# ON A PROBLEM OF M. TALAGRAND 

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#### Abstract

We address a special case of a conjecture of M. Talagrand relating two notions of "threshold" for an increasing family $\mathcal{F}$ of subsets of a finite set $V$. The full conjecture implies equivalence of the "Fractional Expectation-Threshold Conjecture," due to Talagrand and recently proved by the authors and B. Narayanan, and the (stronger) "Expectation-Threshold Conjecture" of the second author and G. Kalai. The conjecture under discussion here says there is a fixed $L$ such that if, for a given $\mathcal{F}, p \in[0,1]$ admits $\lambda: 2^{V} \rightarrow \mathbb{R}^{+}$with $$
\sum_{S \subseteq F} \lambda_{S} \geq 1 \forall F \in \mathcal{F}
$$ and $$
\sum_{S} \lambda_{S} p^{|S|} \leq 1 / 2
$$ (a.k.a. $\mathcal{F}$ is weakly $p$-small), then $p / L$ admits such a $\lambda$ taking values in $\{0,1\}(\mathcal{F}$ is $(p / L)$-small). Talagrand showed this when $\lambda$ is supported on singletons and suggested, as a more challenging test case, proving it when $\lambda$ is supported on pairs. The present work provides such a proof.


## 1. Introduction

Given a finite set $V$, write $2^{V}$ for the power set of $V$ and, for $p \in[0,1], \mu_{p}$ for the product measure on $2^{V}$ given by $\mu_{p}(S)=p^{|S|}(1-p)^{|V \backslash S|}$. An $\mathcal{F} \subseteq 2^{V}$ is increasing if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. For $\mathcal{G} \subseteq 2^{V}$ we use $\langle\mathcal{G}\rangle$ for the increasing family generated by $\mathcal{G}$, namely $\{B \subseteq V: \exists A \in \mathcal{G}, B \supseteq A\}$.

We assume throughout that $\mathcal{F} \subseteq 2^{V}$ is increasing and not equal to $2^{V}, \emptyset$. Then $\mu_{p}(\mathcal{F})\left(=\sum\left\{\mu_{p}(S): S \in\right.\right.$ $\mathcal{F}\}$ ) is strictly increasing in $p$, and we define the threshold, $p_{c}(\mathcal{F})$, to be the unique $p$ for which $\mu_{p}(\mathcal{F})=1 / 2$. (This is finer than the original Erdős-Rényi notion, according to which $p^{*}=p^{*}(n)$ is $\boldsymbol{a}$ threshold for $\mathcal{F}=\mathcal{F}_{n}$ if $\mu_{p}(\mathcal{F}) \rightarrow 0$ when $p \ll p^{*}$ and $\mu_{p}(\mathcal{F}) \rightarrow 1$ when $p \gg p^{*}$. That $p_{c}(\mathcal{F})$ is always an Erdős-Rényi threshold follows from [2].)

Thresholds have been a - maybe the - central concern of the study of random discrete structures (random graphs and hypergraphs, for example) since its initiation by Erdős and Rényi [4], with much of that effort concerned with identifying (Erdős-Rényi) thresholds for specific properties (see [1, 6])-though it was not observed until [2] that every sequence of increasing properties admits such a threshold.

The main concern of this paper is the relation between the following two notions of M. Talagrand $[8,9$, 10]. (Our focus is Conjecture 1.4 and our main result is Theorem 1.6; we will come to these following some motivation.)

Say $\mathcal{F}$ is $p$-small if there is a $\mathcal{G} \subseteq 2^{V}$ such that

$$
\begin{equation*}
\langle\mathcal{G}\rangle \supseteq \mathcal{F} \tag{1}
\end{equation*}
$$

[^0](that is, each member of $\mathcal{F}$ contains a member of $\mathcal{G}$ ) and
\[

$$
\begin{equation*}
\sum_{S \in \mathcal{G}} p^{|S|} \leq 1 / 2 \tag{2}
\end{equation*}
$$

\]

and set $q(\mathcal{F})=\max \{p: \mathcal{F}$ is $p$-small $\}$. Say $\mathcal{F}$ is weakly $p$-small if there is a $\lambda: 2^{V} \rightarrow \mathbb{R}^{+}(:=[0, \infty))$ such that

$$
\begin{equation*}
\sum_{S \subseteq F} \lambda_{S} \geq 1 \forall F \in \mathcal{F} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{S} \lambda_{S} p^{|S|} \leq 1 / 2 \tag{4}
\end{equation*}
$$

and set $q_{f}(\mathcal{F})=\max \{p: \mathcal{F}$ is weakly $p$-small $\}$. As in [5] we refer to $q(\mathcal{F})$ and $q_{f}(\mathcal{F})$ (respectively) as the expectation-threshold and fractional expectation-threshold of $\mathcal{F}$. (Note the former is used slightly differently in [7].) Notice that

$$
\begin{equation*}
q(\mathcal{F}) \leq q_{f}(\mathcal{F}) \leq p_{c}(\mathcal{F}) \tag{5}
\end{equation*}
$$

(The first inequality is trivial and the second holds since, for $\lambda$ as in (3), (4) and $Y$ drawn from $\mu_{p}$,

$$
\begin{equation*}
\left.\mu_{p}(\mathcal{F}) \leq \sum_{F \in \mathcal{F}} \mu_{p}(F) \sum_{S \subseteq F} \lambda_{S} \leq \sum_{S} \lambda_{S} \mu_{p}(Y \supseteq S)=\sum_{S} \lambda_{S} p^{|S|} \leq 1 / 2 .\right) \tag{6}
\end{equation*}
$$

In particular, each of $q, q_{f}$ is a lower bound on $p_{c}$, and these turn out to be easily understood (and to agree up to constant) in many cases of interest; see [5]. The next two conjectures-respectively the main conjecture (Conjecture 1) of [7] and a sort of LP relaxation thereof suggested by Talagrand [10, Conjecture 8.3]—say that these bounds are never far from the truth.

