SHEET 3

1 Classical relative entropy

1. Show that if a function $f : (0, \infty) \to \mathbb{R}$ is convex, then its perspective $g : (0, \infty) \times (0, \infty) \to \mathbb{R}$ defined by $g(x, t) = tf(\frac{x}{t})$ is convex.

2. Deduce that the classical relative entropy defined by $D(P\|Q) = \sum_{x\in\mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}$ is jointly convex.

3. Show that for probability distributions P, Q, $D(P\|Q) \geq 0$ with equality if and only if $P = Q$.

2 Quantum relative entropy: properties and consequences of data processing

1. Positivity of the quantum relative entropy for states:
   (a) Let $\rho, \sigma \in \mathrm{S}(A)$, show that $D(\rho\|\sigma) \geq 0$.
   (b) Show that $D(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$ (you might want to use the classical case).

2. Unital channels increase entropy: Let $\mathcal{N} : \mathrm{L}(A) \to \mathrm{L}(B)$ be a unital quantum channel. Show that $H(B)_{\mathcal{N}(\rho)} \geq H(A)_\rho$.

3. Relative entropy of classical-quantum states. Let $\rho_{XB}, \sigma_{XB} \in \mathrm{S}(X \otimes B)$ be classical-quantum states of the form $\rho_{XB} = \sum_{x\in\mathcal{X}} P(x)|x\rangle\langle x|_X \otimes \rho^x_B$ and $\sigma_{XB} = \sum_{x\in\mathcal{X}} Q(x)|x\rangle\langle x|_X \otimes \sigma^x_B$ for some density operators ρ^x and σ^x. Show that
$$D(\rho_{XB}\|\sigma_{XB}) = D(P\|Q) + \sum_{x\in\mathcal{X}} P(x)D(\rho^x_B\|\sigma^x_B).$$

4. Mutual information and relative entropy: Let $\rho_{AB} \in \mathrm{S}(A \otimes B)$. Show that
$$I(A : B)_\rho = \min_{\omega_A\in\mathrm{S}(A),\sigma_B\in\mathrm{S}(B)} D(\rho_{AB}\|\omega_A \otimes \sigma_B).$$