n}$ is a flat cover of R and so is $R_{f_1} \times \dots \times R_{f_n}$.

Čech non-abelian cohomology II

- Let S is a faithfully flat R–algebra of finite presentation. We denote by $p_i^* : S \longrightarrow S \otimes_R S$ the coprojections $(i = 1, 2)$ and similarly $q_i^* : S \to S \otimes_R S \otimes_R S$ $(i = 1, 2, 3)$, $q_{i,j}^* : S \otimes_R S \longrightarrow S \otimes_R S \otimes_R S$ the partial coprojections $(i < j)$.
- Let G be an R–group scheme. A 1-*cocycle* for G and S/R is an element $g \in G\big(S \otimes_R S\big)$ satisfying

$$q_{1,2}^*(g)\, q_{2,3}^*(g) = q_{1,3}^*(g) \ \ \in \ \ G\big(S \otimes_R S \otimes_R S\big).$$

  We denote by $\mathrm{Z}^1(S/R, G)$ the pointed set of 1-cocycles of S/R with values in G (it is pointed by the trivial 1–cocycle).
- Two such cocycles $g, g' \in G(S)$ are cohomologous if there exists $h \in G(S)$ such that $g = p_1^*(h^{-1})\, g'\, p_2^*(h)$. We denote by $\check{\mathrm{H}}^1(S/R, G) = \mathrm{Z}^1(S/R, G)/\sim$ the pointed set of 1-cocycles up to cohomology equivalence.

Čech non-abelian cohomology III

- In the case of a Zariski cover given by a partition of 1, the definition is the same as before.
- We can pass to the limit on all flat covers of Spec(R) and define $\check{\mathrm{H}}^1_{fppf}(R, G) = \varinjlim \check{\mathrm{H}}^1(S/R, G)$.
- This construction is functorial in R and in the group scheme G.

Torsors

- A (right) G–torsor X (with respect to the flat topology) is an R-scheme equipped with a right action of G which satisfies the following properties :

  (i) the action map $X \times_R G \to X \times_R X$, $(x, g) \mapsto (x, x.g)$, is an isomorphism ;

  (ii) There exists a flat cover R'/R such that $X(R') \neq \emptyset$.
- The first condition reflects the simply transitivity of the action, we mean that $G(T)$ acts simply transitively on $X(T)$ for all R–rings T.
- The second condition is a local triviality condition. An example is $X = G$ with G acting by right translations, it is called the split G–torsor.

Torsors II

- The axioms for a right G-scheme X to be a torsor are :

  (i) the action map $X \times_R G \to X \times_R X$, $(x, g) \mapsto (x, x.g)$, is an isomorphism ;

  (ii) There exists a flat cover R'/R such that $X(R') \neq \emptyset$.
- If $X(R) \neq \emptyset$, a point $x \in X(R)$ defines an morphism $G \to X$, $\phi_x : g \mapsto x.g$ which is an isomorphism by the simple transitive property ; we say that X is trivial and that ϕ_x is a trivialization.
- Condition (ii) shows states an R–torsor X under G is locally trivial for the flat topology.
- A morphism of G–torsors $X \to Y$ is a G–equivariant map ; once again the simple transitivity condition shows that such a morphism is an isomorphism. Thus the category of G–torsors under G is a groupoid.

Torsors and cocycles II

- The R–functor of automorphisms of the trivial G–torsor G is representable by G (acting by left translations).
- We denote by $H^1_{fppf}(R, G)$ the set of isomorphism classes of G–torsors for the flat topology. If S is a flat cover R, we denote by $H^1_{fppf}(S/R, G)$ the subset of isomorphism classes of G–torsors trivialized over S.
- $H^1_{fppf}(R, G)$ the set of isomorphism classes of G–torsors for the flat topology. As in the vector bundle case, we shall construct a class map $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ as follows.

Torsors and cocycles

- Let X be a G–torsor over R equipped with a trivialization $\phi : G \times_R S \xrightarrow{\sim} X \times_R S$. Over $S \otimes_R S$, we have then two trivializations $p_1^*(\phi) : G \times_R (S \otimes_R S) \xrightarrow{\sim} X \times_R (S \otimes_R S)$ and $p_1^*(\phi)$. It follows that $p_1^*(\phi)^{-1} \circ p_2^*(\phi)$ is an automorphism of the trivial G–torsor over $S \otimes_R S$ so is the left translation by an element $g \in G(S \otimes_R S)$.
- A computation shows that g is a 1–cocycle ; also changing ϕ changes g by a cohomologous cocycle. The class map is then well-defined. Its study involves a glueing technique in the flat setting.