Conjecture 1.1. There is a universal $K$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^{V}$,

$$
p_{c}(\mathcal{F}) \leq K q(\mathcal{F}) \log |V|
$$

Conjecture 1.2. There is a universal $K$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^{V}$,

$$
p_{c}(\mathcal{F}) \leq K q_{f}(\mathcal{F}) \log |V|
$$

Talagrand [10, Conjecture 8.5] also proposes the following strengthening of Conjecture 1.2 , in which $\ell(\mathcal{F})$ is the maximum size of a minimal member of $\mathcal{F}$.

Conjecture 1.3. There is a universal $K>0$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^{V}$,

$$
p_{c}(\mathcal{F})<K q_{f}(\mathcal{F}) \log \ell(\mathcal{F})
$$

Conjecture 1.3 is proved in [5], to which we also refer for discussion of the very strong consequences that originally motivated Conjecture 1.1, but follow just as easily from Conjecture 1.2.

Turning, finally, to the business at hand, we are interested in the following conjecture of Talagrand [10, Conjecture 6.3], which says that the parameters $q$ and $q_{f}$ are in fact not very different.

Conjecture 1.4. There is a fixed $L$ such that, for any $\mathcal{F}, q(\mathcal{F}) \geq q_{f}(\mathcal{F}) / L$.
(That is, weakly $p$-small implies $(p / L)$-small.) This of course implies equivalence of Conjectures 1.2 and 1.1, as well as of Conjecture 1.3 and the corresponding strengthening of Conjecture 1.1; in particular, in view of [5], Conjecture 1.4 would now supply a proof of Conjecture 1.1. (Post-[5] this implication is probably the best motivation for Conjecture 1.4, but the authors have long been interested in the conjecture for its own sake, as it would be a striking instance of a broad, natural class of examples where the passage from an integer problem to its fractional counterpart has only a minor effect on behavior.)

The following mild reformulation of Conjecture 1.4 will be convenient.
Conjecture 1.5. There is a fixed $J$ such that for any $V, p \in[0,1]$ and $\lambda: 2^{V} \backslash\{\emptyset\} \rightarrow \mathbb{R}^{+}$,

$$
\begin{equation*}
\left\{U \subseteq V: \sum_{S \subseteq U} \lambda_{S} \geq \sum_{S} \lambda_{S}(J p)^{|S|}\right\} \tag{7}
\end{equation*}
$$

is p-small.

As Talagrand observes, even simple instances of Conjecture 1.4 are not easy to establish. He suggests two test cases, which in the formulation of Conjecture 1.5 become:
(i) $V=\binom{[n]}{2}=E\left(K_{n}\right)$ and (for some $k$ ) $\lambda$ is the indicator of $\left\{\right.$ copies of $K_{k}$ in $\left.K_{n}\right\}$;
(ii) $\lambda$ is supported on 2-element sets.
(He does prove Conjecture 1.5 for $\lambda$ supported on singletons; see Proposition 2.1 for a quantified version that will be useful in what follows.)

The quite specific (i) above was treated in [3]. Here we dispose of the much broader (ii):
Theorem 1.6. Conjecture 1.5 holds when $\operatorname{supp}(\lambda) \subseteq\binom{V}{2}$; in other words, there is a $J$ such that for any graph $G=(V, E), p \in[0,1]$ and $\lambda: E \rightarrow \mathbb{R}^{+}$,

$$
\begin{equation*}
\left\{U \subseteq V: \lambda(G[U]) \geq J^{2} \lambda(G) p^{2}\right\} \tag{8}
\end{equation*}
$$

is $p$-small (where $G[U]$ is the subgraph induced by $U$ ).
(We could of course take $G=K_{n}$, but find thinking of a general $G$ more natural.)
It seems not impossible that the ideas underlying Theorem 1.6 can be extended to give Conjecture 1.4 in full, but we don't yet see this.

The rest of the paper is devoted to the proof of Theorem 1.6. The most important part of this turns out to be a version of the "unweighted" case (that is, with $\lambda$ taking values in $\{0,1\}$ ), though deriving Theorem 1.6 from this still needs some ideas; a precise statement (Theorem 2.2) is given in Section 2, following a few preliminaries. Section 3 then proves Theorem 1.6 assuming Theorem 2.2, and the proof of Theorem 2.2 itself is given in Section 4.

## 2. Orientation

We use $[n]$ for $\{1,2, \ldots, n\}, 2^{X}$ for the power set of $X$, and $\binom{X}{r}$ for the family of $r$-element subsets of $X$, and recall from above that $\langle\mathcal{A}\rangle$ is the increasing family generated by $\mathcal{A} \subseteq 2^{X}$. For a set $X$ and $p \in[0,1]$, $X_{p}$ is the " $p$-random" subset of $X$ in which each $x \in X$ appears with probability $p$ independent of other choices. We assume throughout that $p$ has been specified and often omit it from our notation.

For $\mathcal{A} \subseteq 2^{V}$, the cost of $\mathcal{A}$ (w.r.t. our given $p$ ) is $\mathrm{C}(\mathcal{A})=\sum_{S \in \mathcal{A}} p^{|S|}$. We say $\mathcal{A}$ covers $\mathcal{B} \subseteq 2^{V}$ if $\langle\mathcal{A}\rangle \supseteq \mathcal{B}$, set

$$
\mathrm{C}^{*}(\mathcal{B})=\min \{\mathrm{C}(\mathcal{A}): \mathcal{A} \text { covers } \mathcal{B}\}
$$

and say $\mathcal{B}$ can be covered at cost $\gamma$ if $\mathrm{C}^{*}(\mathcal{B}) \leq \gamma$. So " $\mathcal{B} p$-small" is the same as $\mathrm{C}^{*}(\mathcal{B}) \leq 1 / 2$, and each of Conjecture 1.5, Theorem 1.6 says (roughly) that the collection of subsets $U$ of $V$ for which $\sum_{S \subseteq U} \lambda_{S}$ is much larger than the "natural" value, $\mathbb{E}\left[\sum_{S \subseteq V_{p}} \lambda_{S}\right]=\sum \lambda_{S} p^{|S|}$, admits such a "cheap" cover. Talagrand's proof for singletons, to which we turn next, provides a first, simple illustration of this, and what we do in the rest of the paper amounts to producing such a cover for the collection in (8).

