The Morse complex on singular spaces

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Glimpses of mathematics, now and then:
A celebration of Karen Uhlenbeck's 80th birthday
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Background: Morse Theory

Morse theory relates two seemingly separate topics

- topology (topological invariants of a smooth manifold), and
- analysis (critical points of a smooth function on that manifold).

The function must satisfy certain technical assumptions (e.g. Morse or Morse-Bott).

Motivation.

One might want to compute topological invariants of certain manifolds (e.g. Lie groups, moduli spaces, etc.).

One might want to find critical points of interesting functions (e.g. geodesics on manifolds, solutions to other variational problems).

These ideas have been applied to solve many interesting problems in topology, geometry and analysis.

Background: Morse Theory

Example. Let T denote a 2-torus and let $f\colon T \to \mathbb{R}$ be the height function.

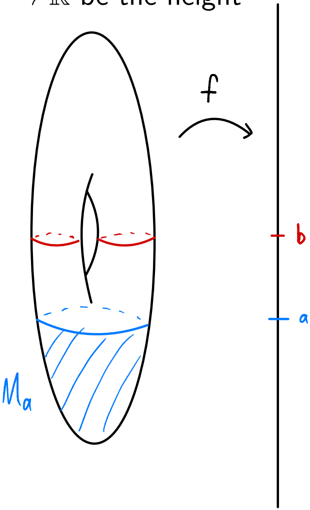
We can see that there are four critical points.

Given $a \in \mathbb{R}$, define

$$M_a = \{x \in M \mid f(x) \leq a\}.$$

Key observation.

The homotopy type of M_a changes when a crosses a critical value.

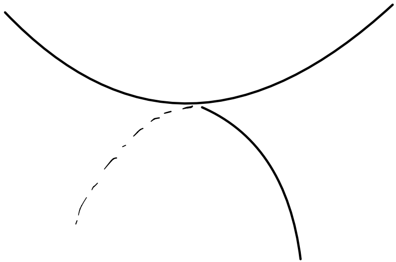


The main theorem of Morse theory makes this precise by describing this change in terms of handle attachments or cell attachments.

Morse Lemma. If M is finite-dimensional and x is a nondegenerate critical point with Morse index λ, then there exist local coordinates $(y_1, \ldots, y_n)$ such that

$$f(y) = -y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2.$$

Critical points of a Morse function locally look like saddle points.



Main Theorem of Morse Theory.

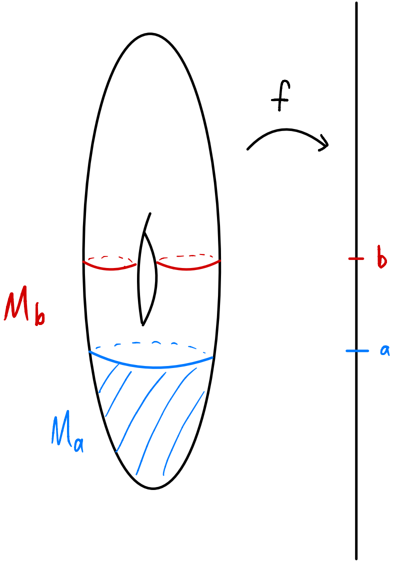
Fix a Riemannian structure on M, suppose that f is proper and that all critical points are nondegenerate.

Let $a, b \in \mathbb{R}$ with $a < b$. Then

(i) If there are no critical values in $(a, b]$, then $M_b \simeq M_a$.

(ii) If there is one critical value $c \in (a, b]$ corresponding to a critical set C, then
$M_b \simeq M_a \cup W_C^-$

where W_C^- is the set of all points that flow up to the critical set C under the gradient flow of f.



Morse theory on infinite-dimensional spaces

By assuming that f is proper, we are excluding many interesting examples.

The original motivating example for Morse theory is to study the path space $\Omega_a^b N$ for $a, b \in N$, a compact manifold.

Given a Riemannian structure on N, there is a natural energy function (or action integral) on paths $\gamma : [0, 1] \to N$ given by

$$J(\gamma) = \int_0^1 \|\gamma'(t)\|^2 \, dt.$$

The critical points are geodesics and the index is determined by counting conjugate points along the geodesic.

BUT the path space is infinite-dimensional and J is non-proper. The previous statement of the main theorem does not apply here.