Interlude : faithfully flat descent

Let T be a faithfully flat extension of the ring R (not necessarily of finite presentation).

- We put $T^{\otimes d} = T \otimes_R T \cdots \otimes_R T$ (d times). One first important thing is that the Amitsur complex

$$0 \longrightarrow M \longrightarrow M \otimes_R T \xrightarrow{d_2} M \otimes_R T \otimes_R T \xrightarrow{d_2} M \otimes_R T^{\otimes 3} \ldots$$

is exact for each R–module M [25, III.1] where

- $d_n(m \otimes t_1 \otimes \cdots \otimes t_n) = \sum_{i=0,\ldots,n} (-1)^i m \otimes t_1 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n.$
- This implies in particular that for any affine R-scheme X, we have an identification

$$X(R) = \big\{ x \in X(T) \mid p_1^*(x) = p_2^*(x) \in X(T \otimes_R T) \big\}$$

which holds actually for any R-scheme.

Descent II

- Given a T–module N we consider the $T \otimes_R T$–modules $p_1^*(N) = T \otimes_R M$ and $p_1^*(N) = M \otimes_R T$.

  A descent data on N is an isomorphism $\varphi : p_1^*(N) \xrightarrow{\sim} p_2^*(N)$ of $T^{\otimes 2}$–modules such that the diagram

$$\begin{array}{ccccc} T \otimes_R T \otimes_R N & & \xrightarrow{\varphi_3} & & N \otimes_R T \otimes_R T \\ & \searrow^{\varphi_2} & & \nearrow_{\varphi_1} & \\ & & T \otimes_R N \otimes_R T & & \end{array}$$

  is commutative where

- • $\varphi_3(t_1 \otimes t_2 \otimes n) = \varphi(t_1 \otimes n) \otimes t_2$;
  • $\varphi_2(t_1 \otimes t_2 \otimes n) = t_2 \otimes \varphi(t_1 \otimes n)$;
  • $\varphi_1(t_1 \otimes n \otimes t_3) = t_1 \otimes \varphi(n \otimes t_3)$

Descent II

- A descent data on N is an isomorphism $\varphi : p_1^*(N) \xrightarrow{\sim} p_2^*(N)$ of $T^{\otimes 2}$–modules such that the diagram commutes

$$\begin{array}{ccc} T \otimes_R T \otimes_R N & \xrightarrow{\varphi_3} & N \otimes_R T \otimes_R T \\ & {\scriptstyle \varphi_2} \searrow \quad \nearrow {\scriptstyle \varphi_1} & \\ & T \otimes_R N \otimes_R T & \end{array}$$

- There is a clear notion of morphisms for T–modules equipped with descent data from T to R.
- If M is an R–module, the identity of M gives rises to a canonical isomorphism $can_M : p_1^*(M \otimes_R T) \xrightarrow{\sim} p_2^*(M \otimes_R T)$, this is a descent data.

Descent III

- **Faithfully flat descent theorem**

  (1) The functor $M \to (M \otimes_R T, can_M)$ is an equivalence of categories between the category of R–modules and that of T–modules with descent data. An inverse functor (the descent functor) is $(N, \varphi) \mapsto \{n \in N \mid n \otimes 1 = \varphi(1 \otimes n)\}$.
- (2) The functor above induces an equivalence of categories between the category of R–algebras (commutative, unital) and that of T–algebras (commutative, unital) with descent data.

Back to vector bundles

- An important example is the extension of Swan-Serre's correspondence. A consequence of the faithfully flat descent theorem (and of the fact that the property to be locally free of rank r is local for the flat topology [39, Tag 05B2]) is the following.
- **Theorem** Let $r \geq 0$ be an integer.