Singletons. In the above language, Talagrand's result for $\lambda$ supported on singletons becomes:
Proposition 2.1. For all $\zeta: V \rightarrow \mathbb{R}^{+}$and $J>2 e$,

$$
\begin{equation*}
\mathrm{C}^{*}(\{U \subseteq V: \zeta(U) \geq J \zeta(V) p\})<2 e /(J-2 e) \tag{9}
\end{equation*}
$$

(The dependence on $J$ is best possible up to constants; e.g. take $|V|=J, p=J^{-2}$ and $\zeta \equiv 1$. The switch from $\lambda$ to $\zeta$ will be convenient when we come to use the proposition; see (17).)

Proof. We may take $V=[n]$ and assume $\zeta$ is non-increasing (and positive) and $J p \leq 1$ (since the statement is trivial when $J p>1$ ). Define $R$ by

$$
\frac{1}{R p}=\left\lceil\frac{1}{J p}\right\rceil=: a
$$

We claim that the collection

$$
\mathcal{A}=\bigcup_{k \geq 1}\binom{[a k]}{k}
$$

covers the family in (9); this gives the proposition since the l.h.s. of (9) is then at most

$$
\mathrm{C}(\mathcal{A})=\sum_{k \geq 1}\binom{a k}{k} p^{k}<\sum_{k \geq 1}\left(\frac{e}{R}\right)^{k}<\frac{e}{R-e}<\frac{2 e}{J-2 e}
$$

(the last inequality holding since $J p \leq 1$ implies $R>J / 2$.)
To see that the claim holds, observe that its failure implies the existence of some $U=\left\{u_{1}<u_{2}<\cdots<\right.$ $\left.u_{\ell}\right\} \subseteq[n]$ with $\zeta(U) \geq J \zeta(V) p$ such that $|U \cap[a k]|<k$ for all $k>0$. But then $u_{i}>i a$ for all $i \in[\ell]$, yielding the contradiction

$$
\zeta(V)>\sum_{i=0}^{\ell-1} \sum_{j \in[a]} \zeta(j+i a) \geq a \zeta(U) \geq \zeta(V)
$$

Toward doubletons. Graphs here are always simple and are mainly thought of as sets of edges; thus $|G|$ is $|E(G)|$. We use $\nabla_{G}(v)$ or $\nabla_{v}$ for $\{e \in E(G): v \in e\}$; so the degree of $v$ is $d_{v}=\left|\nabla_{v}\right|$. (We also use $N_{G}(v)$ for the neighborhood of $v$ in $G$.)

The following convention will be helpful. Given a graph $G$ on $V$, we associate with each $U \subseteq V$ a "weighted subset" $D(U)=D_{G}(U)$ of $E(G)$ that assigns to each $e$ the weight $|e \cap U| / 2$. (We also use $D_{v}$ or $D_{G}(v)$ for $D(\{v\})$. ) We then have (or define), for any $\lambda: G \rightarrow \mathbb{R}^{+}$,

$$
\lambda(D(U))=\frac{1}{2} \sum_{v \in U} \lambda\left(\nabla_{v}\right)
$$

(e.g. $|D(U)|=\frac{1}{2} \sum_{v \in U} d_{v}$ ). Notice that

$$
\mathbb{E} \lambda\left(G\left[V_{p}\right]\right)=\mathbb{E} \lambda\left(D\left(V_{p}\right)\right) p
$$

(e.g. $\left.\mathbb{E}\left|G\left[V_{p}\right]\right|=\mathbb{E}\left|D\left(V_{p}\right)\right| p\right)$, so $\lambda(D(U)) p$ is a natural benchmark against which to measure $\lambda(G[U])$.

As mentioned at the end of Section 1, the heart of our argument deals with the unweighted case of Theorem 1.6, where we are given some (simple) graph $G$ on $V$, and the collection in (8) becomes the set of $U$ 's for which $G[U]$ is atypically large. It is here that we are concerned with the production of covers, which are then available for use in the weighted case.

For the derivation of Theorem 1.6, we will decompose $G$ into subgraphs $G_{1}, G_{2}, \ldots$ so that the $\lambda$-values of the edges within a $G_{i}$ are roughly equal, and show that for each "heavy" $U$ (meaning one with large $\lambda(G[U])$, as in (8)), there is some $i$ for which $G_{i}[U]$ is large. We then plan to appeal to the unweighted case to cover, for each $i$, the $U$ 's that are "heavy" for $G_{i}$-a plan made delicate by the need to sum the contributions of many $G_{i}$ 's to the 1.h.s. of (2).