One solution is to approximate the path space by finite-dimensional manifolds, however it would be more satisfying to do Morse theory directly on the full infinite-dimensional path space.

Morse theory on Hilbert manifolds

In this setting we need a replacement for properness of $f\colon M \to \mathbb{R}$.

Palais-Smale Condition C. If S is any subset of M on which f is bounded, but $\|\nabla f\|$ is not bounded away from zero, then there is a critical point of f in the closure of S.

Palais and Smale proved the following.

Main theorem of Morse Theory. Let $k \geq 1$ and let M be a complete Riemannian manifold of class C^{k+2} and let $f\colon M \to \mathbb{R}$ be a C^{k+2} function. Suppose that all of the critical points of f are non-degenerate and that Condition C holds.
Let $a, b \in \mathbb{R}$ with $a < b$. Then

(i) If there are no critical values in $(a, b]$, then $M_b \simeq M_a$.

(ii) If there is one critical value $c \in (a, b]$ corresponding to critical points $p_1, \ldots, p_r$, then M_b is diffeomorphic to M_a with r handles attached.

Uhlenbeck showed that this theorem extends to the more general setting of Banach manifolds.

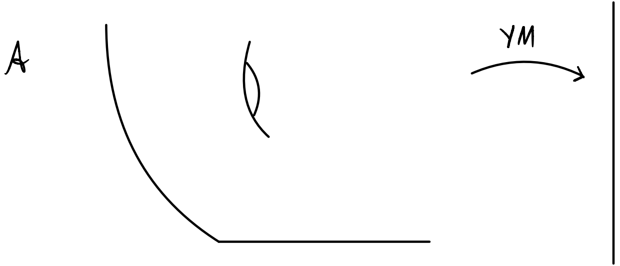
Does Condition C hold for the Yang-Mills functional?

Let X be a compact Riemannian manifold and $E \to X$ a principal bundle with compact structure group. Let $\mathcal{A}$ be the space of connections on E. Consider the Yang-Mills functional

$$\mathrm{YM} : \mathcal{A} \to \mathbb{R}, \quad d_A \mapsto \|F_A\|_{L^2}^2$$

Does Condition C hold for YM? No! Two reasons

(i) YM is invariant by the (infinite-dimensional) gauge group $\mathcal{G}$, therefore the critical sets are infinite-dimensional.



Does Condition C hold for the Yang-Mills functional?

Is it possible that Condition C holds modulo the gauge action?

(ii) When $\dim_{\mathbb{R}} X \geq 4$ then the phenomenon of Uhlenbeck bubbling shows that there are sequences of Yang-Mills minima which become singular and therefore cannot converge strongly to any limiting smooth connection.
Uhlenbeck compactness shows that (modulo gauge) sequences of Yang-Mills minima do converge weakly to an ideal limiting connection (possibly on a different bundle).

When $\dim_{\mathbb{R}} X = 2, 3$ then there is no bubbling. Daskalopoulos '92 and Råde '92 showed that the Yang-Mills flow converges to a smooth limit, justifying the analytic details of Atiyah and Bott's Morse theory for Yang-Mills on Riemann surfaces.

Taubes '88 showed that one can recover Morse theory for $\dim_{\mathbb{R}} X = 4$ after taking bubbling into account.

A class of functions on singular spaces

We would like to extend the above ideas so that the main theorem of Morse theory works for certain examples of functions on singular spaces.

Main Example. Let G be a connected reductive Lie group acting linearly on $\mathbb{C}^n$ (or $\mathbb{C}P^n$) with the standard metric.

Suppose that the action of the maximal compact subgroup K is Hamiltonian with respect to the standard symplectic structure on $\mathbb{C}^n$.

Let $\mu : \mathbb{C}^n \to \mathfrak{k}^*$ be a moment map for the action of K.

Let $Z \subset \mathbb{C}^n$ be a G-invariant algebraic subvariety, and let $f = \|\mu\|^2 : Z \to \mathbb{R}$.

This class of examples includes Nakajima quiver varieties, quivers with more general relations, etc.

A class of functions on singular spaces

Let M be a real analytic manifold and $f\colon M \to \mathbb{R}$ be an analytic function.