  (1) Let M be a locally free R–module of rank r. Then the R–functor
  $S \mapsto \mathrm{Isom}_{S-mod}(S^r, M \otimes_R S)$ is representable by a GL_r–torsor X^M over $\mathrm{Spec}(R)$.
- (2) The functor $M \mapsto X^M$ induces an equivalence of categories between the groupoid of locally free R–modules of rank r and the category of GL_r–torsors over $\mathrm{Spec}(R)$.
- This implies that the GL_r–torsors are the same with flat topology or with Zariski topology.

Hilbert-Grothendieck 90

- The GL_r–torsors are the same with flat topology or with Zariski topology.
- **Corollary** (Hilbert-Grothendieck 90) We have $H^1_{Zar}(R, \mathrm{GL}_r) = H^1_{fppf}(R, \mathrm{GL}_r)$.
- In particular, if R is a local (or semilocal ring), we have $H^1_{fppf}(R, \mathrm{GL}_r) = 1$.

Back to torsors and cocycles

- **Lemma.** The class map $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ is injective.
- We consider only the kernel for simplicity. If (X, ϕ) gives rise to a cocycle which is cohomologous to the trivial cocycle, it means that there exists a trivialization $\phi' : G \times_R S \xrightarrow{\sim} X \times_R S$ such that the associated cocycle is trivial. We put $x = \phi'(1) \in X(S)$. Then $p_1^*(x) = p_2^*(x) = 1$. Since $X(R)$ identifies with $\{x \in X(S) \mid p_1^*(x) = p_2^*(x)\}$, we conclude that $X(R)$ is non-empty.
- **Theorem** If G is affine, the class map $H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ is an isomorphism.
- Note that by passing to the limit on the flat covers, we get a bijection $H^1_{fppf}(R, G) \to \check{H}^1_{fppf}(R, G)$.
- The fact that we can descend torsors under an affine scheme is a consequence of the faithfully flat descent theorem.

Twisting

We sketch the proof of the descent of torsors.

- We are given a cocycle $g \in G(S \otimes_R S)$. We consider the map $L_g^* : (S \otimes_R S)[G] \xrightarrow{\sim} (S \otimes_R S)[G]$ and define φ_g by the diagram

$$\begin{array}{ccc} S \otimes_R S[G] & \xrightarrow[\sim]{\varphi_g} & S[G] \otimes_R S \\ \cong \downarrow \alpha & & \cong \downarrow \beta \\ (S \otimes_R S)[G] & \xrightarrow[\sim]{L_g^*} & (S \otimes_R S)[G] \end{array}$$

  where $\alpha(s_1 \otimes f) = (s_1 \otimes 1)p_2^*(f)$ and $\beta(f \otimes s_2) = p_1^*(f)(1 \otimes s_2)$.

- The cocycle condition implies that ϕ_g is a descent data for the S–algebra $S[G]$. The descent theorem defines an R–algebra $R[X]$ and X is actually a G–torsor denoted by E_g.

Twisting II

- This construction is a special case of *Twisting*. More generally, if Y is an affine R–scheme equipped with a left action of G, then the action map $g : Y \times_R (S \otimes_R S) \xrightarrow{\sim} Y \times_R (S \otimes_R S)$ defines a descent data. This gives rises to the twist of Y_g of Y by the one cocycle g. It is affine over R.

- A special case is the action of G on itself by inner automorphisms, G_g is called the twisted R–group scheme ; it acts on Y_g for Y as above.

- (a) The above construction do not depend of choices of cocycles or of trivializations. We can define for a G–torsor E the twist ${}^E Y$ and ${}^E G$.

- (b) In practice, the affiness assumption is too strong. More generally we can twist G–schemes equipped with an ample invertible G-linearized bundle, see [5, §6, th. 7 and §10, lemma 6] for details).