To deal with this we need a little more than the unweighted version of Theorem 1.6, as follows. Define

$$
\begin{equation*}
\mathrm{C}_{J}^{*}(\mu, T) \tag{10}
\end{equation*}
$$

to be the infimum of those $\gamma^{\prime}$ 's for which, for every $p$ and (simple) graph $G$ (on $V$ ) with $|G| p^{2} \leq \mu$,

$$
\begin{equation*}
\left\{U \subseteq V:|G[U]| \geq \max \left\{T, J\left|D_{G}(U)\right| p\right\}\right\} \tag{11}
\end{equation*}
$$

can be covered at cost $\gamma$.
The technical-looking requirement involving $D_{G}$ is a crucial feature of our argument: for derivation of Theorem 1.6, we will need cost bounds that improve as the "target" $T$ grows, even if $T / \mu$ does not, which need not be the case without this extra condition (e.g. it's not hard to see that if $G$ is the union of $(K p)^{-1}$ disjoint copies of $K_{1, m}$, and $T=m p=K \mu$, then, no matter how large $m$ is, $\{U \subseteq V:|G[U]| \geq T\}$-or the smaller $\left\{U \subseteq V: G[U] \cong K_{1, m p}\right\}$-cannot be covered at cost less than $1 / K$ ). License to use the condition will be provided by the reduction to unweighted in Section 3.

Our central result is:
Theorem 2.2. For any $\mu$ and $T=c J^{2} \mu$ with

$$
\begin{equation*}
c \geq 256 e / J, \quad J \geq 8 e \tag{12}
\end{equation*}
$$

and $J_{1}=J /(8 e)$,

$$
\begin{equation*}
\mathrm{C}_{J}^{*}(\mu, T) \leq 32 c^{-1} \min \left\{J_{1}^{-2}, J_{1}^{-\sqrt{T} / 16}\right\} . \tag{13}
\end{equation*}
$$

(Here and throughout we don't worry about getting good constants, trying instead to keep the argument fairly clean.) As already mentioned, the proof of Theorem 2.2 is given in Section 4, following the derivation of Theorem 1.6, to which we now turn.

## 3. PROOF OF THEOREM 1.6

Here we assume Theorem 2.2 and prove the following quantified version of Theorem 1.6.

Theorem 3.1. For any graph $G$ on $V, \lambda: G \rightarrow \mathbb{R}^{+}$and

$$
\begin{equation*}
R \geq 4096 \sqrt{2} e \tag{14}
\end{equation*}
$$

the set

$$
\mathcal{U}_{0}=\left\{U \subseteq V: \lambda(G[U]) \geq R^{2} \lambda(G) p^{2}\right\}
$$

can be covered at cost $O(1 / R)$.
Proof. We take $G, \lambda, R$ to be as in the theorem, use $D(U)$ for $D_{G}(U)$ (defined in Section 2), and assume throughout that

$$
U \in \mathcal{U}_{0} .
$$

We first observe that it is enough to prove the theorem assuming

$$
\begin{equation*}
\text { the only positive values taken by } \lambda \text { are } \theta_{i}:=2^{-i}, i=1,2, \ldots, \tag{15}
\end{equation*}
$$

with (14) slightly weakened to

$$
\begin{equation*}
R \geq 4096 e . \tag{16}
\end{equation*}
$$

Then for a general $\lambda$ (which we may of course scale to take values in $[0,1]$ ) and $\lambda^{\prime}$ given by

$$
\lambda_{S}^{\prime}=\max \left\{\theta_{i}: \theta_{i} \leq \lambda_{S}\right\},
$$

$\mathcal{U}_{0}$ as in the theorem is contained in the corresponding collection with $\lambda$ and $R^{2}$ replaced by $\lambda^{\prime}$ and $R^{2} / 2$ (which supports (16)), since $U \in \mathcal{U}_{0}$ implies $2 \lambda^{\prime}(G[U])>\lambda(G[U]) \geq R^{2} \lambda(G) p^{2} \geq R^{2} \lambda^{\prime}(G) p^{2}$. So we assume from now on that $\lambda$ and $R$ are as in (15) and (16) (respectively).

Note also that Proposition 2.1, with $\zeta(v)=\lambda\left(D_{v}\right)$ (for which we have $\zeta(V)=\sum \zeta(v)=\frac{1}{2} \sum \lambda\left(\nabla_{v}\right)=$ $\lambda(G)$ and $\zeta(U)=\lambda(D(U)))$, says that the set

$$
\begin{equation*}
\{U \subseteq V: \lambda(D(U)) \geq R \lambda(G) p\} \tag{17}
\end{equation*}
$$

admits a cover of cost less than $6 / R$. So we specify such a cover as a first installment on $\mathcal{G}$ and it then becomes enough to show that

$$
\mathcal{U}^{*}:=\left\{U \in \mathcal{U}_{0}: \lambda(D(U))<R \lambda(G) p\right\}
$$

can be covered at $\operatorname{cost} O(1 / R)$; in fact we will show

$$
\begin{equation*}
\mathrm{C}^{*}\left(\mathcal{U}^{*}\right)=O\left(R^{-2}\right) . \tag{18}
\end{equation*}
$$

We assume from now on that $U \in \mathcal{U}^{*}$.

Set $G_{i}=\left\{e \in G: \lambda(e)=\theta_{i}\right\}$. For (18) we first show that, for each $U \in \mathcal{U}^{*}$, some $G_{i}[U]$ must be "large," meaning $U$ belongs to $\mathcal{U}_{i}$, defined in (25), and then bound the costs of the $\mathcal{U}_{i}$ 's using Theorem 2.2.