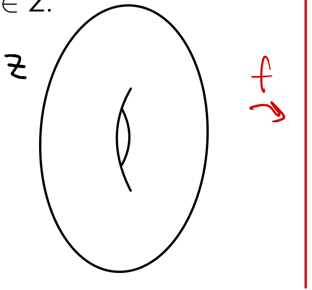
Let $\gamma(x_0, t)$ be the time t negative gradient flow of f with initial condition x_0.

$$\frac{d\gamma}{dt} = -\nabla f(\gamma(x_0, t)) \in T_{\gamma(x,t)} M$$

Let $Z \subset M$ be a closed subset preserved by the flow (if $x \in Z$ then $\gamma(x, t) \in Z$ for all t such that $\gamma(x, t)$ is defined) and suppose a solution exists locally for $t \in (-\varepsilon, \varepsilon)$ and all $x \in Z$.

A critical point of $f : Z \to \mathbb{R}$ is then defined as a fixed point of the flow.

Is there an analog of the main theorem of Morse theory in this setting?



Conditions on the function

Firstly, the Palais-Smale condition C is not satisfied for many examples of interest (since the critical sets may be non-compact).

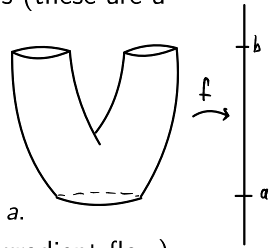
In our examples, we do know a lot about the critical sets and the gradient flow of the function $f : Z \to \mathbb{R}$.

Instead we impose the following weaker conditions (these are a consequence of condition C).

(i) The critical values of f are isolated.

(ii) For any $a < b$ and $x_0 \in f^{-1}(a, b)$, either
   - $\lim_{t \to \infty} \gamma(x_0, t)$ exists in $f^{-1}[a, b]$, or
   - there exists $T_+ > 0$ such that $f(\gamma(x_0, t)) < a$.

   and the same is true for $t < 0$ (the upwards gradient flow).



Condition (ii) says that if the initial condition is in $f^{-1}[a, b]$ then either the flow converges in $f^{-1}[a, b]$ or it flows out of this set.

The flow behaves well near the critical points

Given a critical point c, define the stable and unstable sets

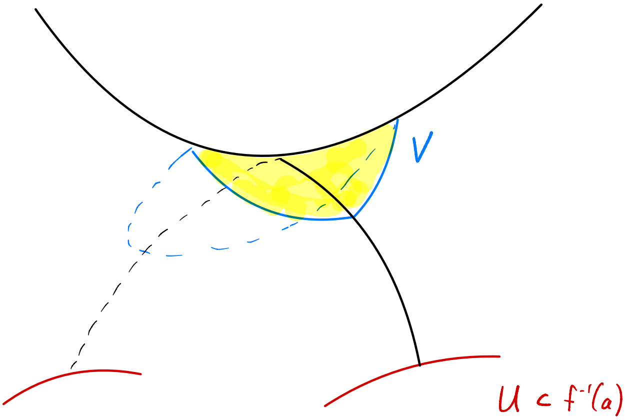
$$W_c^u := \{x \in f^{-1}[f(c), \infty) \mid \lim_{t \to -\infty} \gamma(x,t) = c\}$$
$$W_c^s := \{x \in f^{-1}(-\infty, f(c)] \mid \lim_{t \to \infty} \gamma(x,t) = c\}$$

Similarly, let W_C^u and W_C^s be the stable and unstable sets for a critical set C (all the critical points in a level set of f).

The next condition requires that if the downwards flow starts near the critical set, then it cannot wander too far from the unstable set.

(iii) Let c be any critical point, and let $a < f(c)$ be any real number so that there are no critical values in $[a, f(c))$. Then for each neighbourhood U of $W_c^u \cap f^{-1}(a)$ there exists a neighbourhood V of c such that for each $y \in V \setminus W_C^+$ there exists $T > 0$ such that $\gamma(y,T) \in U$.

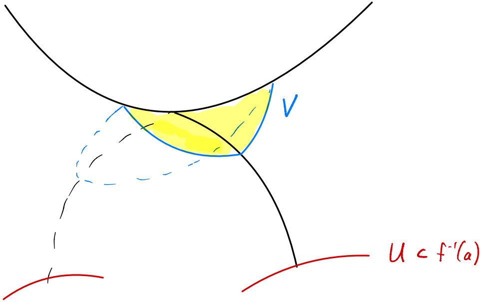
The flow behaves well near the critical points



The final condition imposes some regularity on the singularities of the space Z. A level set of Z has a neighbourhood deformation retract onto a level set of the unstable manifold.

