Monoidal Structures on $GL(2)$-Modules and Abstractly Automorphic Representations

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Introduction

- L-functions are important objects.
- When the L-functions are equal: get deep connections across mathematics.
  - Modularity Theorem $\implies$ Fermat's Last Theorem.
- In this talk: only automorphic L-functions.

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  $\Longrightarrow$ Exotic multiplicative structure on p-adic and automorphic representations.
- Part of PhD thesis, under supervision of J. Bernstein. Relevant part recently uploaded to arXiv as [Dor20].

Overview

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  - For $V \in \mathrm{Mod}(G)$, each $v \in V$ is *smooth*, i.e. fixed by some neighborhood of unity.
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- An irreducible G-module is generic iff it has a Kirillov model.

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  3. Integrate:
$$Z_{\mathrm{GJ}}(\Psi, \beta, s) = \int_{\mathrm{GL}_2(F)} \Psi(g)\beta(g)|\det(g)|^{s+\frac{1}{2}}\,\mathrm{d}^{\times}g.$$
- Get meromorphic zeta integral. Use GCD.

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$$Z_{\mathrm{JL}}(W, v, s) = \int_{F^\times} W_v(y)|y|^{s-\frac{1}{2}}\, \mathrm{d}^\times y.$$

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$$V$$

GJ = JL

Claim

For generic irreducible V, the two spaces are canonically isomorphic:

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- Remarkable – non-linear in V!

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Hidden action exists: Y is a $G \times G \times G$-module.

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- Have Weil representation $S(U \otimes U \otimes e_2) = S(\mathrm{M}_2(F))$.
- Compact induction of $S(\mathrm{M}_2(F))$ from $\mathrm{GL}_2(F)^{3,\det=1}$ to $\mathrm{GL}_2(F)^3$ gives Y.

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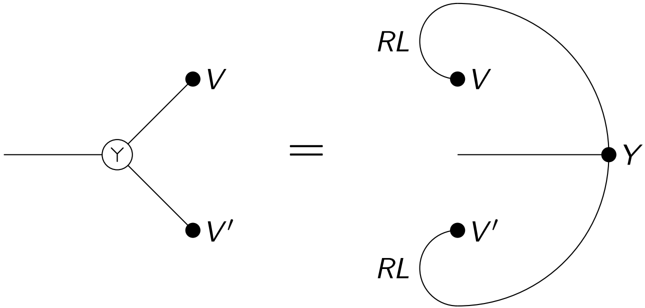
- Think of it as a bi-functor:

$$V \circledY V' = V \otimes_G Y \otimes_G V'$$

- Saw stuff like this before: tensor products.



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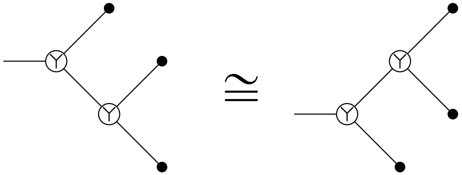

The unit

- Unit is given by Whittaker space:

$$\mathbb{1}_{\curlyvee} = \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is locally const and} \\ \text{compact supp mod } U_2(F) \end{array}, \quad f\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) = e(u) \cdot f(g) \right\}.$$

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- Important takeaway:
  - Irreducible V has $\dim \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) = 0, 1$, exactly if V is generic.
  - Choice of map $\mathbb{1}_{\curlyvee} \to V$ is same data as Kirillov model.

Representations as algebras

- GJ vs JL is now

$$V \otimes \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) = V \circledcurlyvee \mathbb{1}_{\curlyvee} \otimes \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) \to V \circledcurlyvee V$$

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  is an isomorphism.

- Turns out generic V are commutative algebras (in fact, idempotents).
- Follows because $\mathbb{1}_{\curlyvee} \twoheadrightarrow V$ is surjective.

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- New product $\circledY$ behaves like second case, despite $\mathrm{GL}_2(F)$ not being commutative.

Apology

- Global theory deserves a whole lecture on its own.
- We will give a sample instead...

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- Let F be a global function field, $\mathrm{char} F \neq 2$. Let $\mathbb{A} = \mathbb{A}_F$, $G = \mathrm{GL}_2(\mathbb{A})$.
- Category $\mathrm{Mod}(G)$ of smooth G-modules.

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  - Warning: the category is huge, mostly unimportant!
  - Good only as an ambient space for the automorphic representations to live in.
- An irreducible $(\pi, V) \in \mathrm{Mod}(G)$ is *automorphic* iff it is a subquotient of $\widetilde{\mathscr{S}}$, where

$$\mathscr{S} = S(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})).$$

Monoidal structure

- Fix $e : \mathbb{A}/F \to \mathbb{C}^\times$.
- Space $Y = S(\mathrm{M}_2(\mathbb{A}) \times \mathbb{A}^\times)$ is a $G \times G \times G$-module via Weil representation.
- Have global symmetric monoidal $\circledY$ induced from the local theory, given by

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- Unit is global Whittaker space,

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- The theta functional can be used to construct a product
$$\mathcal{J} \circledY \mathcal{J} \to \mathcal{J}.$$
- The product makes $\mathcal{J}$ into a commutative $\circledY$-algebra.
  - Unit map $\mathbb{1}_{\curlyvee} \to \mathcal{J}$ is contragradient to sending an automorphic form to its Whittaker function.

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- Have a category of $\mathcal{J}$-modules in the symmetric monoidal category $\mathrm{Mod}(G)$:

$$\mathrm{Mod}^{\mathrm{aut}}(G),$$

  "abstractly automorphic representations".

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- Decomposes into cuspidal and Eisenstein components, the same way p-adic representations do.
- ... And many more properties!

Questions?

References

Gal Dor.
Exotic Monoidal Structures and Abstractly Automorphic Representations for $\mathrm{GL}(2)$.
arXiv e-prints, page arXiv:2011.03313, November 2020.

Abstract

Consider the function field F of a smooth curve over $\mathbb{F}_q$, with $q \neq 2$. L-functions of automorphic representations of $\mathrm{GL}(2)$ over F are important objects for studying the arithmetic properties of the field F. Unfortunately, they can be defined in two different ways: one by Godement–Jacquet, and one by Jacquet–Langlands. Classically, one shows that the resulting L-functions coincide using a complicated computation.
Each of these L-functions is the GCD of a family of zeta integrals associated to test data. I will categorify the question, by showing that there is a correspondence between the two families of zeta integrals, instead of just their L-functions. The resulting comparison of test data will induce an exotic symmetric monoidal structure on the category of representations of $\mathrm{GL}(2)$.
It turns out that an appropriate space of automorphic functions is a commutative algebra with respect to this symmetric monoidal structure. I will outline this construction, and show how it can be used to construct a category of automorphic representations.