From this point we use $D_{i}(U)$ for $D_{G_{i}}(U)$. We observe that for any $H \subseteq G$,

$$
\lambda(H)=\sum_{i} \theta_{i}\left|H \cap G_{i}\right|,
$$

and abbreviate

$$
\mathrm{w}_{i}=\lambda\left(G_{i}\right)=\theta_{i}\left|G_{i}\right|, \quad \mathrm{w}=\lambda(G)=\sum \mathrm{w}_{i} .
$$

Given $U$, define $L=L(U), K=K(U), L_{i}=L_{i}(U)$ and $K_{i}=K_{i}(U)$ by

$$
\lambda(D(U))=L w p,
$$

$$
\begin{align*}
\lambda(G[U]) & =K L w p^{2} \\
\left|D_{i}(U)\right| & =L_{i}\left|G_{i}\right| p \tag{19}
\end{align*}
$$

and

$$
\begin{equation*}
\left|G_{i}[U]\right|=K_{i} L_{i}\left|G_{i}\right| p^{2} \tag{20}
\end{equation*}
$$

Then

$$
\begin{equation*}
L \mathrm{w} p=\sum \theta_{i}\left|D_{i}(U)\right|=\sum L_{i} \mathrm{w}_{i} p \tag{21}
\end{equation*}
$$

and

$$
K L \mathrm{w} p^{2}=\sum \theta_{i}\left|G_{i}[U]\right|=\sum K_{i} L_{i} \mathrm{w}_{i} p^{2}
$$

Since $U \in \mathcal{U}_{0}$, we have

$$
\begin{equation*}
\sum K_{i} L_{i} \mathrm{w}_{i} \geq R^{2} \mathrm{w} \tag{22}
\end{equation*}
$$

while $U \in \mathcal{U}^{*}$ gives

$$
\begin{equation*}
L<R \tag{23}
\end{equation*}
$$

Note also that, with

$$
I=I(U)=\left\{i: K_{i}>R / 2\right\}
$$

we have

$$
\begin{equation*}
\sum\left\{K_{i} L_{i} \mathrm{w}_{i}: i \in I\right\}>R^{2} \mathrm{w} / 2 \tag{24}
\end{equation*}
$$

as follows from (22) and (using (21) and (23))

$$
\sum\left\{K_{i} L_{i} \mathrm{w}_{i}: i \notin I\right\} \leq(R / 2) L \mathrm{w}<R^{2} \mathrm{w} / 2
$$

Now let $\mathrm{E}_{i}=\left|G_{i}\right| p^{2}\left(=\mathbb{E}\left|G_{i}\left[V_{p}\right]\right|\right)$ and, for integer $\alpha$,

$$
\mathcal{E}_{\alpha}=\left\{i: \mathrm{E}_{i} \in\left(2^{\alpha-1}, 2^{\alpha}\right]\right\}
$$

We arrange the $i$ 's in an array, with columns indexed by $\alpha$ 's (in increasing order) and column $\alpha$ consisting of the indices in $\mathcal{E}_{\alpha}$, again in increasing order. (So $\mathrm{w}_{i}$ 's within a column decrease as we go down. Note column lengths may vary.) Define $\mathcal{B}_{\beta}$ to be the set of indices in row $\beta$.

|  | $\cdots$ | $\alpha-1$ | $\alpha$ | $\alpha+1$ | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |  |
| $\vdots$ |  |  |  |  |  |
| $\beta$ |  |  | $i$ |  |  |
| $\vdots$ |  |  |  |  |  |

TAbLE 1. $i$ is the $\beta$ th smallest index in $\mathcal{E}_{\alpha}$ (when $\left|\mathcal{E}_{\alpha}\right| \geq \beta$ ).

Set $\mathrm{y}_{i}=\theta_{i} 2^{\alpha} / p^{2}\left(\right.$ for $\left.i \in \mathcal{E}_{\alpha}\right)$ and $\mathrm{y}=\sum_{i \geq 1} \mathrm{y}_{i}$, noting that

$$
\mathrm{y}_{i} / 2<\mathrm{w}_{i} \leq \mathrm{y}_{i}
$$

Set

$$
c_{\beta}^{*}=(3 / 2)^{\beta-1} R^{2} / 16 \quad(\beta \geq 1)
$$

and $c_{i}=c_{\beta}^{*}$ if $i \in \mathcal{B}_{\beta}$. Let $\mathrm{w}_{\beta}^{*}$ and $\mathrm{y}_{\beta}^{*}$ be (respectively) the sums of the $\mathrm{w}_{i}$ 's and $\mathrm{y}_{i}$ 's over $i \in \mathcal{B}_{\beta}$, and notice that

$$
\mathrm{y}_{\beta+1}^{*} \leq \mathrm{y}_{\beta}^{*} / 2 \quad \text { for } \beta \geq 1
$$

(since $i=\mathcal{B}_{\beta+1} \cap \mathcal{E}_{\alpha}$ —where we abusively use $i$ for $\{i\}$-implies $i>j:=\mathcal{B}_{\beta} \cap \mathcal{E}_{\alpha}$, whence $2 \mathrm{y}_{i} \leq \mathrm{y}_{j}$ ).
Claim 3.2. For each $U \in \mathcal{U}^{*}$ there is an $i \in I(U)$ with $K_{i}(U) L_{i}(U)>c_{i}$.
Proof. With $\sum^{\star}$ denoting summation over $I(U)$, we have (using (24) at the end)

$$
\begin{aligned}
\sum^{\star} c_{i} \mathrm{w}_{i} \leq \sum c_{\beta}^{*} \mathrm{w}_{\beta}^{*} & \leq \sum c_{\beta}^{*} \mathrm{y}_{\beta}^{*} \\
& \leq \mathrm{y}_{1}^{*}\left(c_{1}^{*}+c_{2}^{*} / 2+c_{3}^{*} / 2^{2}+\cdots\right) \\
& \leq \mathrm{y}\left(c_{1}^{*}+c_{2}^{*} / 2+c_{3}^{*} / 2^{2}+\cdots\right) \\
& \leq\left(R^{2} / 4\right) \mathrm{y}<\left(R^{2} / 2\right) \mathrm{w}<\sum^{\star} K_{i}(U) L_{i}(U) \mathrm{w}_{i}
\end{aligned}
$$