(iv) For each critical set C with $d = f(C)$, there exists $\varepsilon > 0$ such that $f^{-1}(d-\varepsilon)$ has a locally finite triangulation containing $W_C^u \cap f^{-1}(d-\varepsilon)$ as a subtriangulation.

Example: $f^{-1}(d-\varepsilon)$ is an analytic variety and $W_C^u \cap f^{-1}(d-\varepsilon)$ is a subvariety.



The Main Theorem of Morse theory

Theorem 1. (W. 2019) Let M be a real analytic manifold, $f : M \to \mathbb{R}$ an analytic function and Z a subset preserved by the gradient flow of f. Suppose also that conditions (i)-(iv) hold. Then

(a) If there are no critical points in $f^{-1}[a, b]$ then $Z_b \simeq Z_a$.

(b) If there is one critical set C in $f^{-1}[a, b]$ with $a < f(C) < b$ then $Z_b \simeq Z_a \cup W_C^u$.

Theorem 2. (W. 2019) The class of functions where f is the norm square of a moment map on a variety satisfies condition (i)-(iv).

Theorem 3. (Pflaum, W. 2018) If a compact Lie group K acts on Z and f is K-invariant, then all of the above deformation retractions can be chosen to be K-equivariant.

Constructing the Morse-Bott complex

Consider a Morse-Bott function on a compact manifold M.

The gradient flow induces a filtration $M_0 \subset M_1 \subset \cdots \subset M_n = M$ and the Morse-Bott condition means that the terms on the first page of the spectral sequence are

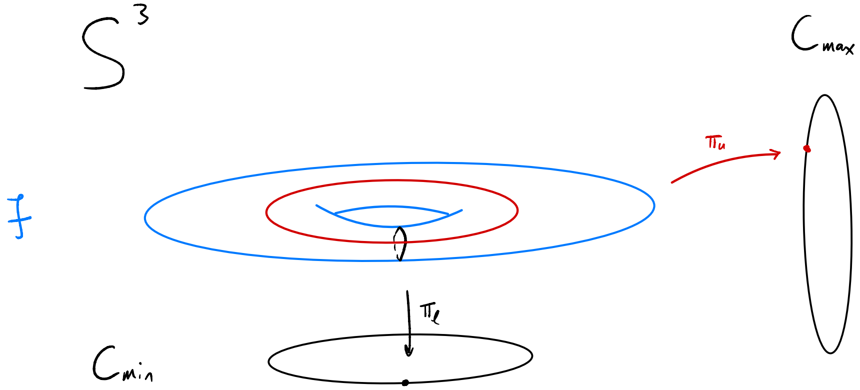
$$H^p(M_k, M_{k-1}) \xrightarrow[\text{Main Thm.}]{\cong} H^p(W_k^u, W_{k,0}^u) \xrightarrow[\text{Thom}]{\cong} H^{p-\lambda_k}(C_k).$$

If the function is Morse-Bott-Smale (the stable and unstable manifolds intersect transversely), then the differentials and cup product can be constructed using spaces of flow lines between an upper critical set C_u and a lower critical set C_ℓ.

$$\begin{tikzcd} & \mathcal{F}_\ell^u \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_u"] & \\ C_\ell & & C_u \end{tikzcd}$$

Constructing the Morse-Bott complex

Example: $f : S^3 \to \mathbb{R}$, $f(z_1, z_2) = |z_1|^2 - |z_2|^2$



Background: Differentials on the Morse-Bott complex

Recall that the differential on the first page of a spectral sequence is given by the composition

$$\begin{tikzcd}
H^p(M_k, M_{k-1}) \arrow[r] \arrow[rr, bend left, "d"] \arrow[d, leftrightarrow, "\cong"] & H^p(M_k) \arrow[r] & H^{p+1}(M_{k+1}, M_k) \arrow[d, leftrightarrow, "\cong"] \\
H^{p-\lambda_k}(C_k) \arrow[rr, dashed] & & H^{p-\lambda_{k+1}+1}(C_{k+1})
\end{tikzcd}$$

For a Morse-Bott-Smale function, the induced homomorphism on critical sets is given by the pullback-pushforward map

$$H^{p-\lambda_k}(C_k) \xrightarrow{\pi_\ell^*} H^{p-\lambda_k}(\mathcal{F}_k^{k+1}) \xrightarrow{(\pi_u)_*} H^{p-\lambda_{k+1}+1}(C_{k+1})$$

For example, see Austin & Braam, "Morse-Bott theory and equivariant cohomology"

Background: Cup product on the Morse-Bott complex

Given $\omega \in H^m(M)$, we can construct the cup product with ω on the terms in the spectral sequence.

