SHEET 5

1 Entangled strategy for a classical channel

Define the classical channel $\{W(y|x)_{x\in\mathcal{X},y\in\mathcal{Y}}\}$ with $\mathcal{X}=\{1,2,3,4\}$ and $\mathcal{Y}=\binom{\mathcal{X}}{2}$ is the subsets of $\mathcal{X}$ of size 2. We let $W(y|x)=\frac{1}{3}$ if $x\in y$ and 0 otherwise. Show that $p_{\mathrm{succ}}(W,2)=\frac{5}{6}$. Give a quantum strategy that beats this value. *Hint:* You might want to take the shared state to be $\frac{1}{2}\sum_{i=1}^{4}|i\rangle\otimes|i\rangle$ and the encoding POVMs of the form $E(x|0)=|x\rangle\langle x|$ and $E(x|1)=U|x\rangle\langle x|U^*$ where $U=\frac{1}{\sqrt{3}}\begin{pmatrix}0&-1&-1&1\\1&0&1&1\\-1&1&0&1\\-1&-1&1&0\end{pmatrix}$.

2 Analyzing the greedy algorithm for submodular functions

1. (Maximization of submodular functions)
   A function $f:2^{\mathcal{X}}\to\mathbb{R}_+$ taking as input a subset $S\subseteq X$ that has the following property.

$$f(S\cup T)+f(S\cap T)\leq f(S)+f(T)\;. \tag{1}$$

   It is said to be monotone if $f(S)\leq f(T)$ whenever $S\subseteq T$.

   (a) Show that an equivalent definition for submodular function is that $f(T\cup\{j\})-f(T)\leq f(S\cup\{j\})-f(S)$ for any $S\subseteq T$ and any $j\in\mathcal{X}-T$. This can be interpreted as a "diminishing returns" property.

   (b) (Remark: this question is independent of the other questions) Let $Z_1,\ldots,Z_n$ be a family of random variables. For a subset $S\subseteq\{1,\ldots,n\}$, let Z_S be the collection of random variables $\{Z_i\}_{i\in S}$. Show that $f(S)=H(Z_S)$ is a submodular and monotone function.

   (c) Let f be a submodular, monotone and nonnegative function and consider the following optimization problem $\max_{S\subseteq\mathcal{X},|S|=M}f(S)$. Let S^* of size M be such that $f(S^*)=\max_{S\subseteq\mathcal{X},|S|=M}f(S)$. Computing such an S^* is computationally hard in general. But there is a natural greedy algorithm for this problem: start with $S_0=\emptyset$, then choose $S_{i+1}=S_i\cup\arg\max\{f(S_i\cup\{j\}):j\in\mathcal{X}-S_i\}$. Show that $f(S^*)\leq f(S_i)+M(f(S_{i+1})-f(S_i))$.

   (d) Prove that $f(S^*)-f(S_{i+1})\leq(1-\frac{1}{M})(f(S^*)-f(S_i))$.

   (e) Conclude that the greedy algorithm gives a approximation factor of $1-\frac{1}{e}$ for this problem.

2. (Channel coding as a submodular optimization problem) Let $p_{\mathrm{succ}}(W,M)$ be the largest average success probability of a code for M messages:

$$p_{\mathrm{succ}}(W,M)=\max_{C\subseteq\mathcal{X},|C|\leq M}\frac{1}{M}f_W(C)\;, \tag{12}$$

   where $f_W(C)=\sum_{i=1}^{M}\max_{x\in C}W(y|x)$.

   (a) Show that f_W is submodular and monotone.