It follows that if, for each $i, \mathcal{G}_{i}$ covers

$$
\begin{equation*}
\mathcal{U}_{i}:=\left\{U \subseteq V: i \in I(U) ; K_{i}(U) L_{i}(U)>c_{i}\right\} \tag{25}
\end{equation*}
$$

then $\cup \mathcal{G}_{i}$ covers $\mathcal{U}^{*}$; so we have

$$
\begin{equation*}
\mathrm{C}^{*}\left(\mathcal{U}^{*}\right) \leq \sum_{i} \mathrm{C}^{*}\left(\mathcal{U}_{i}\right) \tag{26}
\end{equation*}
$$

On the other hand, if $(\alpha, \beta)$ is the pair corresponding to $i$ (that is, $i$ is the $\beta$ th entry in column $\alpha$ of our array), then (see (10), (11) for $\mathrm{C}_{J}^{*}$ )

$$
\mathrm{C}^{*}\left(\mathcal{U}_{i}\right) \leq \mathrm{C}_{R / 2}^{*}\left(2^{\alpha}, T_{\alpha, \beta}\right)
$$

where $T_{\alpha, \beta}=\max \left\{c_{\beta}^{*} 2^{\alpha-1}, 1\right\}$ (since $\left|G_{i}\right| p^{2}=\mathrm{E}_{i} \leq 2^{\alpha}$, while $U \in \mathcal{U}_{i}$ implies, using (19), (20) and $i \in I(U)$,

$$
\left|G_{i}[U]\right|=K_{i}(U) L_{i}(U)\left|G_{i}\right| p^{2}\left\{\begin{array}{l}
>c_{i}\left|G_{i}\right| p^{2}>c_{\beta}^{*} 2^{\alpha-1} \\
\left.=K_{i}\left|D_{i}(U)\right| p>(R / 2)\left|D_{i}(U)\right| p\right)
\end{array}\right.
$$

So, with $\alpha$ and $\beta$ ranging over integers and positive integers respectively, (18) will follow from

$$
\begin{equation*}
\sum \mathrm{C}_{R / 2}^{*}\left(2^{\alpha}, T_{\alpha, \beta}\right)=O\left(R^{-2}\right) \tag{27}
\end{equation*}
$$

Proof of (27). For $T_{\alpha, \beta}=1$ we bound $\mathrm{C}_{R / 2}^{*}\left(2^{\alpha}, T_{\alpha, \beta}\right)$ by $2^{\alpha}$, using the trivial

$$
\begin{equation*}
\mathrm{C}_{J}^{*}(\mu, 1) \leq \mu \tag{28}
\end{equation*}
$$

(since $\{\{x, y\}: x y \in G\}$ itself covers the set in (11)), which—since $T_{\alpha, \beta}=1$ iff $2^{\alpha} \leq 32 R^{-2}(2 / 3)^{\beta-1}$ —bounds the contribution of such pairs to the sum in (27) by

$$
\begin{equation*}
\sum_{\beta} \sum_{\alpha: T_{\alpha, \beta}=1} 2^{\alpha} \leq 64 R^{-2} \sum_{\beta}(2 / 3)^{\beta-1}=3 \cdot 64 R^{-2} \tag{29}
\end{equation*}
$$

For $T_{\alpha, \beta}>1$ we use Theorem 2.2 with $T=T_{\alpha, \beta}\left(=c_{\beta}^{*} 2^{\alpha-1}\right), \mu=2^{\alpha}, J=R / 2$, and (thus)

$$
c=T /\left(\mu J^{2}\right)=c_{\beta}^{*} /\left(2 J^{2}\right)=(3 / 2)^{\beta-1} / 8
$$

Note that (16) gives $J \geq 8 e$ and $c \geq 256 e / J$, so (12) holds.
(Here, finally, we see the role of $C_{J}^{*}$ mentioned in the paragraph following (11): for a given $\beta$ we may be summing over many $\alpha$ 's with the same $T / \mu$, so it's crucial that we have cost bounds that shrink with $T$ when this ratio is held constant.)

For each integer $s \geq 0$ let $\mathcal{T}_{s}=\left\{(\alpha, \beta): T_{\alpha, \beta} \in\left(2^{s}, 2^{s+1}\right]\right\}$. For each $\beta \geq 1$ there is a unique $\alpha$ such that $(\alpha, \beta) \in \mathcal{T}_{s}$, and every $(\alpha, \beta)$ with $T_{\alpha, \beta}>1$ is in some $\mathcal{T}_{s}$. Let $f(s)=\min \left\{J_{1}^{-2}, J_{1}^{-2^{s / 2-4}}\right\}$. Then for fixed $s$, we have (see (13))

$$
\begin{equation*}
\sum_{(\alpha, \beta) \in \mathcal{T}_{s}} \mathrm{C}_{J}^{*}\left(2^{\alpha}, T_{\alpha, \beta}\right) \leq \sum_{\beta} 32 c^{-1} f(s)=\sum_{\beta} 256\left(\frac{2}{3}\right)^{\beta-1} f(s)<3 \cdot 256 f(s) \tag{30}
\end{equation*}
$$

and summing over all $s$ we get

$$
\begin{equation*}
\sum_{T_{\alpha, \beta}>1} \mathrm{C}_{J}^{*}\left(2^{\alpha}, T_{\alpha, \beta}\right)<\sum_{s \geq 0} 768 f(s)=\sum_{s \geq 0} 768 \min \left\{J_{1}^{-2}, J_{1}^{-2^{s / 2-4}}\right\}=O\left(J_{1}^{-2}\right) \tag{31}
\end{equation*}
$$

Finally, combining (31) and (29) gives (27).