For a Morse-Bott-Smale function, the induced homomorphism on critical sets is given by the pullback-pushforward map

$$\begin{tikzcd} & \tilde{\mathcal{F}}_\ell^u \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_u"] & \\ C_\ell & & C_u \end{tikzcd}$$

$$H^{p-\lambda_k}(C_\ell) \xrightarrow{\pi_\ell^*} H^{p-\lambda_k}(\tilde{\mathcal{F}}_\ell^u) \xrightarrow{\smile \omega} H^{p-\lambda+m}(\tilde{\mathcal{F}}_\ell^u) \xrightarrow{(\pi_u)_*} H^{p-\lambda_u+m}(C_u)$$

The moment map examples from before are not Morse-Bott. Instead they are Morse-Kirwan, or minimally degenerate.

We would like to generalise the above construction to work for these examples, first on a smooth space and then after restricting to a singular space.

Returning to the Morse-Bott-Smale case, the differential is constructed as follows.

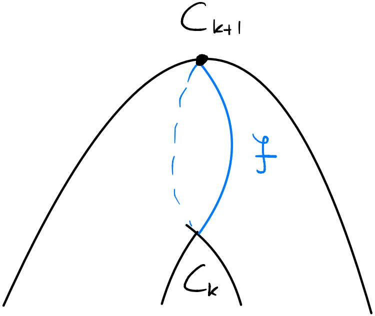
$$
\begin{tikzcd}
H^p(M_k, M_{k-1}) \arrow[rr, "d"] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, "\text{restriction}"] & & H^{p+1}(M_{k+1}, M_k) \arrow[d, leftrightarrow, "\cong"] \\
H^p(W_k^u, W_{k,0}^u) \arrow[d, leftrightarrow, "\cong", "\text{Thom}"'] & H^p(W_{k+1,0}^u, W_{k+1,0}^u \setminus \mathcal{F}_k^{k+1}) \arrow[r] \arrow[d, leftrightarrow, "\cong", "\text{Thom}"'] & H^{p+1}(W_{k+1}^u, W_{k+1,0}^u) \arrow[d, leftrightarrow, "\cong", "\text{Thom}"'] \\
H^{p-\lambda_k}(C_k) \arrow[r, dashed, "\pi_\ell^*"] & H^{p-\lambda_k}(\mathcal{F}_k^{k+1}) \arrow[r, dashed, "(\pi_u)_*"] & H^{p-\lambda_{k+1}+1}(C_{k+1})
\end{tikzcd}
$$

This works because the intersection of stable and unstable manifolds is transverse.

Consequences of transversality

Take the normal bundle of W_k^s inside the ambient manifold M and restrict it to the intersection $W_k^s \cap W_{k+1}^u$.

This is isomorphic to the normal bundle of the intersection $W_k^s \cap W_{k+1}^u$ inside the unstable set W_{k+1}^u.



If the differentials are all zero, then the cup product with $\omega \in H^m(M)$ on the first page of the spectral sequence is constructed as follows.

$$\begin{tikzcd}
H^p(M_k, M_{k-1}) \arrow[rr, "\smile\omega"] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, "\text{restriction}"] & & H^{p+m}(M_{k+1}, M_k) \arrow[d, leftrightarrow, "\cong"] \\
H^p(W_k^u, W_{k,0}^u) \arrow[d, leftrightarrow, "\text{Thom}"', "\cong"] & H^p(W_{k+1,0}^u, W_{k+1,0}^u \setminus \mathcal{F}_k^{k+1}) \arrow[r] \arrow[d, leftrightarrow, "\text{Thom}"', "\cong"] & H^{p+m}(W_{k+1}^u, W_{k+1,0}^u) \arrow[d, leftrightarrow, "\text{Thom}"', "\cong"] \\
H^{p-\lambda_k}(C_k) \arrow[r, dashed, "\pi_\ell^*"] & H^{p-\lambda_k}(\mathcal{F}_k^{k+1}) \arrow[r, dashed, "(\pi_u)_* \circ (\smile\omega)"] & H^{p-\lambda_{k+1}+m}(C_{k+1})
\end{tikzcd}$$

Main example: Representations of a quiver with relations

We would like to understand these pullback-pushforward homomorphisms on the (singular) space of representations of a quiver with relations (such as Nakajima quivers).