Examples

- **(a) Vector group schemes**. Let M be a finite locally free R–module of finite rank, we claim that $\check{H}^1(R, \mathbf{W}(M)) = 0$ so that each $\mathbf{W}(M)$–torsor is trivial.
- We are given a flat cover S/R. Since the complex $M \otimes_R S \xrightarrow{p_1^* - p_2^*} M \otimes_R S \otimes_R S \to M \otimes_R S \otimes_R S \otimes_R S$ is exact, each cocycle $g \in \mathbf{W}(M)(S \otimes_R S) = M \otimes_R S \otimes_R S$ is a coboundary. Thus $\check{H}^1(S/R, \mathbf{W}(M)) = 0$ and $\check{H}^1(R, \mathbf{W}(M)) = 0$.
- (b) An important case is when $G = \Gamma_R$, that is, the *finite constant group scheme* attached to an abstract finite group Γ. We mean that $G(S)$ is the group of locally constant functions $\mathrm{Spec}(S) \to \Gamma$. In other words, $G = \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)_\gamma$ so that its coordinate ring identifies with $R^{(\Gamma)}$.
- In this case a Γ_R–torsor $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the same thing than a Galois Γ–algebra S and is called often a Galois cover. A special case is that of a finite Galois extension L/k of fields of

Examples, II

- (c) As for GL_r, a special nice case is the case of *forms*, that is when G is the automorphism group of some algebraic structure.
  For example, the orthogonal group scheme O_{2n} is the automorphism group of the hyperbolic quadratic form attached to R^n. As regular quadratic forms of rank $2n$ are locally isomorphic to the hyperbolic form for the flat topology, descent theory provides an equivalence of categories between the groupoid of regular quadratic forms of rank $2r$ and $H^1_{fppf}(R, O_{2n})$.

Examples, III

- (d) Another important example is that of the symmetric group S_n. For any R–algebra S, the group $S_n(S)$ is the automorphism group of the S–algebra $S^n = S \times \cdots \times S$ (n–times).
- Since finite étale algebras of degree n are locally isomorphic to R^d for the étale topology, the same yoga shows that there is an equivalence of categories between the category of S_n–torsors and that of finite étale R–algebras of rank n.
- The inverse functor is defined by descent but can be described explicitely. This is the Galois closure construction done by Serre in [33, §1.5], see also [2].

Functoriality issues

Let $G \to H$ be a monomorphism of R–group schemes.

- We say that an R-scheme X equipped with a map $f : H \to X$ is a *flat quotient* of H by G if for each R–algebra S the map $H(S) \to X(S)$ induces an injective map $H(S)/G(S) \hookrightarrow X(S)$ and
- if for each $x \in X(S)$, there exists a flat cover S' of S such that $x_{S'}$ belongs to the image of $H(S') \to X(S')$ (we say that f is "couvrant" in French).
- If it exists, a flat quotient is unique (up to unique isomorphism) ; furthermore, if G is normal in H, then X carries a natural structure of R–group schemes, we say in this case that $1 \to G \to H \to X \to 1$ is an exact sequence of R–group schemes (for the flat topology).

Assume that X is the flat quotient of H by G.

- **Lemma**

  (1) The map $H \to X$ is a G–torsor.

- (2) There is an exact sequence of pointed sets

  $$1 \to G(R) \to H(R) \to X(R) \xrightarrow{\varphi} H^1_{fppf}(R, G) \to H^1_{fppf}(R, H)$$

  where $\varphi(x) = [f^{-1}(x)]$.

- Remark 1. If X is affine (or is equipped with an ample G–linearized invertible sheaf), then the category of G–torsors over Spec(R) is equivalent to the category of couples (F, x) where F is a H–torsor and $x \in ({}^F X)(R)$.

- Remark 2. If G is normal in H, then X has natural structure of R–group scheme. In this case (a) rephrases by saying that the category of G–torsors over Spec(R) is equivalent to the category of couples (F, ϕ) where F is a H–torsor and ϕ a trivialization of the X–torsor ${}^F X$.

Functorialities II

- (c) Using the extended Swan-Serre correspondence, an example is that category of SL_r-torsors is equivalent to the category of pairs (M, θ) where M is a locally free R–module of rank r and $\theta : R \xrightarrow{\sim} \Lambda^r(M)$ is a trivialization of the determinant of M.
- For an integer d, we have the Kummer exact sequence $1 \to \mu_d \to \mathbb{G}_m \xrightarrow{\times d} \mathbb{G}_m \to 1$.
- Similarly the category of μ_d–torsors is equivalent to the category of pairs (M, θ) where M is an invertible R–module and $\theta : R \xrightarrow{\sim} M^{\otimes r}$ a trivialization.