## 4. Proof of Theorem 2.2

Aiming for simplicity, we just bound the cost in (13) assuming

$$
T=2^{2 k+3}
$$

for some positive integer $k$ and

$$
\begin{equation*}
c=T /\left(\mu J^{2}\right) \geq 64 e / J \tag{32}
\end{equation*}
$$

showing that in this case

$$
\begin{equation*}
\mathrm{C}_{J}^{*}(\mu, T) \leq 8 c^{-1} J_{1}^{-2^{k-1}-1} \tag{33}
\end{equation*}
$$

Before proving this, we show it implies Theorem 2.2, which, since $\mathrm{C}_{J}^{*}(\mu, t)$ is decreasing in $t$, just requires showing that the r.h.s. of (13) bounds $\mathrm{C}_{J}^{*}\left(\mu, T_{0}\right)$ for some $T_{0} \leq T$.

If $T<32$ this follows from the trivial (28), since $\mu=T /\left(c J^{2}\right)<32 c^{-1} J_{1}^{-2}$, matching the bound in (13). Suppose then that $T \geq 32$ and let $T_{0}=c_{0} J^{2} \mu$ be the largest integer not greater than $T$ of the form $2^{2 k+3}$ (with positive integer $k$ ). We then have $c_{0}>c / 4$ (supporting (32)) and $2^{k-1}>\sqrt{T_{0}} / 8>\sqrt{T} / 16$, and it follows that the bound on $\mathrm{C}_{J}^{*}\left(\mu, T_{0}\right)$ given by (33) is less than the bound in (13).

Proof of (33). We have $|G| p^{2} \leq \mu, T=2^{2 k+3}$ ( $=c J^{2} \mu$ with $J$ as in (12) and $c$ as in (32)), and, with

$$
\begin{equation*}
\mathcal{U}:=\left\{U \subseteq V:|G[U]|>\max \left\{T, J\left|D_{G}(U)\right| p\right\}\right\} \tag{34}
\end{equation*}
$$

want to show that $\mathrm{C}^{*}(\mathcal{U})$ is no more than the bound in (33).
Here, finally, we come to specification of a cover, $\mathcal{G}$. Each member of $\mathcal{G}$ will be a disjoint union of stars (a.k.a. a star forest), where for present purposes a star at $v$ in $W(\subseteq V)$ is some $\{v\} \cup S \subseteq W$ with $S \subseteq N_{G}(v)$. (Where convenient we will also refer to this as the "star $(v, S) . "$ ) We say such a star is good if

$$
\begin{equation*}
|S| \geq J d_{v} p / 4 \tag{35}
\end{equation*}
$$

Given a positive integer $L$, we define

$$
\begin{equation*}
L^{v}=\max \left\{L,\left\lceil J d_{v} p / 4\right\rceil\right\} \tag{36}
\end{equation*}
$$

and say a $\operatorname{star}(v, S)$ is $L$-special if $|S|=L^{v}$.

For positive integers $b$ and $L$, let $\mathcal{G}(b, L)\left(\subseteq 2^{V}\right)$ consist of all disjoint unions of $b L$-special stars in $G$. We will specify a particular collection $\mathcal{C}$ of pairs $(b, L)$ and set

$$
\mathcal{G}=\cup\{\mathcal{G}(b, L):(b, L) \in \mathcal{C}\}
$$

Theorem 2.2 is then given by the following two assertions.
Claim 4.1. $\mathcal{G}$ covers $\mathcal{U}$.
Claim 4.2. $\mathrm{C}(\mathcal{G})$ is at most the bound in (33).

Set (with $i \in[k]$ throughout) $L_{i}=2^{i-1}$ and

$$
\begin{equation*}
\delta_{i}=\max \left\{2^{-(i+2)}, 2^{i-k-3}\right\} \geq 1 /\left(8 L_{i}\right) \tag{37}
\end{equation*}
$$

and notice that

$$
\begin{equation*}
\sum \delta_{i} \leq \sum 2^{-(i+2)}+\sum 2^{i-k-3} \leq 1 / 2 \tag{38}
\end{equation*}
$$

Let

$$
\begin{equation*}
b_{i}=\delta_{i} 4^{-i} T \geq 2^{k-i} \tag{39}
\end{equation*}
$$

Finally, set

$$
\mathcal{C}=\left\{\left(b_{i}, L_{i}\right): i \in[k]\right\} .
$$

Proof of Claim 4.1. We are given $U \in \mathcal{U}$ and must show it contains a member of $\mathcal{G}$. Let $U_{0}=U$ and for $j=1, \ldots$ until no longer possible do: let $\left(v_{j}, S_{j}\right)$, with $S_{j}=N_{G}\left(v_{j}\right) \cap U_{j-1}$, be a largest good star in $U_{j-1}$, and set $d_{j}=\left|S_{j}\right|$ and $U_{j}=U_{j-1} \backslash\left(\left\{v_{j}\right\} \cup S_{j}\right)$.

The passage from $U_{j-1}$ to $U_{j}$ deletes at most $d_{j}^{2}$ edges that contain vertices of $S_{j}$ of $U_{j-1}$-degree at most $d_{j}$; any other edge deleted in this step contains $u \in S_{j}$ with $U_{j-1}$-degree less than $J d_{u} p / 4$ (or $u$, having $U_{j-1}$-degree greater than $d_{j}$, would have been chosen in place of $v_{j}$ ); and of course each vertex $u$ of the final $U_{j}$ has $U_{j}$-degree less than $J d_{u} p / 4$. We thus have

$$
|G[U]| \leq \sum_{j} d_{j}^{2}+\sum_{v \in U} J d_{v} p / 4 \leq \sum_{j} d_{j}^{2}+|G[U]| / 2
$$

(using the second bound in (34)), so

$$
\begin{equation*}
\sum_{j} d_{j}^{2} \geq|G[U]| / 2 \geq T / 2 \tag{40}
\end{equation*}
$$

Set

$$
B_{i}= \begin{cases}\left\{j: d_{j} \in\left[2^{i-1}, 2^{i}\right)\right\} & \text { if } i \in[k-1] \\ \left\{j: d_{j} \geq 2^{k-1}\right\} & \text { if } i=k\end{cases}
$$

