SHEET 4

1 Relation between maximum and average error probability

Show that for any M-code (E, D) for a channel $\{W_x\}$, there exists an $\lfloor M/2 \rfloor$-code (E', D') for the same channel with $p_{\mathrm{err,max}}(E', D') \leq 2p_{\mathrm{err}}(E, D)$.

2 Channel capacities

1. For a classical-quantum channel $\{W_x\}_{x\in\mathcal{X}}$, we defined the function $f(n) = \sup_{P_{X^n}} I(X^n : B^n)_\rho$ where $\rho_{X^nB^n} = \sum_{x^n\in\mathcal{X}^n} P_{X^n}(x^n)|x^n\rangle\langle x^n|_{X^n} \otimes W_{x^n}$ and $W_{x^n} = W_{x_1} \otimes W_{x_2} \cdots W_{x_n}$. Show that the function f is superadditive.
2. Let $p \in [0, 1]$ and consider the channel W with input alphabet $\mathcal{X} = \{0, 1\}$ defined by $W_0 = (1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1|$ and $W_1 = p|0\rangle\langle 0| + (1 - p)|1\rangle\langle 1|$ (this is called the binary symmetric channel). Compute the capacity $\mathrm{C}(W)$.
3. Let $e \in [0, 1]$ and consider the channel W with input alphabet $\mathcal{X} = \{0, 1\}$ defined by $W_0 = (1 - e)|0\rangle\langle 0| + e|E\rangle\langle E|$ and $W_1 = (1 - e)|1\rangle\langle 1| + e|E\rangle\langle E|$, where $|E\rangle$ is orthogonal to both $|0\rangle$ and $|1\rangle$ (this is called the binary erasure channel). Compute the capacity $\mathrm{C}(W)$.
4. Prove that the von Neumann entropy is strictly concave, i.e., for any distribution $\{p(x)\}$ with support S we have $H(\sum_{x\in S} p(x)W_x) \geq \sum_{x\in S} p(x)H(W_x)$ with equality if and only if $W_x = W$ for all $x \in S$. (You may want to use a question from Sheet 3 for this). Conclude that a classical-quantum channel $\{W_x\}_{x\in\mathcal{X}}$ satisfies $\mathrm{C}(W) = 0$ if and only if there exists W such that $W_x = W$ for all $x \in \mathcal{X}$.

3 Properties of the Holevo information

Recall that the Holevo information $\chi(\{p(x), \sigma_A^x\})$ of an ensemble $\{p(x), \sigma_A^x\}$ is defined as $I(X : A)_\rho$ where $\rho_{XA} = \sum_x p(x)|x\rangle\langle x|_X \otimes \sigma_A^x$.

1. Show that $\chi(\{p(x), \sigma_A^x\}) = H(\sum_x p(x)\sigma_A^x) - \sum_x p(x)H(\sigma_A^x)$
2. Show that χ is concave in $\{p(x)\}$ (for fixed $\{\sigma_A^x\}$) and convex in $\{\sigma_A^x\}$ (for fixed $\{p(x)\}$).
3. The Holevo information of a channel $\mathcal{W} : \mathrm{L}(A) \to \mathrm{L}(B)$ is defined as the supremum over all ensembles $\{p(x), \sigma_A^x\}$ of the Holevo information $\chi(\{p(x), \mathcal{W}(\sigma_A^x)\})$. Show that we may restrict the optimization to ensembles where σ_A^x are all pure states.