Étale covers

- An étale morphism of rings $R \longrightarrow S$ is a smooth morphism of relative dimension zero [27, §I.3].
- There are several alternative definitions, for example, S is a flat R–module such that for each R–field F, then $S \otimes_R F$ is an étale F–algebra (i.e. a finite geometrically reduced F–algebra).
- *Examples.* (a) A localization morphism $R \longrightarrow R_f$ is étale.
- If d is invertible in R, the Kummer morphism $\mathbb{G}_m \longrightarrow \mathbb{G}_m$, $t \mapsto t^d$ is étale.
- More generally, if d is invertible in R and $r \in R^\times$, then $S = R[x]/(x^d - r)$ is a finite étale R–algebra.

Torsors under smooth group schemes

- For an R–group scheme G, we define the subset $H^1_{\text{ét}}(R,G)$ of $\check{H}^1_{fppf}(R,G)$ of classes ot torsors which are trivialized by an étale cover. We define similarly $\check{H}^1_{\text{ét}}(R,G)$
- **Proposition.** If G is (affine) smooth, then we have $H^1_{\text{ét}}(R,G) = H^1_{fppf}(R,G)$.
- Sketch. Smoothness is a local property with respect to flat topology so that any G–torsor E is smooth affine over R. According to the existence of quasi-sections [12, 17.16.3], E admits locally sections with respect of the étale topology.

Isotrivial torsors and Galois cohomology

- We are given a Galois R–algebra S of group Γ. The action isomorphism $\mathrm{Spec}(S) \times_R \Gamma_S \xrightarrow{\sim} \mathrm{Spec}(S) \times_R \mathrm{Spec}(S)$ reads as the isomorphism $S \otimes_R S \xrightarrow{\sim} S \otimes_R R^{(\Gamma)} = S^{(\Gamma)}$.
- A 1-cocycle is then an element $z = (z_\gamma)_{\gamma \in \Gamma} \in G(S \otimes_R S) = G(S)^{(\Gamma)}$ satisfying a certain relation.
- Since Γ acts on the left on S, it acts as well on the left on $G(S)$.

  Lemma A Γ-uple $z = (z_\sigma)_{\sigma \in \Gamma} \in G(S^{(\Gamma)}) = G(S)^{(\Gamma)}$ is a 1–cocycle for S/R if and only if

$$z_{\sigma\tau} = z_\sigma \, \sigma(z_\tau)$$

  for all $\sigma, \tau \in \Gamma$.

Galois cohomology II

- We find that $Z^1(S/R, G)$ is the set of Galois cocycles $Z^1(\Gamma, G(S))$ and that $\check{H}^1(S/R, G)$ is the set of non-abelian Galois cohomology $H^1(\Gamma, G(S)) = Z^1(\Gamma, G(S))/\sim$ where two cocycles z, z' are cohomologous if $z_\gamma = g^{-1}\, z'_\gamma\, \sigma(g)$ for some $g \in G(S)$.
- An interesting case is when G is the constant group scheme associated to an abstract group Θ. In this case, we have $Z^1(S/R, G) = \mathrm{Hom}_{R-gp}(\Gamma_S, \Theta_S)$ and $\check{H}^1(S/R, G) = \mathrm{Hom}_{S-gp}(\Gamma_S, \Theta_S)/\Theta_R(S)$. In particular, if S is connected, we have $Z^1(S/R, G) = \mathrm{Hom}_{R-gp}(\Gamma, \Theta)$ and $\check{H}^1(S/R, G) = \mathrm{Hom}_{gp}(\Gamma, \Theta)/\Theta$.
- Galois descent is then a special case of faithfully flat descent. The reader can check that the category of R-modules is equivalent to the category of couples (N, ρ) where N is a S-module equipped with a semilinear action of Γ (i.e. $\rho(\sigma)(\lambda\,.\,n) = \sigma(\lambda)\,.\,\rho(\sigma)(n)$).

Isotrivial torsors

- We say that torsor E under an R–group scheme G is isotrivial if it is split by a Galois finite étale cover. This is subclass of torsors which can be explicited by Galois cohomology computations and this is a preliminary question is it is the case.
- For example, for the ring of Laurent polynomials in characteristic zero and a reductive group scheme, this is the case [17].
- Special case : loop torsors.

Tomorrow

- Torsors on affine curves over an algebraically closed field ;
- Torsors on the affine line and on $\mathbb{G}_m$.

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