(It may be worth noting that, while the $d_{j}$ 's are decreasing, the degrees corresponding to $B_{i}$ increase with i.) In view of (40), either $\left|B_{k}\right| \geq 1$ or (using (38))

$$
\sum_{i \in[k-1]}\left|B_{i}\right| 4^{i} \geq T / 2 \geq \sum_{i \in[k-1]} \delta_{i} T=\sum_{i \in[k-1]} b_{i} 4^{i}
$$

Since $b_{k}=1$, it follows that for some $i \in[k]$ we have

$$
\begin{equation*}
\left|B_{i}\right| \geq b_{i} \tag{41}
\end{equation*}
$$

On the other hand, since $j \in B_{i}$ implies $\left|S_{j}\right| \geq L_{i}^{v}\left(=\max \left\{L_{i},\left\lceil J d_{v} p / 4\right\rceil\right\}\right)$, the set $\bigcup\left\{S_{j} \cup\left\{v_{j}\right\}: j \in B_{i}\right\}$ contains some $W \in \mathcal{G}\left(b_{i}, L_{i}\right)(\subseteq \mathcal{G})$ whenever $i$ is as in (41). This completes the proof of Claim 4.1.

Proof of Claim 4.2. We first bound the costs, say $\mathrm{C}(b, L)$, of the collections $\mathcal{G}(b, L)$. Given $(b, L)$, set

$$
q_{v}=p\left(\frac{e d_{v} p}{L^{v}}\right)^{L^{v}}
$$

Then $q_{v}$ bounds the total cost of the set of $L$-special stars at $v\left(\right.$ using $\left.\binom{d_{v}}{L^{v}} \leq\left(e d_{v} / L^{v}\right)^{L^{v}}\right)$, and it follows that

$$
\begin{equation*}
\mathrm{C}(b, L) \leq \sum\left\{\prod_{v \in B} q_{v}: B \in\binom{V}{b}\right\} . \tag{42}
\end{equation*}
$$

For a given value of $\varphi:=\sum_{v \in V} q_{v}$, the r.h.s. of (42) is largest when the $q_{v}$ 's are all equal (this just uses $\left.x y \leq[(x+y) / 2]^{2}\right)$, whence

$$
\begin{equation*}
\mathrm{C}(b, L) \leq\binom{|V|}{b}\left(\frac{\varphi}{|V|}\right)^{b} \leq\left(\frac{e \varphi}{b}\right)^{b} \tag{43}
\end{equation*}
$$

Recalling (36), we have

$$
q_{v} \leq d_{v} p^{2} \cdot \frac{e}{L}\left(\frac{4 e}{J}\right)^{L-1}
$$

so (since $|G| p^{2} \leq \mu$ )

$$
\begin{equation*}
\varphi \leq 2 \mu \cdot \frac{e}{L}\left(\frac{4 e}{J}\right)^{L-1} \tag{44}
\end{equation*}
$$

Now using (43) and (44), recalling that $T=c J^{2} \mu, L_{i}=2^{i-1}, b_{i}=\delta_{i} 4^{-i} T=\delta_{i} T /\left(4 L_{i}^{2}\right)$ and $J_{1}=J /(8 e)$, and for the moment omitting the subscript $i$, we have (with the final inequality (45) justified below)

$$
\begin{align*}
\mathrm{C}(b, L) & \leq\left[\frac{2 e^{2} \mu}{L} \frac{4 L^{2}}{\delta T}\left(\frac{4 e}{J}\right)^{L-1}\right]^{b} \\
& =\left[8 e^{2} L \cdot \frac{1}{c J^{2} \delta}\left(\frac{4 e}{J}\right)^{L-1}\right]^{b} \\
& =\left[c^{-1} \frac{L}{2 \delta}\left(\frac{4 e}{J}\right)^{L+1}\right]^{b} \\
& \leq\left[\frac{c}{4} \cdot J_{1}^{L+1}\right]^{-b} \tag{45}
\end{align*}
$$

For (45), or the equivalent

$$
\begin{equation*}
2^{L+4} \delta \geq L \tag{46}
\end{equation*}
$$

it is enough to show $2^{L+1} \geq L^{2}$ (since $\delta \geq 1 /(8 L)$; see (37)), which is true for positive integer $L$.

Finally, returning to Claim 4.2 (and recalling that $L$ and $b$ in the display ending with (45) are really $L_{i}$ and $b_{i}$ ), we have

$$
\begin{equation*}
\mathrm{C}(\mathcal{G})=\sum_{i=1}^{k} \mathrm{C}\left(b_{i}, L_{i}\right) \leq \sum_{i=1}^{k}\left[\frac{c}{4} \cdot J_{1}^{L_{i}+1}\right]^{-b_{i}} \tag{47}
\end{equation*}
$$

We use $b_{i} \geq 2^{k-i}$ (see (39)) and $L_{i}=2^{i-1}$ to bound the r.h.s. of (47) by

$$
\begin{equation*}
\sum_{i=1}^{k}\left[\frac{c J_{1}^{2^{i-1}+1}}{4}\right]^{-2^{k-i}}=\sum_{i=1}^{k} J_{1}^{-2^{k-1}}\left[\frac{c J_{1}}{4}\right]^{-2^{k-i}}=\sum_{j=0}^{k-1}\left(\frac{c}{4} J_{1}^{2^{k-1}+1}\right)^{-1}\left[\frac{c J_{1}}{4}\right]^{1-2^{j}} \tag{48}
\end{equation*}
$$

but, since $c J_{1} / 4 \geq 2$ (using (32) and $J_{1}=J /(8 e)$ ), the last expression in (48) is less than the bound $8 c^{-1} J_{1}^{-2^{k-1}-1}$ in (33).

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