1 Entangled strategy for a classical channel

Define the classical channel \( \{W(y|x)_{x \in \mathcal{X}, y \in \mathcal{Y}}\} \) with \( \mathcal{X} = \{1, 2, 3, 4\} \) and \( \mathcal{Y} = \binom{\mathcal{X}}{2} \) is the subsets of \( \mathcal{X} \) of size 2. We let \( W(y|x) = \frac{1}{3} \) if \( x \in y \) and 0 otherwise. Show that \( p_{\text{succ}}(W, 2) = \frac{5}{9} \). Give a quantum strategy that beats this value. \textit{Hint:} You might want to take the shared state to be \( \frac{1}{2} \sum_{i=1}^{4} |i \rangle \otimes |i \rangle \) and the encoding POVMs of the form 
\[ E(x|0) = |x \rangle \langle x | \text{ and } E(x|1) = U|x \rangle \langle x |U^* \text{ where } U = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}. \]

2 Analyzing the greedy algorithm for submodular functions

1. (Maximization of submodular functions)
   A function \( f : 2^\mathcal{X} \to \mathbb{R}_+ \) taking as input a subset \( S \subseteq \mathcal{X} \) that has the following property.
   \[
   f(S \cup T) + f(S \cap T) \leq f(S) + f(T) .
   \]
   It is said to be monotone if \( f(S) \leq f(T) \) whenever \( S \subseteq T \).
   (a) Show that an equivalent definition for submodular function is that \( f(T \cup \{j\}) - f(T) \leq f(S \cup \{j\}) - f(S) \) for any \( S \subseteq T \) and any \( j \in \mathcal{X} - T \). This can be interpreted as a “diminishing returns” property.
   (b) (Remark: this question is independent of the other questions) Let \( Z_1, \ldots, Z_n \) be a family of random variables. For a subset \( S \subseteq \{1, \ldots, n\} \), let \( Z_S \) be the collection of random variables \( \{Z_i\}_{i \in S} \). Show that \( f(S) = H(Z_S) \) is a submodular and monotone function.
   (c) Let \( f \) be a submodular, monotone and nonnegative function and consider the following optimization problem \( \max_{S \subseteq \mathcal{X}, |S| = M} f(S) \). Let \( S^* \) of size \( M \) be such that \( f(S^*) = \max_{S \subseteq \mathcal{X}, |S| = M} f(S) \). Computing such an \( S^* \) is computationally hard in general. But there is a natural greedy algorithm for this problem: start with \( S_0 = \emptyset \), then choose \( S_{i+1} = S_i \cup \arg\max \{f(S_i \cup \{j\}) : j \in \mathcal{X} - S_i\} \). Show that \( f(S^*) \leq f(S_i) + M(f(S_{i+1}) - f(S_i)) \).
   (d) Prove that \( f(S^*) - f(S_{i+1}) \leq (1 - \frac{1}{M})(f(S^*) - f(S_i)) \).
   (e) Conclude that the greedy algorithm gives an approximation factor of \( 1 - \frac{1}{e} \) for this problem.

2. (Channel coding as a submodular optimization problem) Let \( p_{\text{succ}}(W, M) \) be the largest average success probability of a code for \( M \) messages:
   \[
   p_{\text{succ}}(W, M) = \max_{C \subseteq \mathcal{X}, |C| \leq M} \frac{1}{M} f_W(C) ,
   \]
   where \( f_W(C) = \sum_{i=1}^{M} \max_{y \in C} W(y|x) \).
   (a) Show that \( f_W \) is submodular and monotone.