In this case the total space is a contractible subvariety $Z \subset \mathbb{C}^n$ and the equivariant cohomology ring $H_K^*(\mathbb{C}^n)$ is a polynomial ring.

The critical sets are quiver varieties (for example, moduli of framed instantons on ALE four manifolds) and the pullback-pushforward homomorphisms are part of Nakajima's constructions in geometric representation theory.

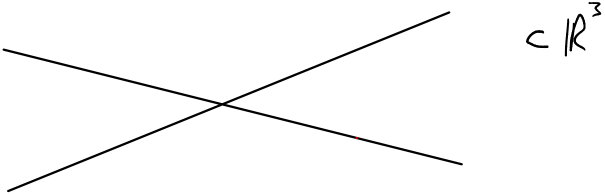
We would like to apply the above theory on the ambient manifold $\mathbb{C}^n$ and then pull back the cohomology classes to $Z \subset \mathbb{C}^n$, however the moment map energy function is not Morse-Bott (instead it is minimally degenerate/Morse-Kirwan).

Moreover, the stable and unstable manifolds do not intersect transversely, however this is not fatal, as it is analogous to a transverse intersection on a submanifold.

Recovering transversality for moment map functions

Take the normal bundle of W_k^s inside the ambient manifold M and restrict it to the intersection $W_k^s \cap W_{k+1}^u$.

Transversality fails and this is no longer isomorphic to the normal bundle of the intersection $W_k^s \cap W_{k+1}^u$ inside the unstable set W_{k+1}^u. Instead it contains this normal bundle as a subbundle.



Main example: Representations of a quiver with relations

This means that we need an extra step to construct the cup product for this minimally degenerate example. This corresponds to taking cup product with the pullback of the Euler class of the quotient bundle described on the previous slide.

$$\begin{tikzcd}
H^p(M_k, M_{k-1}) \arrow[rrr, "\smile\omega"] \arrow[d, leftrightarrow, "\cong"] \arrow[rd, "\text{restriction}"] & & & H^{p+m}(M_{k+1}, M_k) \arrow[d, leftrightarrow, "\cong"] \\
H^p(W_k^u, W_{k,0}^u) \arrow[dd, leftrightarrow, "\cong"', "\text{Thom}"] & H^p(W_{k+1,0}^u, W_{k+1,0}^u \setminus \mathcal{F}_k^{k+1}) \arrow[rr, "c(\omega)"] \arrow[d, leftrightarrow, "\cong"', "\text{Thom}"] & & H^{p+m}(W_{k+1}^u, W_{k+1,0}^u) \arrow[d, leftrightarrow, "\cong"', "\text{Thom}"] \\
& H^{p-\nu_k}(\mathcal{F}_k^{k+1}) \arrow[rr, dashed, "(\pi_u)_* \circ (\smile\omega)"] & & H^{p-\lambda_{k+1}+m}(C_{k+1}) \\
H^{p-\lambda_k}(C_k) \arrow[r, dashed, "\pi_k^*"] & H^{p-\lambda_k}(\mathcal{F}_k^{k+1}) \arrow[u, "\smile i^* e"] & &
\end{tikzcd}$$

The equivariant Euler class $i^* e$ is computable and so we can construct the cup product for this class of examples.

Application to Nakajima's constructions

Motivation. Nakajima constructs representations of certain algebras (e.g. enveloping algebras of Kac-Moody algebras) via convolution (pullback-pushforward) in Borel-Moore homology.

$$\begin{tikzcd} & \mathcal{B} \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_u"] & \\ \mathcal{M}_1 & & \mathcal{M}_2 \end{tikzcd}$$

Theorem. (W. 2017) For moment map flows on spaces of quivers with relations, the Hecke correspondence is homeomorphic to a space of flow lines modulo the group action.

Application to Nakajima's constructions